Mass functional and mass-angular momenta inequality for $U(1)^2$-invariant black holes

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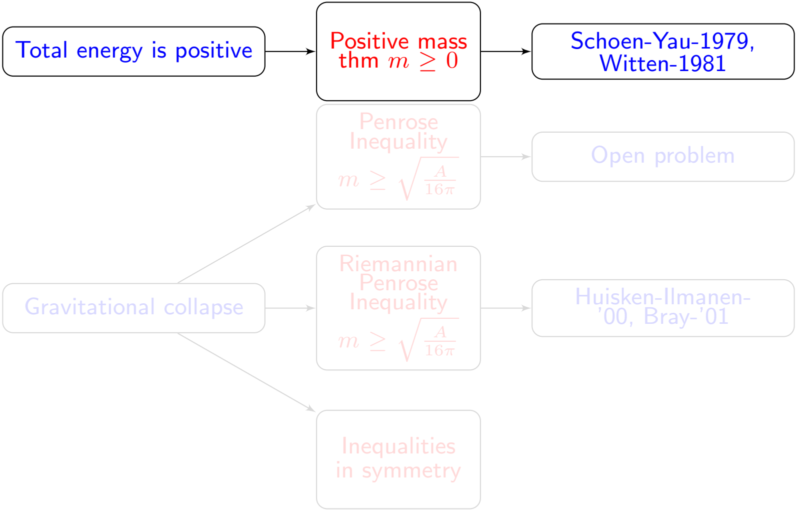
(joint work with Dr. Hari K. Kunduri)

Motivation: Geometric Inequalities



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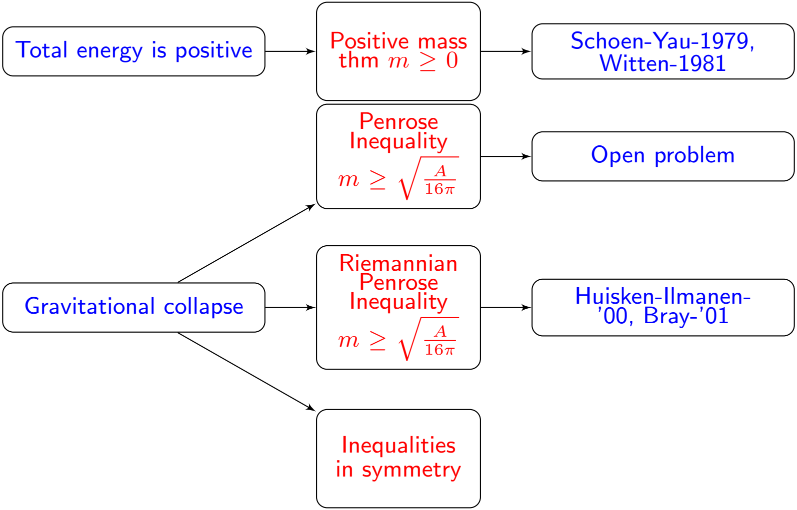

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Inequalities in symmetry

- **Mass angular momentum**
  - $m \geq \sqrt{J}$ (Dain-'07)
  - charges (Chrusciel-Costa-'09)
  - multiple ends(Chruciel-Weinstein-'08)
  - Higher D?
- **Area angular momentum**
  - $A > 8\pi |J|$ (Acena,Dain,Clement-10)
  - charges (Clement,Jaramillo,Reiris-'12)
  - cosmo. const. (Clement,Reiris,Simon-'15)
  - Higher D (Hollands-'11)

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Outline

- Mass and symmetry
- Review 4D mass-angular inequality
- Initial data for 5D BHs
- Main results
- Current and Future projects

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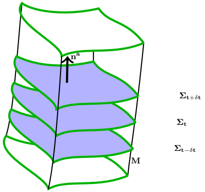
Asymptotically flat and Constraint equation

- Assume (Σ, h) is smooth n-dimensional Riemannian manifold and C is a compact sub-manifold, (Σ, h) is AF if $\Sigma \backslash C$ is diffeomorphic to $\mathbb{R}^n \backslash B^n(0)$ for large $r+$ fall-offs

