Surprising consequences of tiny positive cosmological constant in Bondi-Sachs formalism

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Based on: Phys.Rev.D 103 (2021) 6, 064008 with Piotr T. Chruściel, Tomasz Smolka,

EPJC, 81, 696(2021) with Piotr T. Chruściel, Tomasz Smolka, Maciej Maliborski

A brief history

- Well-defined notion of gravitational wave in full non-linear theory of general relativity for asymptotically flat space-time.

- Bondi and his collaborators provided a detailed analysis of gravitational waves in full non-linear theory along with a definition of energy carried away from an isolated system by gravitational radiation.

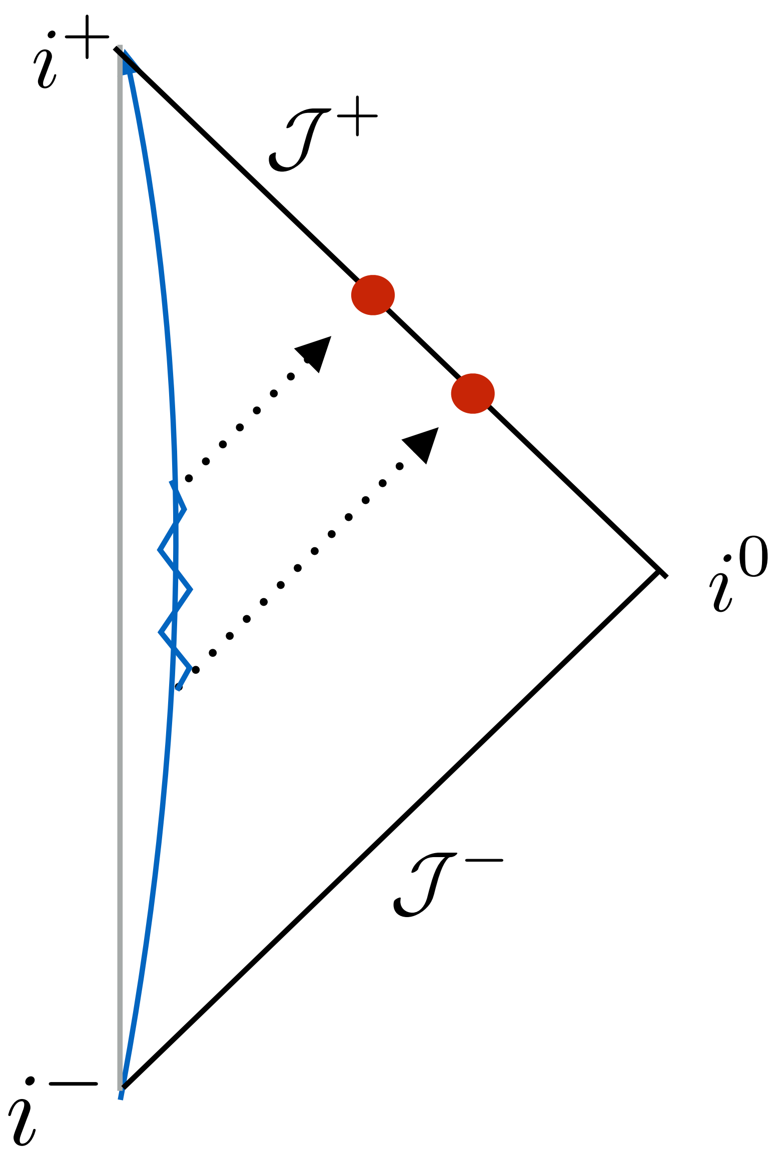
H. Bondi, M.G.J. Vander Burg, A. W. K. Metzner - 1962

- One of the remarkable milestones in gravitational radiation theory!

- Bondi's analysis has a lot of application in mathematical relativity, numerical relativity, asymptotic symmetries, recently on cosmology!!

A brief history

- Bondi and his collaborators performed a systematic expansions of axis symmetric gravitational wave metric along outgoing null directions.
- Given a boundary data on null hypersuface, solve Einstein's equations.
- Deduce the asymptotic fall-off condition for the gravitational fields.
- As a supplementary condition obtain mass-loss formula.
- 'News function' - absolute square integrated over the sphere at infinity, measures the rate of energy loss by an isolated system.



$$\frac{dM}{du} = -\int |\partial_u \overset{(-1)}{\check{h}}{}_{AB}|^2 \sin\theta d\theta d\phi$$

Questions…

- Given that cosmological observations suggest a positive Λ,

  how to generalise Bondi-Sachs's formalism?

  Asymptotic fall-off condition for the gravitational fields?

  Asymptotic symmetries?

  Mass-loss formula for $\Lambda > 0$?

  Analogous 'News tensor'?

Surprisingly this remained unsolved for 60 years!! Lots of Non-trivialities !!

