Norm convergence of the resolvent for wild perturbations

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framework

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Energy form $\mathfrak{q}(f) = \int_X |df|^2 dvol_g$ for $f \in C_0^\infty(X)$.
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This form is closable and defines by polarization the operator Laplacian, with local expression

$$\Delta(f) = -\sum_{1 \le i,j \le m} \frac{1}{\rho} \partial_{x_i}(\rho g^{ij} \partial_{x_j} f)$$

if $dvol_g = \rho . dx_1 \otimes \cdots \otimes dx_n$.
Δ is selfadjoint, non negative.

Wild Perturbations

JOURNAL OF FUNCTIONAL ANALYSIS **18**, 27–59 (1975)

Potential and Scattering Theory on Wildly Perturbed Domains*

JEFFREY RAUCH AND MICHAEL TAYLOR

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48104

Communicated by Ralph Phillips

Received February 1, 1974

We study the potential, scattering, and spectral theory associated with boundary value problems for the Laplacian on domains which are perturbed in very irregular fashions. Of particular interest are problems in which a "thin" set is deleted and the behavior of the Laplace operator changes very little, and problems where many tiny domains are deleted. In the latter case the "clouds" of tiny obstacles may tend to disappear, to solidify, or to produce an intermediate effect, depending on the relative numbers and sizes of the tiny domains. These phenomena vary according to the specific boundary value problem and in many cases their behavior is contrary to crude intuitive guesses.

INTRODUCTION

In many situations one studies the behavior of elliptic boundary a typical result of this paper : $\Omega \subset \mathbb{R}^m$ open bounded with some regularity : $H_0^1(\Omega) = \{u \in H^1(\mathbb{R}^m), \mathrm{supp}(u) \subset \overline{\Omega}\}$. K a compact set included in Ω.

$\Omega_n \xrightarrow[n\to\infty]{} \Omega \setminus K$ metrically (every compact in Ω is in Ω_n for n large enough, every compact outside $\bar{\Omega}$ is outside $\bar{\Omega}_n$ for n large enough)

$\Delta_\Omega, \Delta_{\Omega_n}$ Laplacians with Dirichlet boundary conditions

Theorem ([RT])

If K has capacity zero then for any real continuous and bounded function F and any $u \in L^2(\Omega)$ $F(\Delta_{\Omega_n})P_n u \xrightarrow[n\to\infty]{} F(\Delta_\Omega)u$ in $L^2(\mathbb{R}^m)$.

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*1this assure convergence of the discret spectrum. examples : small holes. question: can we have convergence in norm which works also for unbounded domains ?

O. Post, *Spectral analysis on graph-like spaces*, Lecture Notes in Mathematics, **2039**, Springer, Heidelberg, 2012.

$\mathcal{H}$ $(\mathcal{H}_\varepsilon, \varepsilon > 0)$, separable Hilbert spaces
closed non negative quadratic forms $(\mathfrak{q}, \mathcal{H}^1)$ in $\mathcal{H}$ and $(\mathfrak{q}_\varepsilon, \mathcal{H}^1_\varepsilon)$ in $\mathcal{H}_\varepsilon$.
$\Delta_0, \Delta_\varepsilon$ corresponding selfadjoint operators.
In $\mathcal{H}$ Sobolev type spaces $\mathcal{H}^k$ with norms $\|f\|_k = \|(\Delta_0 + 1)^{k/2} f\|$.
$(\mathcal{H}^1, \|.\|_1)$ and $(\mathcal{H}^1_\varepsilon, \|.\|_1)$ are complete by assumption.

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$$J : \mathcal{H} \to \mathcal{H}_\varepsilon \qquad J_1 : \mathcal{H}^1 \to \mathcal{H}^1_\varepsilon$$
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2. $\|f - J'Jf\| \leq \delta_\varepsilon \|f\|_1$ and $\|u - JJ'u\| \leq \delta_\varepsilon \|u\|_1$
3. $\|(J_1 - J)f\| \leq \delta_\varepsilon \|f\|_1$ and $\|(J_1' - J')u\| \leq \delta_\varepsilon \|u\|_1$
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Moreover, if $\delta_\varepsilon \to 0$ we obtain the convergence of functions of the operators in norm, of the spectrum, and of the eigenfunctions, in energy norm. (Δ_ε converge to Δ_0 in the resolvent sense).
for us, k=2

(X, g) complete Riemannian manifold of dimension $m \geq 2$
$X_\varepsilon = X - B_\varepsilon$ with $B_\varepsilon = \cup_{j \in \mathcal{J}_\varepsilon} B(x_j, \varepsilon)$.*2
$\varepsilon > 0, (x_j)_{j \in \mathcal{J}_\varepsilon}$ such that $\mathrm{d}(x_j, x_k) \geq 2b_\varepsilon \gg \varepsilon$
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χ_ε cut-off function on $\varepsilon < r < \varepsilon^+$, $\varepsilon \ll \varepsilon^+ \ll b_\varepsilon$,
$\chi_\varepsilon(r) = \frac{1/r^{m-2} - 1/\varepsilon^{m-2}}{1/\varepsilon^{+(m-2)} - 1/\varepsilon^{m-2}}$ resp. $\chi_\varepsilon(r) = \frac{\log(r/\varepsilon)}{\log(\varepsilon^+/\varepsilon)}$.