- Consider spacetime (M, g) foliated by spacelike leaves (slices) (Σ, h_{ab}, K_{ab}), where h_{ab} is induced metric on Σ and K_{ab} is extrinsic curvature tensor



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- Vacuum Einstein equations equivalent to evolution equations for initial data (Σ, h_{ab}, K_{ab})+2 constraint equations on Σ

$$R_h - K^{ab}K_{ab} + \mathrm{Tr}_h\, K = 0 \qquad \text{Hamiltonian Constraint} \tag{1}$$

$$\nabla^b \left[K_{ab} - (\mathrm{Tr}_h\, K)h_{ab}\right] = 0 \qquad \text{Momentum Constraint} \tag{2}$$

Mass formulas

- In GR $\Longrightarrow$ mass has complicated quasi-local definition
  - Komar mass $\Longrightarrow$ AF and stationary sp-t at spacelike infinity

$$\boxed{M_{\mathsf{K}} = -\frac{1}{8\pi}\lim_{r\to\infty}\oint_{S_r} \star d\xi} \quad \leftrightsquigarrow \quad \boxed{\xi \text{ timelike KVF}}$$

  - ADM mass $\Longrightarrow$ AF sp-t and at spacelike infinity

$$\boxed{M_{\mathsf{ADM}} = \frac{1}{16\pi}\lim_{r\to\infty}\oint_{S_r} (\partial_c h_{ac} - \partial_c h_{aa})\, n^c dS}$$

  - Bondi-Sachs mass $\Longrightarrow$ AF sp-t and at null infinity= ADM for stationary sp-t
  - Hawking mass $\Longrightarrow$ AF sp-t and is not necessarily positive
  - Wang and Chrusciel-Herzlich mass $\Longrightarrow$ AH sp-t

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Symmetries on data

- An D dimensional initial data (Σ, h_{ab}, K_{ab}) is called **axisymmetric** if there exist $D-2$ rotational KVFs ϕ^i which generates $U(1)^{D-2}$ isometry group on Riemannian manifold Σ and

$$\mathcal{L}_{\phi^i} h_{ab} = \mathcal{L}_{\phi^i} K_{ab} = 0$$

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Dain's mass functional

- Dain ('05) If $(\Sigma, h_{ab}, \bar{K}_{ab})$ is complete, $t-\phi$ symmetric, maximal, AF data in vacuum with two ends and rescaling

$$h_{ab} = e^{4v}\ \tilde{h}_{ab} \qquad \tilde{h}_{ab} = e^{2q}\left(d\rho^2 + dz^2\right) + \rho^2\, d\phi^2 \longleftarrow \boxed{\text{2 functions}}$$

then

1. $\bar{K}_{ab}$ can be represented by an scalar Y
2. positive definite mass functional is

$$\mathcal{M}(v,Y) = \frac{1}{32\pi}\int_{\mathbb{R}^3}\left(16(\mathrm{d}v)^2 + \rho^{-4}e^{-8v}(\mathrm{d}Y)^2\right)\mathrm{d}\mu_0 \geq 0$$

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4. mass of any axisymmetric data $\geq \mathcal{M} \geq 0$
5. critical points of $\mathcal{M}$ are stationary sp-t

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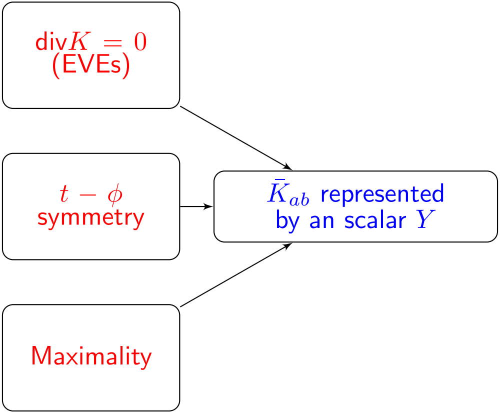
Fundamental assumptions



Orbit space of 3D initial data of BHs

- Σ with isometry group $U(1)$: $\dfrac{\Sigma}{U(1)} \cong \mathcal{B}$ orbit space which
- $\mathcal{B}$ is, two dimensional manifold with boundary, homemoriphic to upper-half plane.

(a) Orbit space as half plane

(b) Orbit space as infinite strip

- Orbit space of data is *unique*!!!

