Inverse Mean Curvature Flow And The Proof Of The Riemannian Penrose Inequality

Brian Allen

Advised by Dr. Alexandre Freire
Department of Mathematics
University of Tennessee, Knoxville

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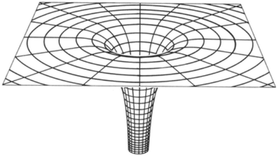
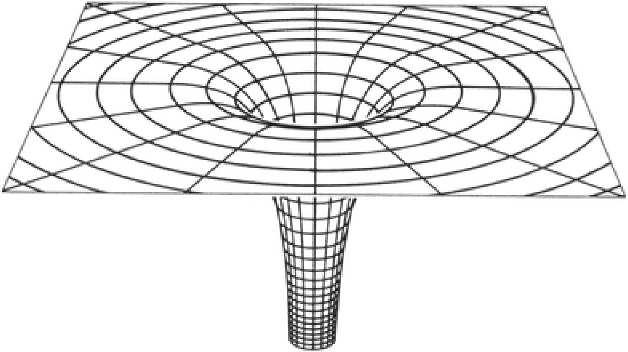


Introduction To Inverse Mean Curvature Flow



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- Assume that φ satisfies the following equation

$$\frac{\partial \varphi}{\partial t}(p, t) = \frac{\nu(p, t)}{H(p, t)} \quad (1)$$

where $p \in M$, $t \in [0, T)$ and $\nu(p, t)$ is the outward pointing unit normal vector to $\varphi_t(M)$. **Note:** $H > 0$

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We say that $M_t := \varphi(M, t)$ is a solution of IMCF.

Star-shaped Hypersurfaces

We say that a hypersurface $M^n \subset \mathbb{R}^{n+1}$ is star-shaped if it can be written as a graph over a sphere S^n ($w(x) = \langle \nu, x \rangle > 0$ for all $x \in \Sigma$).

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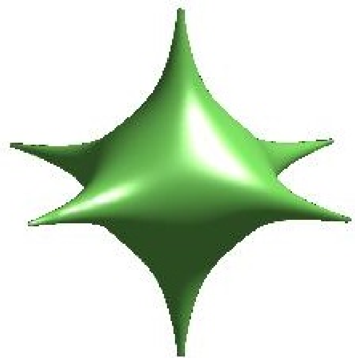

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which has the solution

$$r(t) = r_0 e^{t/n}$$

defined on the time interval $[0, \infty)$.

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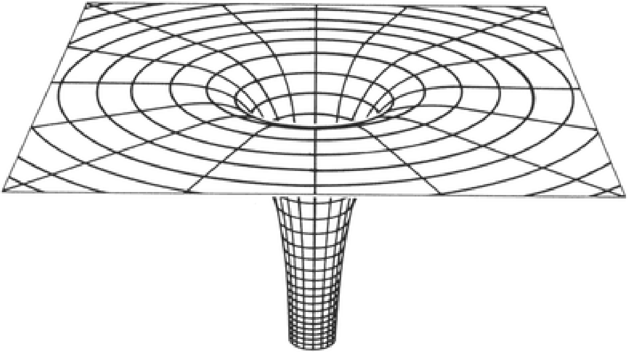
Then the rescaled embeddings $\tilde{\varphi}(t) = e^{-t/n}\varphi(t)$ converge to a smooth embedding $\tilde{\varphi}_\infty$ so that $\tilde{\varphi}_\infty(M) = S^n_{r_\infty} \subset \mathbb{R}^{n+1}$ where $r_\infty = \left(\frac{|M_0|}{|S^n|}\right)^{1/n}$.

Penrose Inequality



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$$|g_{ij} - \delta_{ij}| \leq \frac{C}{|x|}, \quad |g_{ij,k}| \leq \frac{C}{|x|^2}, \quad \bar{Rc} \geq -\frac{Cg}{|x|^2}$$

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- The scalar curvature $\bar{R}$ of (M, g) satisfies

$$\int_M |\bar{R}| d\mu < \infty$$

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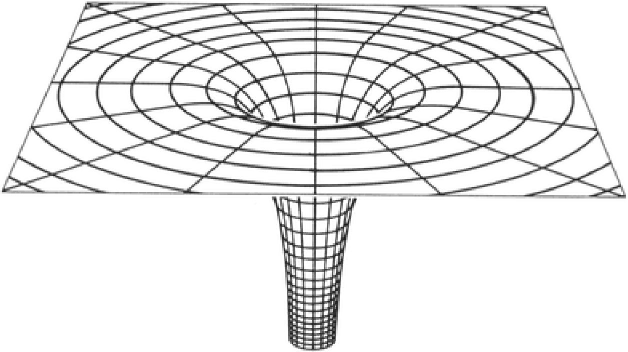
Equality holds iff M is isometric to one-half of the spatial Schwarzschild manifold.

Weak Solutions Of IMCF



Why Are Weak Solutions Necessary?

