

# Shnol-type theorem for the Agmon ground state

Siegfried Beckus

Technion - Israel Institute of Technology

(joint work with Y. Pinchover)

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## A short historical review

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$$\begin{aligned} & H \geq 0 \text{ and} \\ & \text{for all } 0 \leq W \in L^p_{loc}(\Omega, \mathbb{R}) : \\ & H - W \not\geq 0 \end{aligned}$$

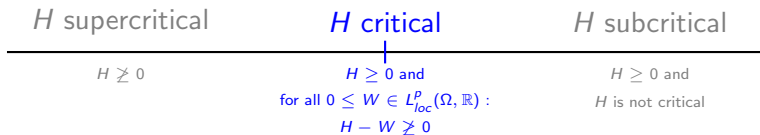
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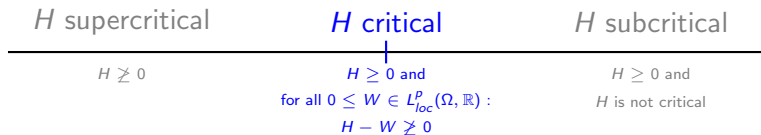
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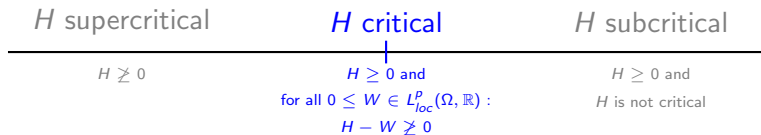
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- $H$  critical  $\Leftrightarrow H$  admits an (Agmon) ground state  $\varphi$   
(harmonic & minimal growth at infinity)

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Consider a general Schrödinger-type operator  $H+W$  on  $\Omega$  being critical with (Agmon) ground state  $\varphi$  where  $W \in L^\infty(\Omega, \mathbb{R})$ . If  $Hu = Eu$  is a generalized eigenfunction for  $E \in \mathbb{R}$  and  $|u| \leq \varphi$ , then  $E \in \sigma(H)$ .

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