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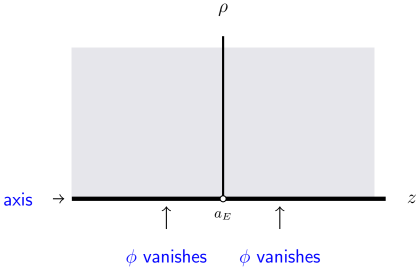
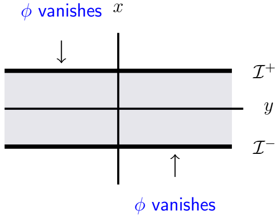


(c) Orbit space as half plane



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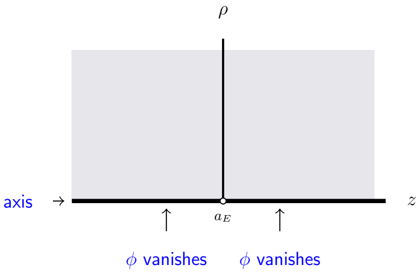

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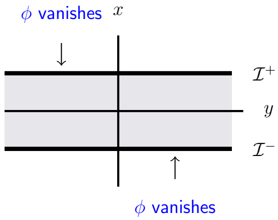

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Results of Dain's mass functional

- Let (Σ, h, K) be a vacuum, AF, maximal, initial data set with appropriate decay. Then

$$m \geq \sqrt{J} = m_{\text{ex}} \qquad \text{(Dain [local-'06,global-'07])} \tag{3}$$

  Moreover, the equality holds if and only if the data are a slice of the extreme Kerr spacetime.

- (Dain '08) $\mathcal{M}$ is a conserved quantity under axisymmetric evolution of Einstein equations.

- (Dain '14) Axisymmetric linear gravitational perturbations of the extreme Kerr black hole is stable.

Can we prove similar inequalities in higher dimensions?

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Restriction of method on dimension

- One of the restriction of Dain's method is existence of $U(1)^{D-2}$-isometry on D-dimensional initial data (Why?)
- If we have an n-dimensional sp-t with $U(1)^{n-3}$-isometry group $\Longrightarrow$ only for $n = 4, 5$ sp-t is AF
- We only consider $D = 4$ dimensional initial data (Σ, h, K)
- Is there any uniqueness theorem for 5D sp-t which represent the minimum mass?

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Topology of 5D black holes

- Consider AF 5D Black hole satisfies DEC: then

Horizon topology theorem (Galloway,Schoen-'06)

$H \cong S^3$ (and quotients), $H \cong S^1 \times S^2$ and connected sums of these two cases

Topology of Stationary Rotating BH (Hollands,Holland,Ishibashi-'10)

1. $H \cong \#k(S^1 \times S^2)\#l$ Lens space, $k \geq 0$, $l \geq 1$
2. $H \cong$ quotients of S^3

Topological censorship (Galloway, Friedmann, Schleich, Witt)

Domain of outer communication is simply connected

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Domain of outer communication (DOC)

- (Orlik,Raymond-'70) If Σ is a 4D simplify connected manifold with action $U(1)^2$ then
$$\Sigma \cong \#n\,(S^2 \times S^2)\#n'\,(\pm\mathbb{CP}^2)$$

- (Hollands,Holland,Ishibashi,'12)If Σ is a 4D simplify connected AF manifold with action $U(1)^2$ then
$$\Sigma \cong \mathbb{R}^4\#n\,(S^2 \times S^2)\#n'\,(\pm\mathbb{CP}^2)$$

Domain of outer communication (DOC)

Theorem(Hollands,Holland,Ishibashi,'12): Consider stationary, rotating vacuum black hole. Then DOC has topology $DOC \cong \mathbb{R} \times \Sigma$ where

$$\Sigma \cong \mathbb{R}^4 \# n\,(S^2 \times S^2) \# n'\,(\pm\mathbb{CP}^2) \setminus B \tag{4}$$

where B is compact, $\partial B \cong H$. B is **'black hole region '**.

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Remark Alaee, Kunduri, Martinez-Pedroza ('13)

Question Does topology of H uniquely determine Σ ?