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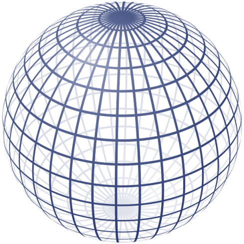

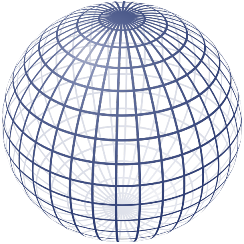

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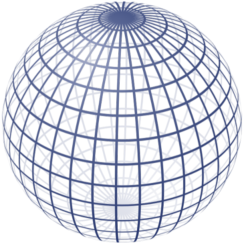
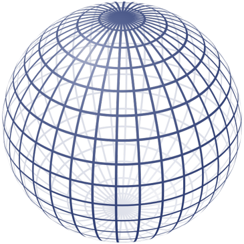
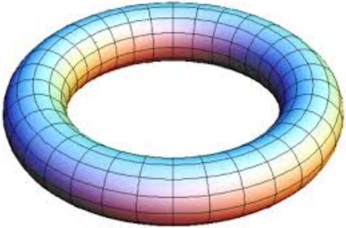




Thin Torus:



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Let $u : \Omega \subset M \to \mathbb{R}$ be a function which satisfies the following degenerate elliptic PDE

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To get weak solutions the idea is to set $\Omega = M \setminus \bar{E}_0$ and minimize a certain functional.

Level Set Solutions II

Then if we regularize the degenerate PDE we find

$$\operatorname{div}\left(\frac{\nabla u_\epsilon}{\sqrt{|\nabla u_\epsilon|^2+\epsilon^2}}\right)=\sqrt{|\nabla u_\epsilon|^2+\epsilon^2} \qquad \tilde{\Sigma}_t^\epsilon := \text{graph}\left(\frac{u_\epsilon}{\epsilon}-\frac{t}{\epsilon}\right)$$

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- One can show, for a subsequence ϵ_i, that $\tilde{\Sigma}_t^{\epsilon_i} \to \Sigma_t \times \mathbb{R}$ as $\epsilon_i \to 0$.

Variational Level Set Solutions

- Define weak solutions to be (self) minimizers of the following functional

$$J_u^K(v) = \int_K |\nabla v| + v|\nabla u| dx$$

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- Bounded sequences of solutions defined in this way have a compactness property.

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Let $\Omega \subset M$ be open, then we say that E is a **minimizing hull** if E minimizes area on the outside in Ω

$$|\partial^* E \cap K| \leq |\partial^* F \cap K|$$

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- For $t \geq 0$, $|\partial E_t| = |\partial E_t^+|$, provided that E_0 is a minimizing hull.

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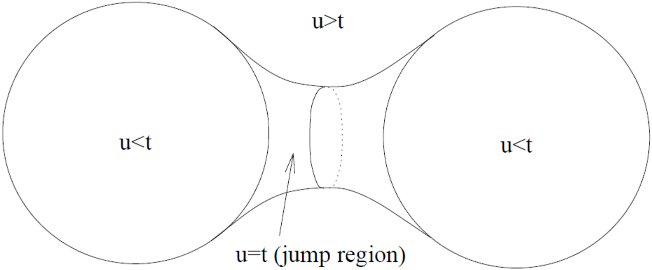
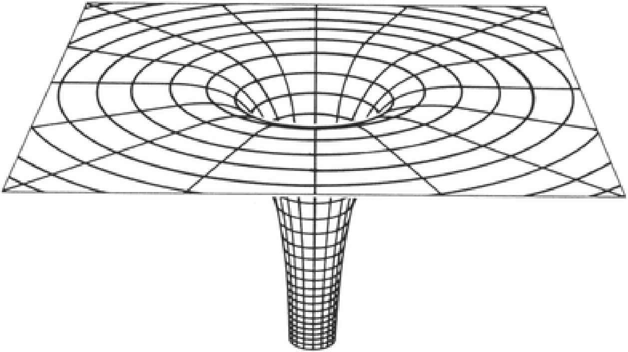


** Picture source Huisken and Ilmanen*

Monotonicity Of Hawking Mass



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- $m_H(\Sigma_t)$ is non-decreasing along IMCF.
- This proof method was proposed by Geroch and further developed by Jang and Wald when the flow remains smooth.

Important Equations For Monotonicty

We will need the following evolution equations under IMCF

$$\left(\partial_t - \frac{1}{H^2}\Delta\right) H = -2\frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H} - \frac{\bar{R}c(\nu,\nu)}{H}$$

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As well as the following consequence of the Gauss equation

$$\sigma_\Sigma = \bar{\sigma}_\Sigma + \lambda_1\lambda_2 = \frac{\bar{R}}{2} - \bar{R}c(\nu,\nu) + \frac{1}{2}(H^2 - |A|^2)$$

Where σ_Σ is the sectional curvature of $T_x\Sigma$ in Σ or the Gauss curvature of Σ, $\bar{\sigma}_\Sigma$ is the sectional curvature of $T_x\Sigma$ in M, $\bar{R}$ and $\bar{R}c(\cdot,\cdot)$ are the scalar and ricci curvature of M, and λ_1, λ_2 are the principal curvatures of Σ in M.

