The viscous Burgers Equation on locally metric spaces

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Burgers Equation (Burgers '48)

Viscous Burgers Equation (BE):

$$\frac{\partial u}{\partial t} + \underbrace{(u \cdot \nabla)\, u}_{convection} = \nu \Delta u, \qquad \nu > 0$$

(BE) describes laminar flow in fluid dynamics.

Aim:

1. Formulation on X
2. Existence of solutions

For related results (existence, uniqueness and regularity of the solution for (BE)) we refer to Liu and Qian [LQ].

Starting point

Consider the Cauchy problem for the Heat Equation (HE):

$$\begin{cases} w_t(x,t) = \nu \Delta w(x,t), & t > 0 \\ w(x,0) = w_0(x) \end{cases}$$

with $\operatorname{ess\,sup} w_0(x) > c_0 > 0$, $w_0 \in L^2(X,\mu)$.

Idea:

Use knowledge about (HE) and Cole Hopf Transformation [Col51, Hop50]

$$u(x,t) := -2\nu \frac{(w(x,t))_x}{w(x,t)}$$

to proof existence of solutions!

Setup

- X locally compact separable metric space
- μ Radon measure on X s.t. $\mu(U) > 0\ \forall U \subset X$ open, $U \neq \emptyset$
- $(\mathcal{E}, \mathcal{F})$ symmetric local regular Dirichlet form on $L^2(X, \mu)$

$\mathcal{C}_b := \mathcal{F} \cap C_b(X)$.
Endowed with the norm $\| f \|_{\mathcal{C}_b} := \mathcal{E}_1(f)^{1/2} + \sup_{x \in X} | f(x) |, \quad f \in \mathcal{C}_b$
$\Rightarrow \mathcal{C}_b$ becomes an algebra, see [BH91, Cor. I.3.3.2], and it holds

$$\mathcal{E}(fg)^{\frac{1}{2}} \leq \|f\|_\infty \mathcal{E}(g)^{\frac{1}{2}} + \|g\|_\infty \mathcal{E}(f)^{\frac{1}{2}} \qquad \forall f, g \in \mathcal{F}$$

$\mathcal{C}_b^*$ - dual space of $\mathcal{C}_b$, normed by

$$\| g \|_{\mathcal{C}_b^*} = \sup\{| g(f) |: f \in \mathcal{C}_b, \| f \|_{\mathcal{C}_b} \leq 1\}.$$

Abstract Derivation And Divergence

According to Ionescu, Rogers and Teplyaev [IRT12] we use the framework of 1-forms and derivations introduced by Cipriani and Sauvageot [CS03].

Definition

A *derivation operator* $\partial : \mathcal{F} \to \mathcal{H}$ can be defined by setting

$$\partial f := f \otimes \mathbb{1}, \qquad f \in \mathcal{F}.$$

Remark

It is a bounded linear operator satisfying the Leibniz property

$$\partial(fg) = (\partial f)g + f(\partial g).$$

The operator $\partial : \mathcal{F} \to \mathcal{H}$ can be extended to a closed linear operator $\partial_\mu : L^2(X, \mu) \to \mathcal{H}$ with domain dense in $\mathcal{F}$, satisfying

$$\|\partial f\|^2_{\mathcal{H}} = \mathcal{E}(f), \qquad f \in \mathcal{F}.$$

Corollary

Let $(\mathcal{E}, \mathcal{F})$ be a strong local Dirichlet form. For $f \in \mathcal{F}$ and $F \in C^1(\mathbb{R})$ the chain rule is also satisfied

$$\partial F(f) = F'(f)\partial f.$$

Definition (divergence)

The *divergence* $\partial_\mu^* : \mathcal{H} \to L^2(X,\mu)$ is defined as -adjoint operator to ∂_μ, equipped with the domain

$$\mathcal{D}(\partial_\mu^*) := \left\{ v \in \mathcal{H} : \exists u \in L^2(X,\mu) : \langle u, \phi \rangle_{L^2(X,\mu)} = -\langle v, \phi \rangle_{\mathcal{H}} \, \forall \phi \in \mathcal{F} \right\}.$$

For $v \in \mathcal{D}(\partial_\mu^*)$ set $\partial_\mu^* v := u$.