1. Choice of H does not specify a unique smooth submanifold B in $\mathbb{R}^4 \Longrightarrow$ different B can yield different Σ.
2. They are unique up to homological level.

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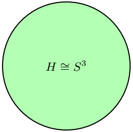
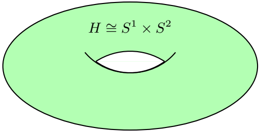
- If $H \cong S^3$ the there is unique possibility for black hole region $B \cong B^4$ (4-ball)
- If $H \cong S^1 \times S^2$ the standard choice $B \cong S^1 \times B^3$
- For multiple black hole one can find standard choice!

Slice of 5D BH: Alaee, Kunduri, Martinez-Pedroza ('13)



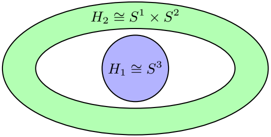
Myers-Perry BH

$\Sigma \cong \mathbb{R} \times S^3$, $\chi = 0$



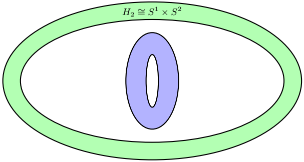
Black Ring

$\Sigma \cong (S^2 \times D^2)\#\mathbb{R}$, $\chi = 1$



Black Saturn

$\Sigma \cong \left(S^2 \times D^2\right)\#\mathbb{R}\#B^4$, $\chi = 0$



Bicycling Black Rings

$\Sigma \cong \#2(S^2 \times D^2)\#\mathbb{R}$, $\chi = 1$

Topology of Σ and Orbit space of 5D BHs

- Σ with isometry group $U(1)^2$: $\dfrac{\Sigma}{U(1)^2} \cong \mathcal{B}$ orbit space
- $\mathcal{B}$ is two dimensional manifold with boundary and corners (Hollands-Yazadjiev ('08) for sp-t).

(g) Orbit space as half plane

(h) Orbit space as infinite strip

- Orbit space structure of Σ is related to topology of H and bubbles

$$\mathcal{B}_{MP} \ncong \mathcal{B}_{BR}$$

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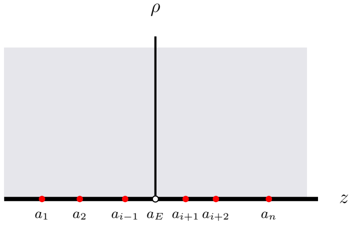
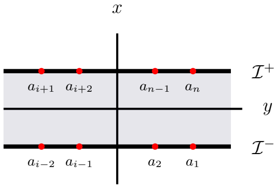


(i) Orbit space as half plane



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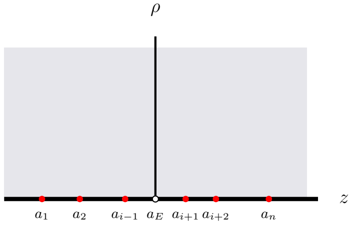

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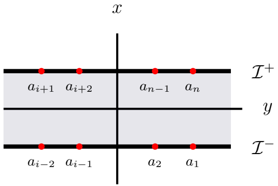

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Result

- Alaee-Kunduri ('14) If $(\Sigma, h_{ab}, \bar{K}_{ab})$ is complete, $t-\phi^i$ symmetric, maximal, AF data in vacuum with rescaling

$$h_{ab} = \Phi^2 \; \tilde{h}_{ab} \qquad \tilde{h}_{ab} = e^{2U}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + \lambda_{ij}\, \mathrm{d}\phi^i \mathrm{d}\phi^j \longleftarrow \boxed{\text{4 functions}}$$

where $\rho^2 = \det \lambda_{ij}$. Then

1. $\bar{K}_{ab}$ is divergence-less and traceless 2 tensor and can be represented by two scalars Y^i and λ_{ij}

$$\boxed{\bar{K}_{ab} \equiv P^1_{(a}\phi^1_{b)} + P^2_{(a}\phi^2_{b)}}$$

$$P \equiv \lambda^{-1} S \quad \text{where} \quad S^i_a \equiv \frac{1}{2\rho^2} i_{\phi^2} i_{\phi^1} \star \mathrm{d}Y^i, \qquad d \star S^i = 0$$