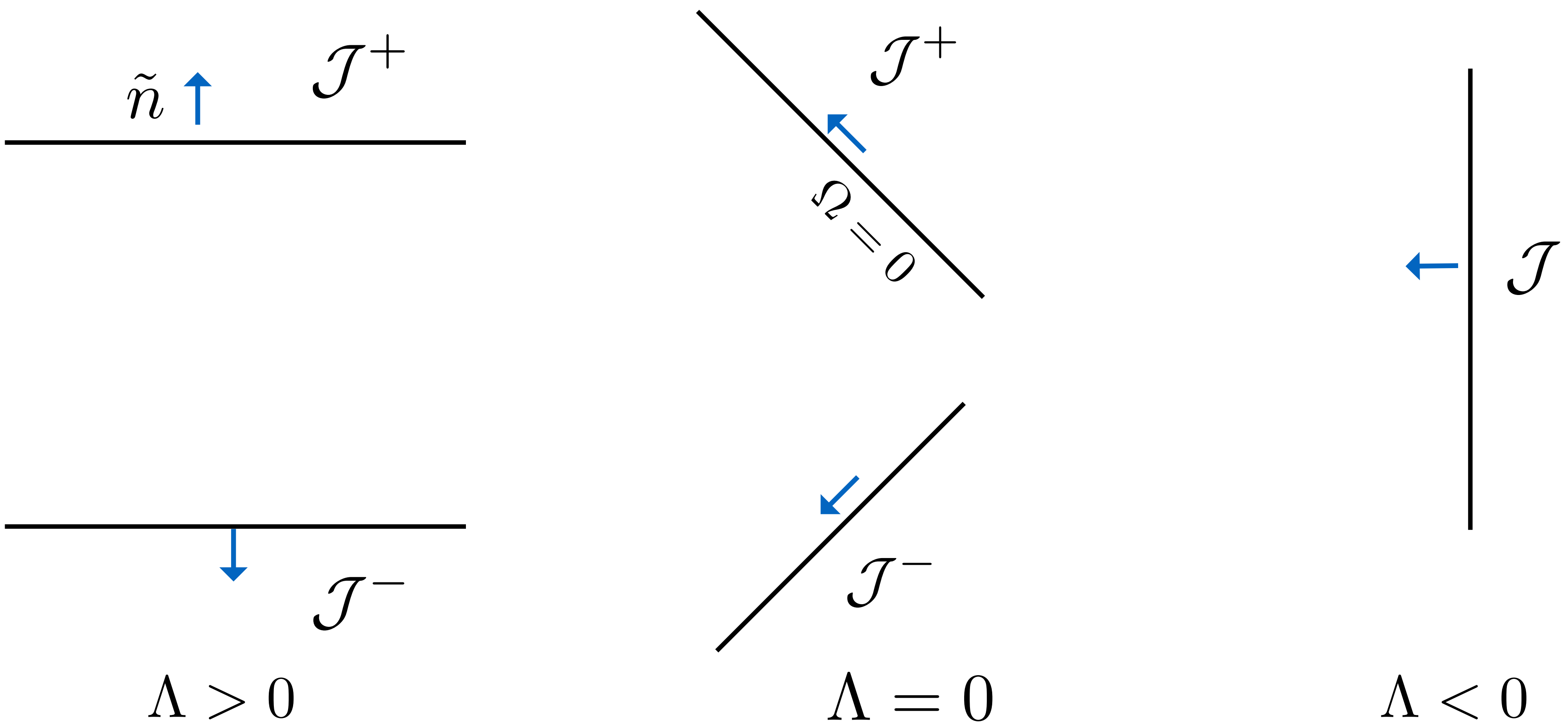
- A simpler version of the problem - Bondi-Sach's formalism for linearised gravitational fields on de Sitter background.

Non-triviality for positive Λ

- Standard framework does not extend from $\Lambda = 0$ to $\Lambda > 0$.