Remark

For $f \in \mathcal{C}_b$ is the following true:

$$\partial_\mu f \in \mathcal{D}(\partial_\mu^*) \qquad \text{and} \qquad \Delta_\mu f = \partial_\mu^* \partial_\mu f.$$

In our set-up we will consider $f \in \mathcal{D}(\Delta_\mu)$ such that

$$\Delta_\mu f \in C(X) \subset L^2(X,\mu). \tag{1}$$

Definition

We define the space of test vector fields as

$$\mathcal{D}_{\mathcal{H}\to C_b(X)}(\partial_\mu^*) := \left\{v \in \mathcal{D}(\partial_\mu^*) : \partial_\mu^* v \in C_b(X)\right\}.$$

For $u \in \mathcal{H}$, $v \in \mathcal{D}_{\mathcal{H}\to C_b(X)}(\partial_\mu^*)$

$$\Delta_{\mu,1} u(v) := \left(\partial_\mu \partial_\mu^* u\right)(v) := -(\partial_\mu^* u)(\partial_\mu^* v),$$
$$\partial_\mu \langle u, u\rangle_{\mathcal{H}} := -\langle(\partial_\mu^* v)u, u\rangle_{\mathcal{H}}$$

For f as in (1), we have $\partial f \in \mathcal{D}_{\mathcal{H}\to C_b(X)}(\partial_\mu^*)$.

Existence of weak solution

Definition

Let $u_0 \in \mathcal{H}$. We say that a function $u : [0, \infty) \to \mathcal{H}$ with initial condition u_0 is a *weak solution of the abstract Burgers Equation*, if the function is differentiable on $(0, \infty)$ and obeys for all $v \in \mathcal{D}_{\mathcal{H} \to C_b}(\partial^*)$

$$\begin{cases} \Delta_{\mu,1} u(v) - \partial \langle u(t), u(t) \rangle_{\mathcal{H}}(v) & = \langle u_t(t), v \rangle_{\mathcal{H}}, \qquad t > 0 \\ \lim_{t \to 0} \langle u(t) - u_0, v \rangle_{\mathcal{H}} & = 0. \end{cases} \tag{2}$$

Theorem

Let $w_0 \in C_b$ a positively function with $w_0(x) \geq c_0$, $x \in X$, for a fixed constant $c_0 > 0$.

$$u(t) := -\partial(\log w(t)), \qquad t > 0, \qquad \textit{with } u_0 = -\partial \log w_0$$

is a weak solution of the initial problem (2).

Application: Burgers Equation On Metric Graphs

Based on Boutet de Monvel, Lenz and Stollmann [BdMLS09] and Haeseler [Hae], we define the notion of metric graphs and a topology on it.

Definition

A metric graph is $\Gamma = (E, V, i, j)$ where

- E (edges) is a countable family of open intervals $(0, l(e))$ and V (vertices) is a countable set.
- $i : E \to V$ defines the initial point of an edge and $j : \{e \in E \mid l(e) < \infty\} \to V$ the end point for edges of finite length.

Set $X_e := \{e\} \times e$, $X = X_\Gamma = V \cup \bigcup_{e \in E} X_e$ and $\bar{X}_e := X_e \cup \{i(e), j(e)\}$.