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$$|q_\varepsilon(J_1 f, u) - q(f, J_1' u)| = \Big| \int_{X_\varepsilon} (d(\chi_\varepsilon f) - df, du) \Big|$$

$$= \Big| \int_{B_{\varepsilon^+}} (\chi_\varepsilon - 1)(df, du) + \int_{B_{\varepsilon^+}} f(d\chi_\varepsilon, du) \Big|$$

$$\leq \|u\|_1 \Big(\sqrt{\int_{B_{\varepsilon^+}} |df|^2} + \sqrt{\int_{B_{\varepsilon^+}} f^2 |d\chi_\varepsilon|^2} \Big)$$

*3

assumption of bounded geometry

(X, g) has *bounded geometry*: there exist, $i_0 > 0$ and k_0 such that

$$\forall x \in X, \mathrm{Inj}(x) \geq i_0 \quad \mathrm{Ric}(x) \geq k_0.g$$

E. Hebey, *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, Courant Lecture Notes in Mathematics **5**, American Mathematical Society, Providence, 1999.

- this assure the existence of a uniform harmonic radius
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Theorem (Dirichlet fading)

If $b_\varepsilon = \varepsilon^\alpha, 0 < \alpha < \frac{m-2}{m}$ (and for m=2 $b_\varepsilon = |\log \varepsilon|^{-\alpha}$, $0 < \alpha < 1/2$) The Laplacian with Dirichlet boundary conditions on X_ε converge (in the resolvent sense) to the Laplacian on X.

crushed ice problem

also results for the *solidifying* situation: $X_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} X \setminus \Omega_0$
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$$\Omega_{\alpha_\varepsilon} = \{x \in X,\ d(x, \Omega_0) < \alpha_\varepsilon\} \subset B_{b_\varepsilon^+}$$
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Theorem

If $\lim_{\varepsilon\to 0} \alpha_\varepsilon \lambda_\varepsilon = +\infty$, the Laplacian Δ_ε with Dirichlet boundary conditions on $X_\varepsilon = X - B_\varepsilon$ converge in the sense of the resolvent, to the Laplacian Δ_0 with Dirichlet boundary conditions on $X - \Omega_0$.

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$\lambda_\varepsilon = \lambda_1(\Delta_D^N(B_{\mathbb{R}^m}(0, b_\varepsilon^+) - B_{\mathbb{R}^m}(0, \varepsilon)) \geq \frac{C\varepsilon^m}{b_\varepsilon^{+(m-2)}}$. [RT]

Adding handles

Adding handles

Comment. Math. Helvetici **56** (1981) 83–102

0010–2571/81/001083–20$01.50+0.20/0

Spectra of manifolds with small handles

I. CHAVEL[1] and E. A. FELDMAN[1]

To H. E. RAUCH, in memoriam

In this paper we consider a compact connected C^∞ Riemannian manifold M of dimension $n \geq 2$ and study the effect, on the spectrum of the associated Laplace–Beltrami operator Δ acting on functions, of adding a "small" handle to M.

The handles we consider are defined as follows: Fix two distinct points p_1, p_2 in M and for $\varepsilon > 0$ define

$B_\varepsilon \equiv$: union of the open geodesic disks about p_1, p_2 of radius ε,
$\Omega_\varepsilon \equiv: M - \overline{B}_\varepsilon$,
$\Gamma_\varepsilon \equiv$: common boundary of B_ε and Ω_ε,
$S_\varepsilon \equiv$: $(n-1)$-sphere in R^n of radius ε,
$S \equiv: S_1$.

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- $\varepsilon > 0, (x_j^{\pm})_{j\in\mathcal{J}}$ such that $\mathrm{d}(x_j^{\pm}, x_k^{\pm}) \geq 2b_\varepsilon \gg \varepsilon$
- We define $X_\varepsilon = X - B_\varepsilon$ with $B_\varepsilon = \cup_{s=\pm, j\in\mathcal{J}} B(x_j^s, \varepsilon)$
- a set of handles of length $l_\varepsilon > 0$:

$$C_\varepsilon = \cup_{j\in\mathcal{J}}[0, l_\varepsilon] \times \varepsilon\mathbb{S}^{m-1}.$$

- $\forall j \in \mathcal{J}_\varepsilon$, we glue $[0, l_\varepsilon] \times \varepsilon\mathbb{S}^{m-1}$ between x_j^- and x_j^+ (almost isometric):*5