- Structure of null infinity alters, for $\Lambda > 0$: space-like

  $\Lambda < 0$: time-like



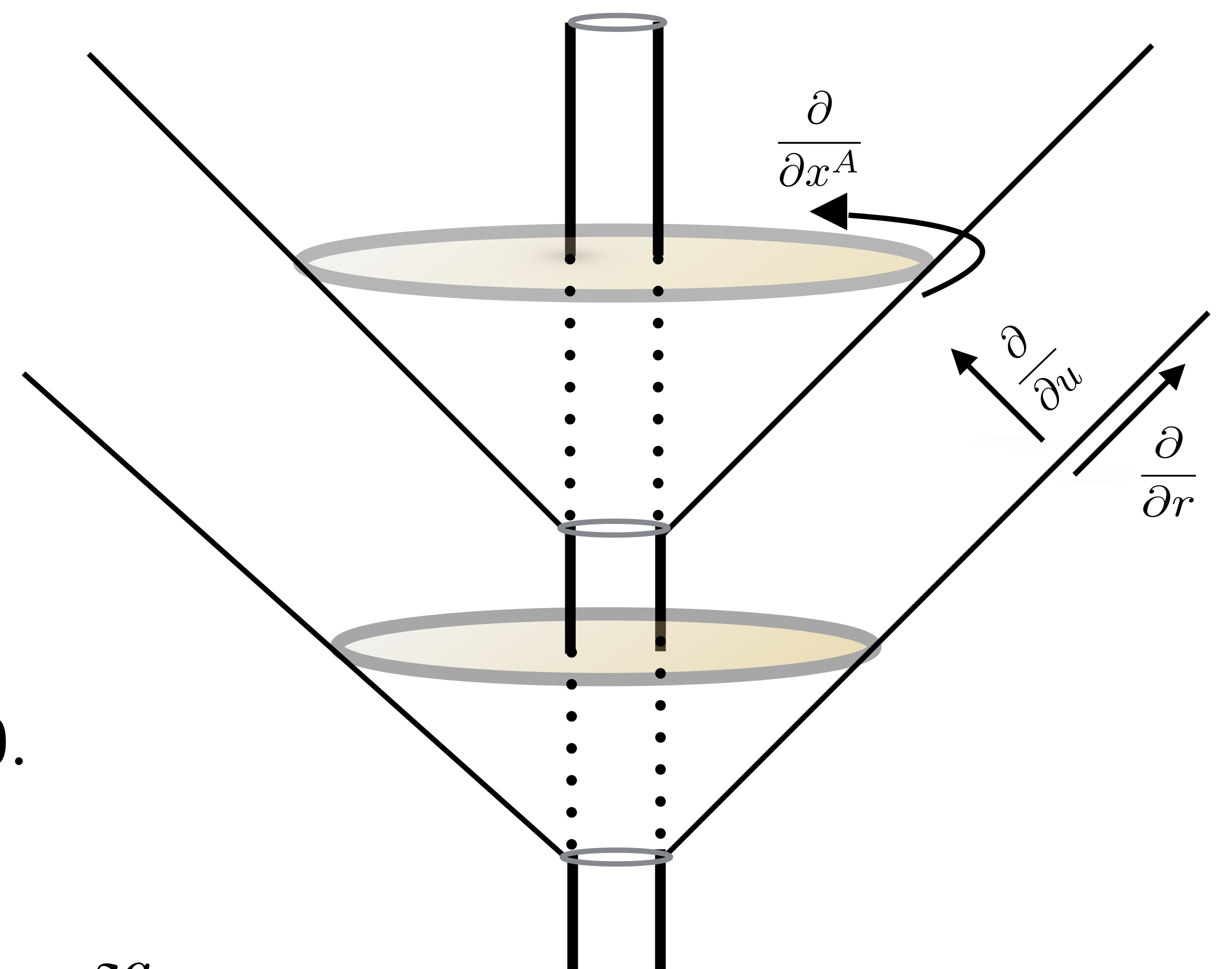
Set up for Bondi coordinates

- We construct Bondi coordinates for de Sitter.

- Bondi coordinates are based on a family of outgoing null hypersurfaces.

- Hypersurfaces $u = const$ are null.

$$\implies \quad g^{ab}\partial_a u \partial_b u = 0 \quad \implies \quad g^{uu} = 0$$



- Two angular coordinates x^A, are constant along null rays.

$$\implies \quad g^{ab}\partial_a u \partial_b x^A = 0 \quad \implies \quad g^{uA} = 0.$$

- g^{ab} and g_{ab} are related by $g^{ac}g_{cb} = \delta^a_b$

$$\implies \quad g_{rr} = 0 = g_{rA}$$

Metric in Bondi-Sachs coordinates

- Metric in Bondi-Sachs coordinates,

$$ds^2 = -\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2\gamma_{AB}(dx^A - U^A du)(dx^B - U^B du)$$

- r varies along null rays, chosen to be an areal coordinate;

$$\mathrm{det} g_{AB} = r^4 \sin^2\theta$$

- We will explore Einstein equations for linearized fields on de Sitter background.

background $g_{ab}(\lambda = 0) := \bar{g}_{ab}$; perturbation $h_{ab} := \left.\frac{dg_{ab}(\lambda)}{d\lambda}\right|_{\lambda=0}$

Bondi gauge for linearized theory

- In Bondi coordinates de Sitter Background metric takes the form,

$$\bar{ds}^2 = -\left(1 - \frac{\Lambda r^2}{3}\right)du^2 - 2dudr + r^2\mathring{\gamma}_{AB}dx^A dx^B$$

- Bondi gauge condition for linearized fields

$$h_{rr} = 0 = h_{rA}, \qquad \mathring{\gamma}^{AB}h_{AB} = 0$$

- We wish to explore linearised Einstein equation with Bondi metric,

$$E_{ab} := R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 0$$

Einstein equations: systems of hierarchical PDEs

- Four independent hyper surface equations, $E^u_a = 0$

- $$E^u_r = 0 \implies \partial_r \beta = \frac{r}{16}\gamma^{AC}\gamma^{BD}(\partial_r \gamma_{AB})(\partial_r \gamma_{CD})$$

  Linearisation: $\partial_r \delta\beta = 0 \implies \delta\beta = \delta\beta(u, x^A)$

Using gauge $\delta\beta = 0 = \delta g_{ur}$, $h_{ab} \mapsto h_{ab} + \mathcal{L}_\xi \bar{g}_{ab}$

- $$E^u_A = 0 \implies \partial_r[r^4 e^{-2\beta}\gamma_{AB}(\partial_r U^B)] = 2r^4 \partial_r \left(\frac{1}{r} D_A \beta\right) - r^2 \gamma^{EF} D_E(\partial_r \gamma_{AF})$$

  Linearisation: $\partial_r[r^4 \partial_r(r^{-2}\delta g_{uA})] = r^2 \mathring{D}^F \partial_r(r^{-2}\delta g_{AF})$

- $$\begin{aligned} E^u_u = 0 \implies 2e^{-2\beta}(\partial_r V) &= \mathcal{R} - 2\gamma^{AB}[D_A D_B \beta + D_A \beta D_B \beta] \\ &+ \frac{e^{-2\beta}}{r^2} D_A[\partial_r(r^4 U^A)] - \frac{r^4}{2} e^{-2\beta}\gamma_{AB}(\partial_r U^A)(\partial_r U^B) - 2\Lambda r^2 \end{aligned}$$

  Linearisation: $2\partial_r \delta V = \delta\mathcal{R} - \frac{1}{r^2}\mathring{D}^A[\partial_r(r^2 \delta g_{uA})]$