Monotonicity Of Hawking Mass

$$\frac{\partial}{\partial t}\int_{\Sigma_t} H^2 d\mu_t = \int_{\Sigma_t} -2H\Delta\left(\frac{1}{H}\right) - 2|A|^2 - 2\bar{R}c(\nu,\nu) + H^2 d\mu_t$$

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Then from $|\Sigma_t|^{1/2} = |\Sigma_0|^{1/2} e^{t/2}$ we see $m_H(\Sigma_t)$ is non-decreasing.

Weak Monotonicity

Heuristically we expect the following when a surface jumps

$$\int_{\partial E_t^+} H^2 d\mu_t \leq \int_{\partial E_t} H^2 d\mu_t \quad |\partial E_t^+| = |\partial E_t|$$

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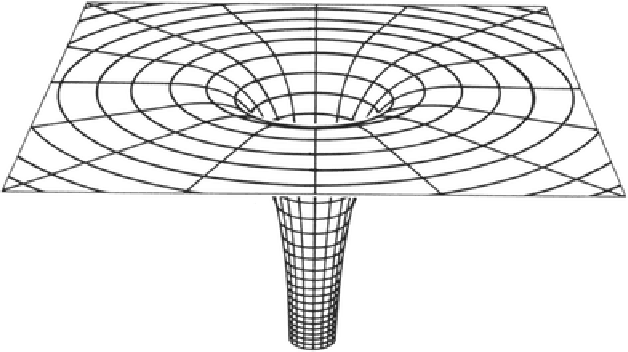
Then weak montonicty follows, with some work, by taking a limit as $\epsilon \to 0$.

Asymptotic Analysis



Weak Convergence To A Sphere

Remember the definition of the ADM mass

$$m := \lim_{r\to\infty} \frac{1}{16\pi} \int_{\partial B_r(0)} (g_{ij,i} - g_{ii,j})\nu^j d\mu$$

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For any surface $\Sigma \subset \mathbb{R}^{n+1}$ we define the eccnetricity

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This follows from rescaling and the estimate $|\nabla u(x)| \leq \frac{C}{|x|}$ for all $|x| \geq R_0$.

Asymptotic Comparison Of Hawking And ADM Mass

- Let M be an asymptotically flat manifold
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Then

$$\left(\sqrt{\frac{|\Sigma_0|}{16\pi}} = m_H(\Sigma_0)\right) \leq \lim_{t \to \infty} m_H(\Sigma_t) \leq m_{ADM}(M)$$

where the part in parenthesis is only true for minimal surfaces Σ_0.

Proof Of Asymptotic Comparison I

Let $r(t)$ be s.t. $|\Sigma_t| = 4\pi r^2$ then a previous slide implies that

$$\frac{1}{r(t)}\Sigma_t \to \partial B_1(0) \text{ in } C^1 \text{ as } t \to \infty \tag{2}$$

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Consider g, the metric of M and the corresponding quantities $H, A, \nu, d\mu$ and δ, the metric of $\mathbb{R}^{n+1}$ and the corresponding quantities $\bar{H}, \bar{A}, \bar{\nu}, d\bar{\mu}$.

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Obtain expressions for the following quantities

$$H - \bar{H} \quad d\mu - d\bar{\mu}$$

in terms of $p_{ij} = g_{ij} - \delta_{ij}$.

Proof Of Asymptotic Comparison II

$$H - \bar{H} = -h^{ik} p_{kl} h^{lj} A_{ij} + \frac{1}{2} H \nu^i \nu^j p_{ij} - h^{ij} \nabla_i p_{jl} \nu^l + \frac{1}{2} h^{ij} \nabla_l p_{ij} \nu^l$$
$$\pm C|p||\nabla p| \pm C|p|^2 |A|$$

$$\bar{H}^2 (d\mu - d\bar{\mu}) = \left(\frac{1}{2} H^2 h^{ij} p_{ij} \pm |p|^2 |A|^2 \pm C|\nabla p|^2 \right) d\mu$$

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where h is the metric of Σ_t.

Rewrite part of $m_H(\Sigma_t)$ in the following way

$$\int_{\Sigma_t} H^2 d\mu_t = \int_{\Sigma_t} \bar{H}^2 d\bar{\mu}_t + \bar{H}^2(d\mu_t - d\bar{\mu}_t) + 2H(H - \bar{H}) - (H - \bar{H})^2 d\mu_t$$

Proof Of Asymptotic Comparison III

Use the inequality for $\mathbb{R}^3$ that implies

$$\int_{\Sigma} \bar{H}^2 d\bar{\mu} \geq 16\pi$$

to cancel the 16π that shows up in $m_H(\Sigma_t)$.

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Then using the relation

$$\nabla p = \bar{\nabla} p \pm C|p||\nabla p|$$

one can write $m_H(\Sigma_t)$ in the following way

$$32\pi m_H(\Sigma_t) \leq 2\int_{\Sigma_t} (g_{ij,i} - g_{ii,j})\nu^j d\mu_t + \epsilon(t)$$

where $\epsilon(t)$ represents the error and $\epsilon(t) \to 0$.

The End!

Questions?

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