2. If (Σ, h_{ab}, K_{ab}) is a AF, $U(1)^2$-invariant, maximal data in vacuum then

$$K_{ab}K^{ab} \geq \bar{K}_{ab}\bar{K}^{ab}$$

maximal, AF, $t-\phi^i$ symmetric initial data → 6 functions with a constraint → Mass functional $\mathcal{M}$

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4 dimensional mass functional

- Alaee-Kunduri ('14) If $(\Sigma, h_{ab}, \bar{K}_{ab})$ is complete, $t - \phi^i$ symmetric, maximal, AF data in vacuum, then
  - Mass functional is

$$\begin{aligned} \mathcal{M} &= \frac{\pi}{4}\int_{\mathcal{B}}\left(-\frac{\det \mathrm{d}\lambda}{\rho^2} + e^{-6v}\frac{\mathrm{d}Y^t\lambda^{-1}\mathrm{d}Y}{2\rho^2} + 6\,(\mathrm{d}v)^2\right)\,\rho\mathrm{d}\rho\mathrm{d}z \\ &- \frac{\pi}{4}\sum_{\text{rods}}\int_{I_i}\log V_i\,\mathrm{d}z \end{aligned}$$

where $v = \log\Phi$ and I_i are intervals with direction vector v^i on $\partial\mathcal{B}$

$$V_i = \frac{2\sqrt{\rho^2+z^2}\lambda_{ij}v^iv^j}{\rho^2} \qquad \text{where} \quad z\in(a_i, a_{i+1}) \text{ and } \rho\to 0$$

Results of $\mathcal{M}$

1. $\mathcal{M}$ encodes information of rod structure
2. $\mathcal{M}$ is a positive definite functional for a large class(Which?)
3. $\mathcal{M}$ evaluates ADM mass of these class
4. mass of any axisymmetric data $\geq \mathcal{M}$
5. critical points of $\mathcal{M}$ are stationary sp-t
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Question

- Geometric inequalities in vacuum for two known extreme AF solutions

$$m^3 \geq \frac{27\pi}{32}\left(|J_1| + |J_2|\right)^2 \qquad \text{(Myers-Perry)}$$
$$m^3 \geq \frac{27\pi}{4}|J_1|(|J_2| - |J_1|) \qquad \text{(black ring)}$$

- But which one is minimum?

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Uniquness of extreme $5D$ BHs

Theorem [Figueras,Lucietti-'10] Consider a five dimensional, AF, stationary black hole solution of the vacuum Einstein equations, with $U(1)^2$ isometry and a connected degenerate horizon (with non-toroidal sections). There exists at most one such solution with given angular momenta J_1, J_2 and a given interval structure.

- The non-extreme version was proved by Hollands-Yazadjiev ('08)

Extreme class of data-Alaee-Kunduri 15

Definition: The set of *extreme class E* is the collection of data arising from extreme, vacuum, AF, $\mathbb{R} \times U(1)^2$ invariant black holes which consist of triples $u_0 = (v_0, \lambda_0', Y_0)$ where v_0 is a scalar, $\lambda_0' = [\lambda_{ij}]$ is a positive definite 2×2 symmetric matrix, and Y_0 is a column vector with the appropriate bounds.

Main result-Alaee-Kunduri '15

Theorem Let (Σ, h_{ab}, K_{ab}) be an AF, maximal, $U(1)^2$-invariant, vacuum initial data with mass m and fixed angular momenta J_1 and J_2 and fixed orbit space $\mathcal{B}$. Then in small neighborhood

$$m \geq f(J_1, J_2)$$

for some f which depends on the orbit space $\mathcal{B}$. Moreover, $m = f(J_1, J_2)$ in the neighborhood if and only if the data are extreme data.

- Currently we are investigating the global mass-angular momenta for the large class of data(including MP data)

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Open Problems for 5D BHs

- Positivity of mass functional $\mathcal{M}$ for all orbit spaces and extend the global proof to all orbit spaces.
- Mass-angular momenta-charge?
- Mass-angular momenta with multiple ends?
- Is $\mathcal{M}$ a conserved quantity under $U(1)^2$-invariant evolution of Einstein equations?
- Study stability (or instability) of $U(1)^2$-invariant linear gravitational perturbations of the extreme MP and extreme doubly spinning black ring.

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Thank you

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