Solution for h_{uA}

Given the ansatz : $$h_{AB} = r^2\left(\overset{(0)}{\check{h}}_{AB} + \frac{\overset{(-1)}{\check{h}}_{AB}}{r} + \frac{\overset{(-2)}{\check{h}}_{AB}}{r^2} + \frac{\overset{(-3)}{\check{h}}_{AB}}{r^3} + \dots\right)$$

solve $$\partial_r[r^4\partial_r(r^{-2}\delta g_{uA})] = r^2\mathring{D}^F\partial_r(r^{-2}\delta g_{AF})$$?

$$h_{uA} = r^2\left(\overset{(0)}{\check{h}}_{uA} + \frac{1}{2}\mathring{D}^B\overset{(-1)}{\check{h}}_{AB}\, r^{-2} + \left(\overset{(-3)}{\check{h}}_{uA} + \frac{2}{9}\mathring{D}^B\overset{(-2)}{\check{h}}_{AB}(3\ln r + 1)\right)r^{-3} + \dots\right)$$

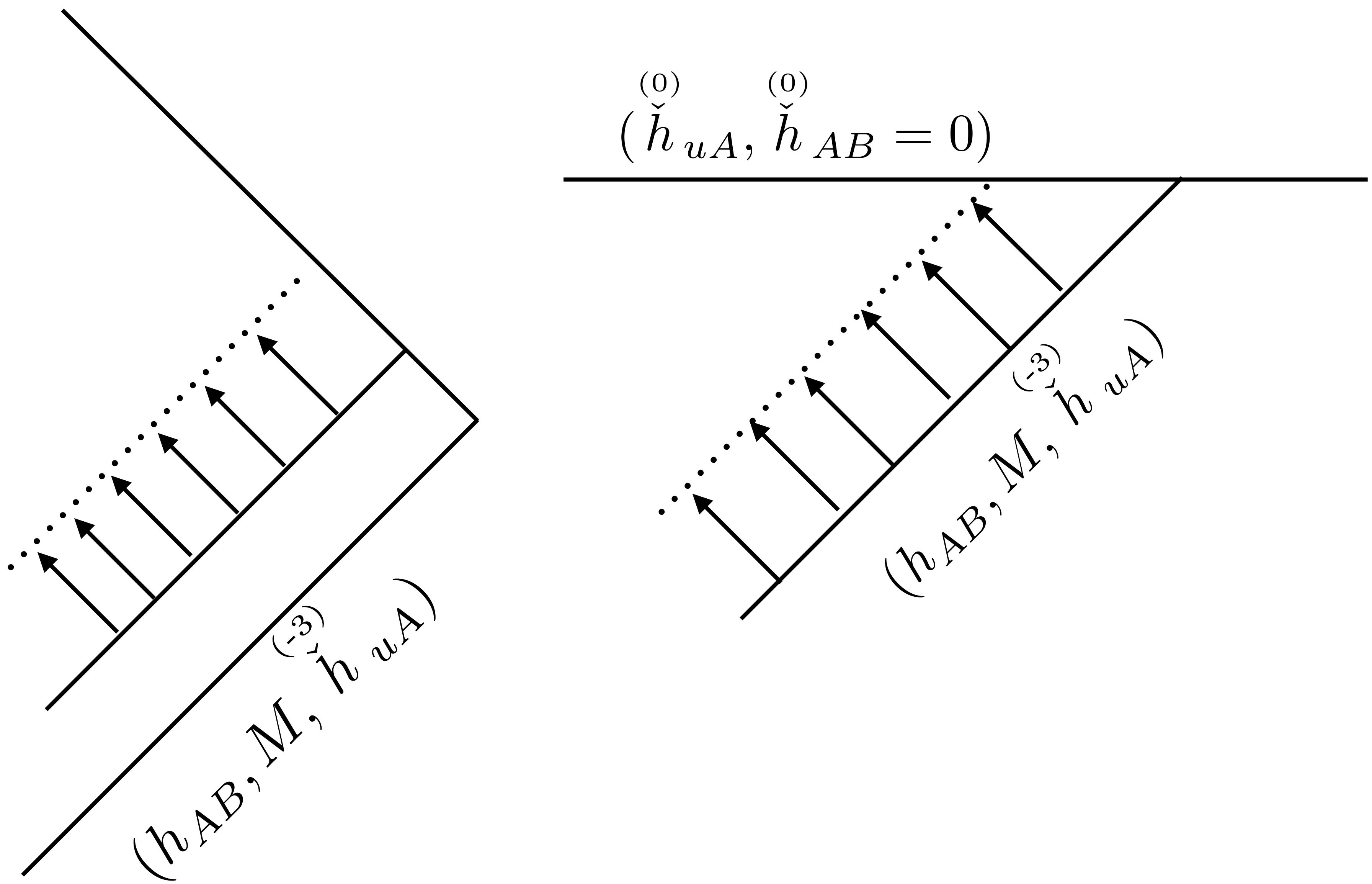
- Similarly, solve for V

- Given the ansatz h_{AB}, hypersuface equations $E^u_a = 0$ fix the asymptotic fall off condition for other components of field.