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quadratic form on study:

$$\mathfrak{q}_\varepsilon(u,h) = \int_{X_\varepsilon} |du|^2 + \sum_{j\in\mathcal{J}} \int_C \left(\frac{1}{l_\varepsilon^2} |\partial_s h_j|^2 + \frac{1}{\varepsilon^2} |\partial_\theta h_j|^2 \right)$$

with domain:

$$\mathcal{D}(\mathfrak{q}_\varepsilon) = \Big\{ (u,h) \in H^1(X_\varepsilon) \times H^1(C);\ \forall j \in \mathcal{J}_\varepsilon$$
$$h_j(0,\theta) = \sqrt{\varepsilon^{m-1} l_\varepsilon}\, u(x_j^-, \varepsilon\theta),\ h_j(1,\theta) = \sqrt{\varepsilon^{m-1} l_\varepsilon}\, u(x_j^+, \varepsilon\theta) \Big\}$$

fading condition

Theorem

If $b_\varepsilon = \varepsilon^\beta$ $(\beta < 1/2)$, $\lim_{\varepsilon \to 0} l_\varepsilon = 0$ and $\lim_{\varepsilon \to 0} \frac{\varepsilon^{1-2\beta}}{l_\varepsilon} = 0$, then

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$$\begin{aligned} J : \mathcal{H} &\to \mathcal{H}_\varepsilon & J_1 : \mathcal{H}^1 &\to \mathcal{H}^1_\varepsilon \\ f &\mapsto (\mathbf{1}_{X_\varepsilon}.f, 0) & f &\mapsto (\mathbf{1}_{X_\varepsilon}.f, \Phi_\varepsilon(f))(\text{harmonic on } C_\varepsilon) \\ J' = J^* : \mathcal{H}_\varepsilon &\to \mathcal{H} & J'_1 : \mathcal{H}^1_\varepsilon &\to \mathcal{H}^1 \\ u &\mapsto \mathbf{1}_{X_\varepsilon}.u & u &\mapsto \tilde{u}(\text{harmonic on } B_\varepsilon) \end{aligned}$$

gluing condition

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$\Omega^{\pm}$ regular isometric $(\mathrm{d}(\Omega^{+},\Omega^{-})>0)$, let:

$$\Omega_{\alpha}=\Omega_{\alpha}^{-}\cup\Omega_{\alpha}^{+},\quad \Omega=\Omega^{-}\cup\Omega^{+}.$$

$$\exists\alpha_0>0\quad \phi:\Omega_{\alpha_0}^{-}\to\Omega_{\alpha_0}^{+}\text{ isometry such that }\phi(\Omega^{-})=\Omega^{+}$$

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Theorem

If $B_\varepsilon^{\pm}$ solidify in $\Omega^{\pm}$ and if

$$\lim_{\varepsilon\to 0}\alpha_\varepsilon+\frac{b_\varepsilon^{+m}(l_\varepsilon+\varepsilon)}{\alpha_\varepsilon\varepsilon^{m-1}}+\frac{\varepsilon l_\varepsilon}{b_\varepsilon^{+2}}=0$$

then the Laplacian on $X_\varepsilon\cup C_\varepsilon$ converges, in the resolvent sense, to the Laplacian on functions on X which coincide on Ω^+ and Ω^- : $\mathcal{D}(\Delta_0)=\{f\in\mathcal{H}^2,(f-f\circ\phi)_{|\Omega^-}=0\}$

Thank you!

complements

$\Phi_\varepsilon(f) = (h_j)_{j\in\mathcal{J}}$ satisfies $-\partial_s^2(h_j) + (\frac{l_\varepsilon}{\varepsilon})^2\Delta_{\mathbb{S}^{m-1}}(h_j) = 0$.

Gluing condition applies for $l_\varepsilon = \varepsilon$ if there exists β, $\frac{m-2}{m-1} < \beta < 1$ and γ, $\beta < \gamma < \beta m - (m-2)$ such that $\Omega^\pm_{\alpha(\varepsilon)} \subset B^\pm_{\varepsilon^+}$ as a k-regular cover, for $\alpha(\varepsilon) = \varepsilon^\gamma$ and $\varepsilon^+ = \varepsilon^\beta$.

recall

transplantations:

$$J : \mathcal{H} \to \mathcal{H}_\varepsilon \qquad J_1 : \mathcal{H}^1 \to \mathcal{H}^1_\varepsilon$$

$$J' : \mathcal{H}_\varepsilon \to \mathcal{H} \qquad J'_1 : \mathcal{H}^1_\varepsilon \to \mathcal{H}^1.$$

assumption of quasi-unitary equivalence:
$\forall f \in \mathcal{H}^1$, $\forall u \in \mathcal{H}^1_\varepsilon$:

1. $| < J'u, f > - < u, Jf > | \leq \delta_\varepsilon \|f\|_1 \, \|u\|_1$
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4. $|q_\varepsilon(J_1 f, u) - q(f, J'_1 u)| \leq \delta_\varepsilon \|f\|_2 \, \|u\|_1$

conclusion:

$$\|(\Delta_\varepsilon + 1)^{-1} J - J(\Delta_0 + 1)^{-1}\|_{0 \to 0} \leq 4\delta_\varepsilon.$$