# Schrödinger operators over dynamical systems

**Winter semester 2020** **Dr. S. Beckus**

**Sheet 4** **Due on Thursday 12/03/2020 at 10.00 am**

**Exercise 1** (4 points)**.** Let $\mathcal{A} := \{a, b\}$ and $\omega \in \mathcal{A}^{\mathbb{Z}}$ be the one-defect, namely

$$\omega(n) := \begin{cases} a, & n \neq 0, \\ b, & n = 0, \end{cases} \qquad n \in \mathbb{Z}.$$

Prove that $\overline{Orb(\omega)}$ is uniquely ergodic.

**Exercise 2** (4 points)**.** Let $G := \mathbb{C} \times \mathbb{R}$ be the topological group from Sheet 1 equipped with the composition and inverse defined by

$$(v, s)(w, t) := (v + w, s + t - \Im(v \cdot \overline{w})),$$
$$(v, s)^{-1} := (-v, -s),$$

for $(v, s), (w, t) \in G$. Prove the following statements.

(a) The set

$$\Gamma := \{(m + il, k) \,|\, m, l \in \mathbb{Z}, k \in \mathbb{Z}\}$$

is a discrete subgroup of $G$.

(b) The discrete group $\Gamma$ is amenable with Følner sequence

$$F_n := \{(m + il, k) \in \Gamma \,|\, |m| \leq n,\ |l| \leq n, |k| \leq n^2\}, \qquad n \in \mathbb{N}.$$

Hint: For (b), it suffices to consider only compact sets $K$ that are singletons (why?).

**Exercise 3** (4 points)**.** Let $(X, G)$ be a dynamical system and $\mathcal{J}$ be the space of dynamical subsystems. Consider a uniquely ergodic $Y \in \mathcal{J}$ and a sequence $Y_n \in \mathcal{J},\ n \in \mathbb{N}$. Sow that if $Y_n \to Y$ in $\mathcal{J}$, then

$$\lim_{n\to\infty} \mathcal{M}^1(Y, G) = \limsup_{n\to\infty} \mathcal{M}^1(Y_n, G) = \mathcal{M}^1(Y, G),$$

where the limit is taken in the Chabauty-Fell topology on $\mathcal{K}(\mathcal{M}^1(X, G))$.

**Exercise 4** (4 points)**.** Let $X$ be a compact metric space and $\mu \in \mathcal{M}(X)$. Prove the following assertions.

(a) The support $supp(\mu) \subseteq X$ is closed.

(b) $\mu(A) = \mu(A \cap supp(\mu))$ for all $A \subseteq X$ measurable.

(c) If $(X, G)$ is a dynamical system and $\mu$ is $G$-invariant, then $supp(\mu)$ is invariant.

**Bonus exercise 1.** Let $X$ be topological space. Let $Y \subseteq X$ be a clopen subset. Prove that $\chi_Y \in C(X)$, where

$$\chi_Y(x) := \begin{cases} 1, & x \in Y, \\ 0, & x \notin Y, \end{cases} \qquad x \in X.$$

**Bonus exercise 2** (1 point)**.** Let $\mathcal{A} := \{a, b\}$ and $\omega, \rho \in \mathcal{A}^{\mathbb{Z}}$ be defined by

$$\omega(n) := \begin{cases} a, & n \leq 0, \\ b, & n \geq 1, \end{cases} \qquad \rho(n) := \begin{cases} b, & n \leq 0, \\ a, & n \geq 1, \end{cases} \qquad n \in \mathbb{Z}.$$

Prove that the subshift $\Omega = \overline{Orb(\omega)} \cup \overline{Orb(\rho)}$ is countable. Determine all point $\Omega$ that are not isolated.