Evolution equation for $\check{h}_{AB}$

- Traceless symmetric parts of $E_{AB}=0$ gives evolution equation for $\check{h}_{AB} := r^{-2} h_{AB}$

$$r\partial_r[r(\partial_u \check{h}_{AB})] + \frac{1}{2}\partial_r[r^2(\frac{\Lambda}{3}r^2 - 1)(\partial_r \check{h}_{AB})] - TS[\mathring{D}_A(\partial_r(r^2\check{h}_{AB}))] = 0$$



Non-polyhomogenous de Sitter

$$r\partial_r[r(\partial_u \check{h}_{AB})] + \frac{1}{2}\partial_r[r^2(\frac{\Lambda}{3}r^2 - 1)(\partial_r \check{h}_{AB})] - TS[\mathring{D}_A(\partial_r(r^2 \check{h}_{AB}))] = 0$$

Asymptotic analysis of this equation gives non trivial equations

$$\frac{\Lambda}{3} \overset{(-2)}{\check{h}}_{AB} = 0$$

- NO log term in de Sitter. De Sitter is non-polyhomogenous!!
- To get rid of log term one needs to set $\overset{(-2)}{\check{h}}_{AB} = 0,$ for flat space-time. In Bondi's paper this condition is termed as outgoing radiation condition.
- For de Sitter this is a consequence of equation of motion.
- This result is true for full non-linear theory also.

G. Compère, A. Fiorucci, R Ruzziconi - 2019

A. Pole, K. Skenderis, M. Taylor -2019

Asymptotic symmetry group is NOT BMS

$$r\partial_r[r(\partial_u \check{h}_{AB})] + \frac{1}{2}\partial_r[r^2(\frac{\Lambda}{3}r^2 - 1)(\partial_r \check{h}_{AB})] - TS[\mathring{D}_A(\partial_r(r^2\check{h}_{AB}))] = 0$$

Asymptotic analysis of this equation also gives,

$$\partial_u \overset{(0)}{\check{h}}_{AB} = \frac{\Lambda}{3}\overset{(-1)}{\check{h}}_{AB} + (\mathring{D}_A \overset{(0)}{\check{h}}_{uB} + \mathring{D}_B \overset{(0)}{\check{h}}_{uA} - \mathring{\gamma}_{AB}\mathring{D}^C \overset{(0)}{\check{h}}_{uC})$$

- $\overset{(0)}{\check{h}}_{AB}$ and $\overset{(0)}{\check{h}}_{uA}$ can not be zero simultaneously by a gauge transformation!! Asymptotic symmetry group of de Sitter is not BMS.

- Whether this gauge condition is achieved by any physical space-time is difficult.

S. J. Hoque, A. Virmani - 2021

Asymptotic expansion of linearised fields

$$h_{AB} = r^2\Big(\underbrace{\overset{(0)}{\check{h}}{}_{AB}}_{=0} + \frac{\overset{(-1)}{\check{h}}{}_{AB}}{r} + \underbrace{\overset{(-2)}{\check{h}}{}_{AB}}_{=0} r^{-2} + \frac{\overset{(-3)}{\check{h}}{}_{AB}}{r^3} + \ldots\Big),$$

$$h_{uA} = r^2\Big(\overset{(0)}{\check{h}}{}_{uA} + \frac{1}{2}\mathring{D}^B \overset{(-1)}{\check{h}}{}_{AB}\, r^{-2} + \overset{(-3)}{\check{h}}{}_{uA} r^{-3} + \ldots\Big),$$

$$h_{uu} = r\mathring{D}^A \overset{(0)}{\check{h}}{}_{uA} + \frac{M}{r} - \frac{1}{2r^2}\mathring{D}^A \overset{(-3)}{\check{h}}{}_{uA} + \ldots$$

$$h_{ur} = 0$$

Evolution equations for integration constant

- $E_{uu} = 0$, gives the evolution equation for h_{uu}

$$2\partial_u M = \partial_u \mathring{D}^A \mathring{D}^B \overset{(-1)}{\check{h}}{}_{AB} - \Lambda \mathring{D}^A \overset{(-3)}{\check{h}}{}_{uA}$$

- $E_{uA} = 0$, gives the evolution equation for h_{uA}

$$3\partial_u \overset{(-3)}{\check{h}}{}_{uA} = \mathring{D}_A M + \frac{1}{2}(\mathring{D}^B \mathring{D}_A \mathring{D}^C \overset{(-1)}{\check{h}}{}_{CB} - \triangle_{\mathring{\gamma}} \mathring{D}^C \overset{(-1)}{\check{h}}{}_{CA}) - \Lambda \mathring{D}^B \overset{(-3)}{\check{h}}{}_{AB}$$

Summary and outlook

- Bondi-Sachs coordinates are constructed for de Sitter.
- NO log term in de Sitter
- Asymptotic fall off condition for linearised gravitational field have been obtained in Bondi frame. Qualitatively different from $\Lambda = 0$ case.
- Due to different fall-off asymptotic symmetry group is not BMS
- Relation with other radiative solution in de Sitter? Relation with quadrupole formula in de Sitter?

A. Ashtekar, B. Bonga, A. Kesavan - 2015

G. Date, Sk J. Hoque - 2015

- Interesting to generalise Bondi-Sachs formalism for FLRW case.

B. Bonga, K. Prabhu - 2020

- Relation with BMS charges and charges in double null formalism?

Thank you

Energy in the linearised theory

- The Hamiltonian for the linearised theory associated with a hyper surface Σ, and a vector field X reads,

$$\tilde{\mathcal{H}}[\Sigma, X] := \int_\Sigma \tilde{\mathcal{H}}^\mu d\Sigma_\mu$$
$$= \frac{1}{2}\left(\int_\Sigma \omega^\mu(\tilde{\phi}, \mathcal{L}_X \tilde{\phi}) \; d\Sigma_\mu - \int_{\partial\Sigma} X^{[\sigma} \tilde{\pi}_A^{\mu]} \tilde{\phi}^A d\Sigma_{\sigma\mu} \right)$$

$\tilde{\phi}$ is linearised field, $\tilde{\pi}^\mu$ is associated canonical conjugate momenta.