The topology on X_Γ will be such that the mapping $\pi_e : X_e \to (0, l(e))$, $(e, t) \mapsto t$ extends to a homeomorphism again denoted by $\pi_e : \bar{X}_e \to (0, \bar{l}(e))$ that satisfies $\pi_e(i(e)) = 0$ and $\pi_e(j(e)) = l(e)$.

measure on X_Γ: for $Y \subset X_\Gamma$

$$\int_Y u(x) d\mu(x) := \sum_{e \in E} \int_{e \cap Y} u(x) d\mu(x),$$

where μ is the measure induced by the images of the Lebesgue measure on each $(0, l(e))$.

$$L^2(X_\Gamma, \mu) = \bigoplus_{e \in E} L^2(0, l(e))$$

$$\mathcal{D}(\mathcal{E}) = W_0^{1,2}(X_\Gamma), \qquad \mathcal{E}(u, v) := \sum_{e \in E} \int_0^{l(e)} u_e'(x) v_e'(x) dx,$$

where $u_e := u \circ \pi_e^{-1}$ defined on $(0, l(e))$,

$$W^{1,2}(X_\Gamma) = \{u \in C(X_\Gamma) \mid \sum_{e \in E} \| u_e \|_{W^{1,2}}^2 =: \| u \|_{W^{1,2}}^2 < \infty\},$$

$$W_0^{1,2}(X_\Gamma) := W^{1,2}(X_\Gamma) \cap C_c(X_\Gamma).$$

Because of energy measure

$$d\Gamma(u(x)) = \mid u'(x) \mid^2 \, d\mu(x) = \sum_{e \in E} \mid u_e'(x) \mid^2 \, dx$$

the mapping $g\partial f \in \mathscr{H}$ to gf' can be extended to an isometric isomorphism $\mathscr{H} \cong L^2(X_\Gamma, \mu)$ in that

$$\| \, g\partial f \, \|_{\mathscr{H}}^2 = \| \, gf' \, \|_{L^2(X_\Gamma, \mu)}^2 \, .$$

Proposition [IRT12], see also [BK]

Identifying $\mathscr{H}$ and L^2 as above, the derivation $\partial : \mathcal{D}(\mathcal{E}) \to \mathscr{H} \cong L^2(X_\Gamma, \mu)$ is the usual derivative (which takes orientation of edges into account).

Similarly, we obtain the divergence operator ∂^*.

$$u \mapsto \partial^* v(u) := -\langle \partial u, v \rangle_{\mathscr{H}} = -\sum_{e \in E} \int_0^{l(e)} u_e'(x) v_e(x) dx \qquad \forall u \in \mathcal{C}_b.$$

We consider $f \in \mathcal{D}(\Delta_\mu)$ such that

$$\partial f \in \mathcal{D}(\partial^*) \qquad \text{and} \qquad \partial^* \partial f = \Delta_\mu f \text{ in } L^2(X_\Gamma, \mu).$$

$$\tilde{\mathcal{D}} := \{ v \in \mathscr{H} \mid v = \partial f + \eta : f \in \mathcal{D}(\Delta_\mu), \eta \in Ker\, \partial^* \}$$

subspace of the space $\mathcal{D}_{\mathscr{H} \to \mathcal{C}_b}(\partial^*)$.
For $v \in \tilde{\mathcal{D}}$

$$(\partial \partial^* u)(v) := -(\partial^* u)(\partial^* v) = -\partial^*(\Delta_\mu f) = -\langle \partial^* u, \Delta_\mu f \rangle_{L^2(X_\Gamma, \mu)},$$

$$\partial \langle u, u \rangle_{\mathscr{H}} = -\langle (\Delta_\mu f) u, u \rangle_{L^2(X_\Gamma, \mu)} = -\sum_{e \in E} \int_0^{l(e)} (\Delta_\mu f_e(x)) u_e(x)^2 dx,$$

Application: Burgers Equation On Sierpinski Gasket

Setup

- $X = SG$ Sierpinski gasket
- μ finite Borel measure s.t. $\mu(U) > 0 \ \forall U \subset SG$ open, $U \neq \emptyset$
- $(\mathcal{E}, \mathcal{F})$ standard resistence form
- Δ_μ Laplacian, defined by

$$\mathcal{E}(u, v) = -\int v \Delta_\mu u d\mu$$

for all $v \in \mathcal{F}$ vanishing on the boundary

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