C*-Algebras

Winter semester 2016/17 **Prof. Dr. M. Keller**

Sheet 4

(1) Let A be a unital Banach algebra. Show that the inversion

$$\mathrm{Inv} : GL(A) \to GL(A),\ x \mapsto x^{-1}$$

is differentiable in every point a with derivative

$$D_a\mathrm{Inv} : A \to A, x \mapsto -a^{-1}xa^{-1}$$

i.e.,

$$\frac{1}{\|b-a\|}\|\mathrm{Inv}(b) - \mathrm{Inv}(a) - D_a\mathrm{Inv}(b-a)\| \to 0, \quad b \to a$$

(2) Let A be a unital Banach algebra. Show that the connected components of $\mathrm{GL}(A)$ are given by the path components. (The path component of an element $a \in \mathrm{GL}(A)$ is the set of all $b \in \mathrm{GL}(A)$ such that there is a continuous $\gamma : [0,1] \to \mathrm{GL}(A)$ such that $\gamma(0) = a$ and $\gamma(1) = 1$.)

(3) Let A be a unital Banach algebra. Let $a, b \in A$ and $\lambda \in \sigma(a) \cap \sigma(b)$ and $\mu \in \sigma(a)$.

(a) Show

$$(a-\lambda)^{-1} - (b-\lambda)^{-1} = (a-\lambda)^{-1}(b-a)(b-\lambda)^{-1} = (b-\lambda)^{-1}(b-a)(a-\lambda)^{-1}.$$

(b) Show

$$(a-\lambda)^{-1} - (a-\mu)^{-1} = (\mu-\lambda)(a-\lambda)^{-1}(a-\mu)^{-1} = (\mu-\lambda)(a-\mu)^{-1}(a-\lambda)^{-1}.$$

(4) Let A be a unital Banach algebra and $a, b \in A$. Show

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}.$$

Optional Problems

(OP1) Let K be a compact metric space. Determine the connected components of $\mathrm{GL}(C(K))$.