- When X is a time-translational symmetry of background, the numerical value of the integration is identified with the total energy of the field contained in Σ.

Energy flux in the linearised theory

- Consider a family of hyper surfaces labelled by τ and define,

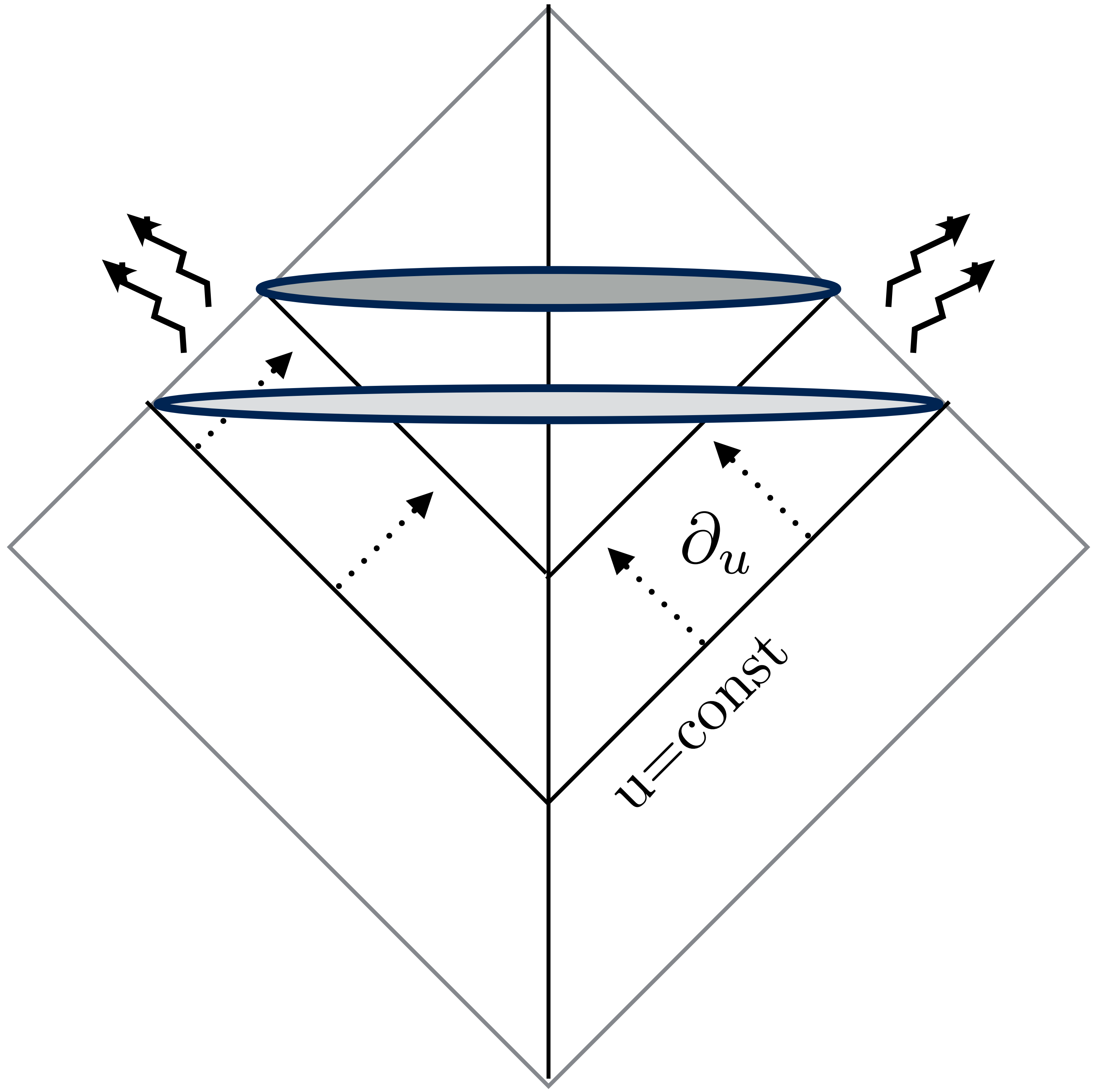
$$\tilde{\mathcal{H}}(\Sigma_\tau, X) := \int_{\Sigma_\tau} \tilde{\mathcal{H}}^\mu dS_\mu$$

$$\begin{aligned}
\frac{d\tilde{\mathcal{H}}[\Sigma_\tau, X]}{d\tau} &= \frac{1}{2}\frac{d}{d\tau}\int_\Sigma \omega^\mu(\tilde{\phi}, \mathcal{L}_X\tilde{\phi})\, d\Sigma_\mu - \frac{1}{2}\int_{\partial\Sigma} \mathcal{L}_X\left(X^{[\sigma}\tilde{\pi}_A{}^{\mu]}\tilde{\phi}^A\right) d\Sigma_{\sigma\mu} \\
&= -\frac{1}{2}\int_{\partial\Sigma} X^{[\sigma}\omega^{\mu]}(\tilde{\phi}, \mathcal{L}_X\tilde{\phi})\, d\Sigma_{\sigma\mu} \\
&\quad -\frac{1}{2}\int_{\partial\Sigma}\left(X^{[\sigma}\mathcal{L}_X\tilde{\pi}_A{}^{\mu]}\tilde{\phi}^A + X^{[\sigma}\tilde{\pi}_A{}^{\mu]}\mathcal{L}_X\tilde{\phi}^A\right) d\Sigma_{\sigma\mu}\,.
\end{aligned}$$

Using, $\omega^\mu(\tilde{\phi}, \mathcal{L}_X\tilde{\phi}) := \mathcal{L}_X\tilde{\phi}^A\tilde{\pi}_A{}^\mu - \tilde{\phi}^A\mathcal{L}_X\tilde{\pi}_A{}^\mu$

$$\frac{d\tilde{\mathcal{H}}(\Sigma_\tau, X)}{d\tau} = -\int_{\partial\Sigma_\tau} X^{[\sigma}\tilde{\pi}_A{}^{\mu]}\pounds_X\tilde{\phi}^A d\Sigma_{\sigma\mu}.$$

The integrand represents the flux of the energy through $\partial\Sigma$ when Σ is dragged along the flow of X.



