Γ-Calculus of Cones

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Analysis and Geometry on Graphs and Manifolds

July 31, 2017

Setting

1. Given an unweighted, undirected, connected, finite, simple graph without any loop-edges, $G = (V, E)$.
2. Consider the cone, $C(G) = (V^c, E^c)$, over the vertices of G.
3. The Curvature-Dimension inequality for the Γ-Calculus of Bakry and Émery at the cone point is:

$$\Gamma_2^c(f)(p) \geq \frac{1}{N^c} (\Delta f)^2 (p) + K_p^c \Gamma_1^c(f)(p)$$

4. Where $\Gamma_1^c(f,g)(x) = \frac{1}{2}\{\Delta(f \cdot g)(x) - f(x)\Delta g(x) - g(x)\Delta f(x)\}$, and $\Gamma_1^c(f,g)(x) = \frac{1}{2}\sum_{y\sim x}(f(y) - f(x))(g(y) - g(x))$ and so we say $\Gamma_1^c := \frac{1}{2}\langle \nabla f, \nabla g \rangle$.
5. $\Gamma_2^c(f,g)(x) = \frac{1}{2}\{\Delta\Gamma_1^c(f,g)(x) - \Gamma_1^c(f,\Delta g)(x) - \Gamma_1^c(\Delta f, g)(x)\}$
6. Note: this formulation is for the un-normalized graph laplacian, $\Delta f(x) = \sum_{y\sim x}(f(y) - f(x))$.

Main Corollary

Then, given any function $f : V \to \mathbb{R}$ with $avg(f) = 0$,

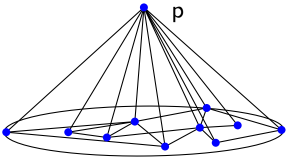
$$\|\nabla f\|_2^2 \geq \frac{2K_p^c + |V| - 3}{2} \|f\|_2^2$$

This inequality comes from a slightly more involved one, namely,

$$\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{2 - N^c}{2N^c} \left(\sum_{y \in V} f(y) \right)^2 + \frac{2K_p^c + |V| - 3}{4} \sum_{y \in V} f^2(y)$$

Notice that, save for the constant, N^c and K_p^c, this inequality relies only on the underlying graph $G = (V, E)$.

Cone over the vertices of G



Given $f : V \to \mathbb{R}$, let $f^c : V^c \to \mathbb{R}$ such that for all $v \in V$, $f^c(v) = f(v)$ and $f^c(p) = 0$.

Note that:

$$\Delta^c f(x) = \begin{cases} \Delta f(x) - f(x), & x \sim p \\ \sum_{y \in V} f(y), & x = p \end{cases}$$

and

$$\Gamma_1^c(f)(x) = \begin{cases} \Gamma_1(f)(x) + \frac{f^2(x)}{2}, & x \sim p \\ \frac{1}{2} \sum_{y \in V} f^2(y), & x = p \end{cases}$$

So, given

$$\Gamma_2^c(f)(p) \geq \frac{1}{N^c} \left(\Delta^c f\right)^2 (p) + K_p^c \Gamma_1^c(f)(x)$$

Substituting yields:

$$\frac{1}{2}\Delta^c \Gamma_1^c(f)(p) - \Gamma_1^c(f, \Delta^c f)(p) \geq \frac{1}{N^c} \left(\sum_{y \in V} f(y) \right)^2 + \frac{K_p^c}{2} \sum_{y \in V} f^2(y)$$

$$\frac{1}{2}\sum_{y\in V\sim p}(\Gamma_1^c(f)(y)-\Gamma_1^c(f)(p))-\frac{1}{2}\sum_{y\in V\sim p}(f(y)-f(p))(\Delta^c f(y)-\Delta^c f(p))$$

$$\geq \frac{1}{N^c}\left(\sum_{y\in V} f(y)\right)^2+\frac{K_p^c}{2}\sum_{y\in V} f^2(y)$$

$$\frac{1}{2}\sum_{y\in V\sim p}\Gamma_1^c(f)(y)-\frac{|V|}{4}\sum_{y\in V} f^2(y)-\frac{1}{2}\sum_{y\in V\sim p} f(y)(\Delta f(y)-f(y)-\sum_{z\in V} f(z))$$

$$\geq \frac{1}{N^c}\left(\sum_{y\in V} f(y)\right)^2+\frac{K_p^c}{2}\sum_{y\in V} f^2(y)$$

$$\frac{1}{2}\sum_{y\in V\sim p}\Gamma_1(f)(y)-\frac{1}{2}\sum_{y\in V} f(y)\Delta f(y)-\frac{|V|-3}{4}\sum_{y\in V} f^2(y)+\frac{1}{2}\left(\sum_{y\in V} f(y)\right)^2$$

$$\geq \frac{1}{N^c}\left(\sum_{y\in V} f(y)\right)^2+\frac{K_p^c}{2}\sum_{y\in V} f^2(y)$$

A very useful divergence identity:

$$\sum_{y \in V \sim p} \Gamma_1^c(f)(y) = -\sum_{y \in V} f(y) \Delta f(y)$$

Thus,

$$\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{2 - N^c}{2N^c} \left(\sum_{y \in V} f(y) \right)^2 + \frac{2K_p^c + |V| - 3}{4} \sum_{y \in V} f^2(y)$$

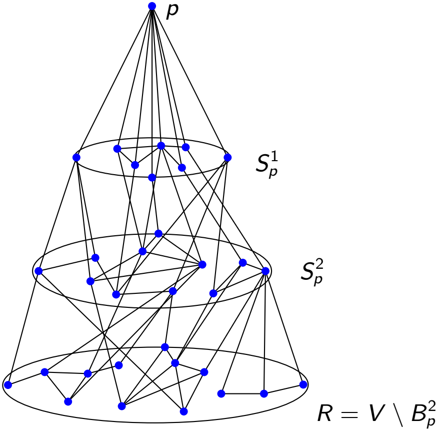
In our paper what we actually do is slightly more general. We consider the cone over not the entire vertex set of the graph G, but instead for some subset $X \subseteq V$ and derive Δ, Γ_1, and Γ_2 for the cone over X in terms of those operators on G plus some extra terms. When taking $X = V$, those extra terms simplify quite nicely to the corollary above.

To do this, we play a game common for this area where all the sums in the definitions above are split into terms that go toward the point of interest, terms that stay equidistant to the point of interest and terms that go away from the point of interest.

e.g. For $x \sim p$,

$$\begin{aligned}\Gamma_1(f, \Delta f)(x) =& \frac{1}{2}(f(p) - f(x))(\Delta f(p) - \Delta f(x)) \\ &+ \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))(\Delta f(y) - \Delta f(x)) \\ &+ \frac{1}{2} \sum_{y \in S_p^2 \sim x} (f(y) - f(x))(\Delta f(y) - \Delta f(x))\end{aligned}$$

The Tower



Recall that one may calculate the first non-trivial eigenvalue of the graph laplacian via a Rayleigh quotient:

$$\lambda_1 = inf \left\{ \frac{\|\nabla f\|^2}{\|f\|^2} \ : \ avg(f) = 0 \right\}$$

Hence,

$$\|\nabla f\|_2^2 \geq \frac{2K_p^c + |V| - 3}{2} \|f\|_2^2$$

yields that

$$\lambda_1 \geq K_p^c + \frac{\|V\| - 3}{2}$$

There is more to this story but here is likely best to stop for a 15 minute talk.

Thank you all for listening and thank you again to the organizers.

For anyone interested, the paper behind this talk is on ArXiv.org:

$$\text{a}rXiv.org - math - arXiv : 1605.05432version2$$