Shnol-type theorem for the Agmon ground state

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(joint work with Y. Pinchover)

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A short historical review

- study Schrödinger operator $H = -\Delta + V$ defined by

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- H critical $\Leftrightarrow$ H admits an (Agmon) ground state φ
  (harmonic & minimal growth at infinity)

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Consider a general Schrödinger-type operator H on Ω being critical with (Agmon) ground state φ.

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Theorem (B/Pinchover 2017)

Consider a general Schrödinger-type operator $H+W$ on Ω being critical with (Agmon) ground state φ where $W \in L^\infty(\Omega, \mathbb{R})$. If $Hu = Eu$ is a generalized eigenfunction for $E \in \mathbb{R}$ and $|u| \leq \varphi$, then $E \in \sigma(H)$.

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