∂_u
u=const

Canonical energy for gravitational field

$$
\begin{aligned}
E_c[h,\mathcal{C}_{u,R}] &= \frac{1}{64\pi}\int_{\mathcal{C}_{u},R} \bar{g}^{BE}\bar{g}^{FC}(\partial_u h_{BC}\partial_r h_{EF} - h_{BC}\partial_r\partial_u h_{EF})r^2\sin\theta dr d\theta d\phi \\
&- \frac{1}{32\pi}\int_{S(R)} \bar{P}^{r(bc)d(ef)}h_{bc}\bar{\nabla}_d h_{ef}\ r^2\sin\theta d\theta d\phi
\end{aligned}
$$

where:

h_{ab} solution of linearised vacuum Einstein equations,

$\mathcal{C}_u$ light cone $u = \mathrm{const}$ emanating from $r = 0$,

$\mathcal{C}_{u,R}$ light cone truncated at radius $r = R$,

$S(R)$ sphere of radius R

Boundary term in canonical energy

- In Bondi gauge boundary integral of $E_C(h, \mathcal{C}_{u,R})$ becomes,

$$-\frac{\Lambda R}{192\pi}\int_{S^2}\mathring{\gamma}^{AB}\mathring{\gamma}^{CD}\overset{(-1)}{\check{h}}{}_{AC}\overset{(-1)}{\check{h}}{}_{BD}\ \sin\theta d\theta d\phi$$

$$-\frac{1}{64\pi}\int_{S^2}(\mathring{\gamma}^{AB}\mathring{\gamma}^{CD}\overset{(-1)}{\check{h}}{}_{AC}\overset{(-1)}{\check{\partial}}{}_{u}\check{h}_{BD}-6\mathring{\gamma}^{AB}\overset{(0)}{\check{h}}{}_{uA}\overset{(-3)}{\check{h}}{}_{uB})\sin\theta d\theta d\phi$$

Renormalised energy and flux

We propose to introduce a renormalised canonical energy

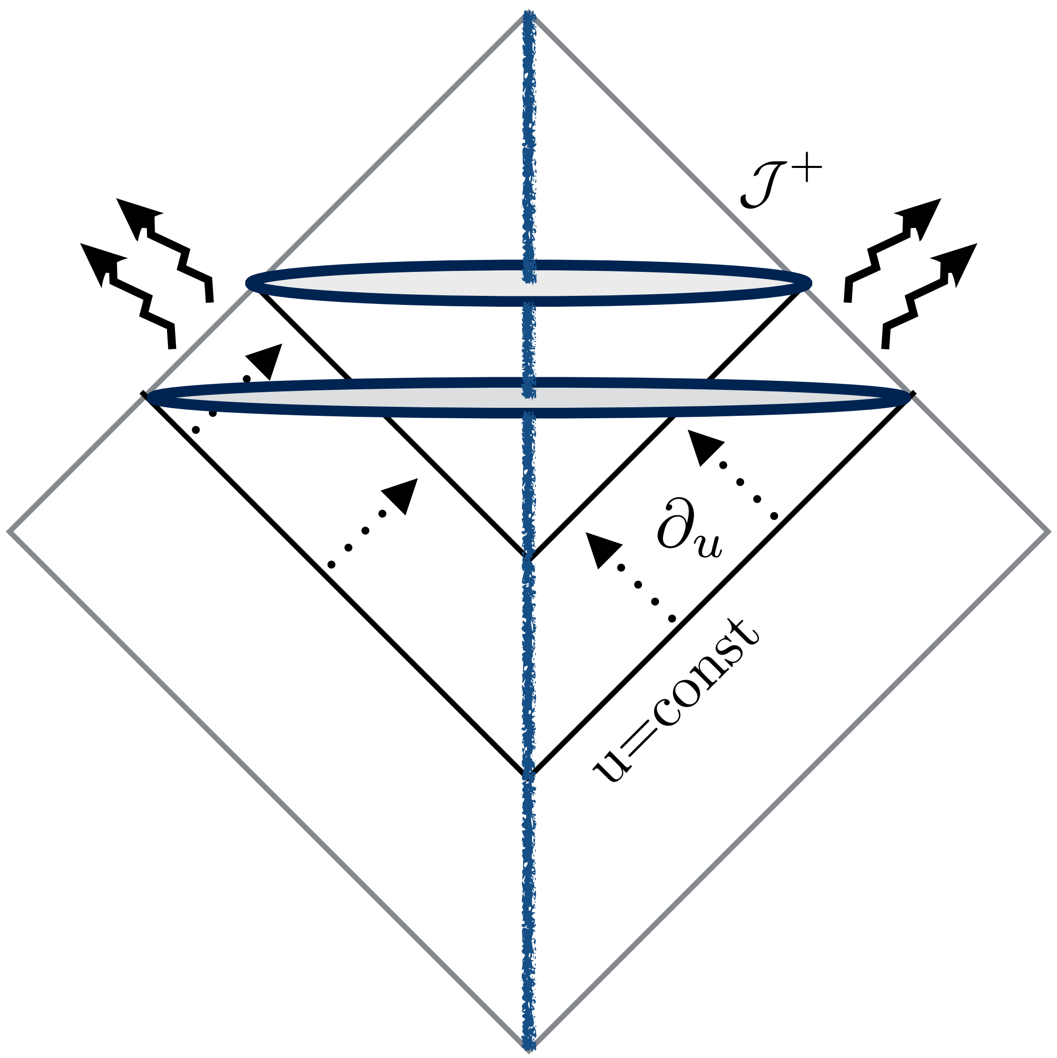
$$\hat{E}_c[h,\mathcal{C}_u] := \frac{1}{64\pi}\int_{\mathcal{C}_u} g^{BE}g^{FC}(\partial_u h_{BC}\partial_r h_{EF} - h_{BC}\partial_r\partial_u h_{EF})r^2 \sin\theta dr d\theta d\phi$$

$$- \frac{1}{64\pi}\int_{S^2}(\mathring{\gamma}^{AB}\mathring{\gamma}^{CD}\overset{(-1)}{\check{h}}_{AC}\partial_u \overset{(-1)}{\check{h}}_{BD} - 6\mathring{\gamma}^{AB}\overset{(0)}{\check{h}}_{uA}\overset{(-3)}{\check{h}}_{uB})\sin\theta d\theta d\phi$$

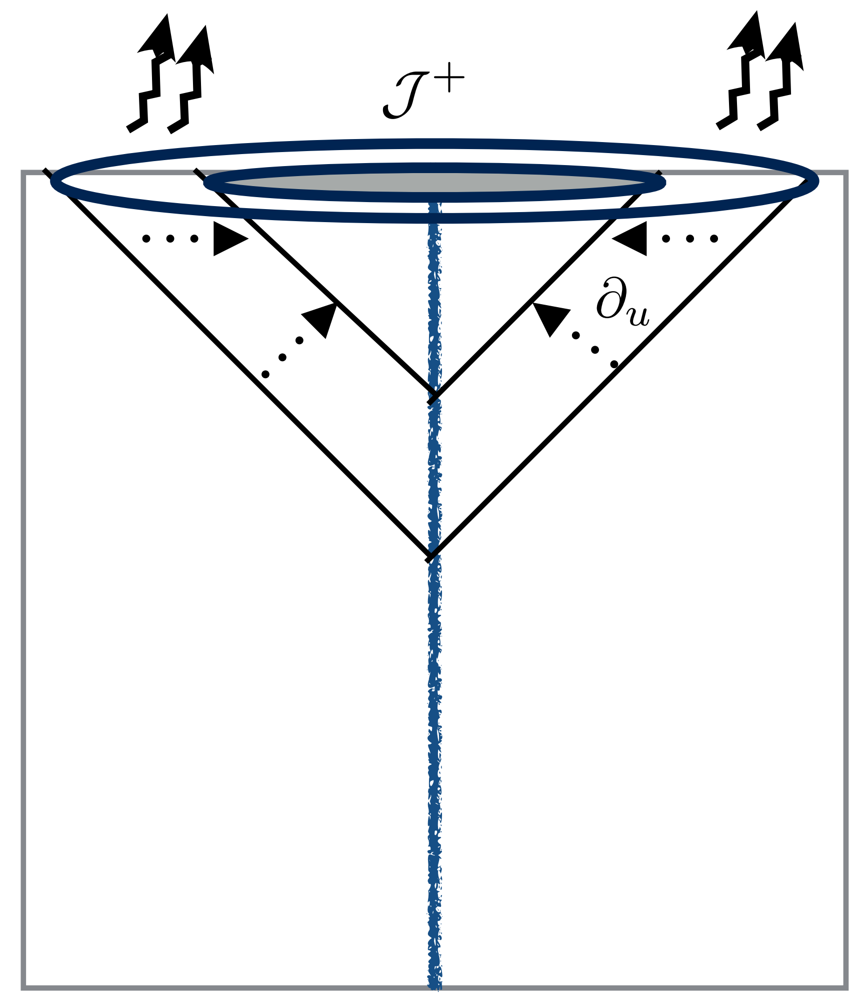
which has its own flux formula

$$\frac{d\hat{E}_c[h,\mathcal{C}_u]}{du} = -\frac{1}{32\pi}\int_{S^2}(\mathring{\gamma}^{AB}\mathring{\gamma}^{CD}\partial_u\overset{(-1)}{\check{h}}_{AC}\partial_u \overset{(-1)}{\check{h}}_{BD} - 6\mathring{\gamma}^{AB}\overset{(-3)}{\check{h}}_{uA}\partial_u\overset{(0)}{\check{h}}_{uB})\sin\theta d\theta d\phi$$

For $\Lambda = 0$, we obtain linearised version of Bondi's mass-loss formula.



$\Lambda = 0$



$\Lambda > 0$

Summary

- Bondi-Sachs coordinates are constructed for de Sitter.
- NO log term in de Sitter
- Asymptotic fall off condition for linearised gravitational field have been obtained in Bondi frame. Qualitatively different from $\Lambda = 0$ case.
- Due to different fall-off asymptotic symmetry group is not BMS
- Proposed renormalised energy and flux in the limit $\Lambda = 0$ become classical Bondi quantities.