Schrödinger operators over dynamical systems

Winter semester 2020 **Dr. S. Beckus**

Sheet 2 **Due on Thursday 11/19/2020 at 10.00 am**

Exercise 1 (4 points)**.** Let (X, d) be a compact and complete metric space. Prove that $\mathcal{K}(X)$ equipped with the Hausdorff metric

$$\delta_H(F, K) := \max\left\{ \sup_{x\in F} \inf_{y\in K} d(x, y), \sup_{y\in K} \inf_{x\in F} d(x, y) \right\}$$

is a totally bounded metric space.
Hint: You can assume that $(\mathcal{K}(X), \delta_H)$ is a metric space.

Exercise 2 (4 points)**.** Let X, Y be locally compact spaces. Prove the following statements.

(a) If $f : X \to Y$ is continuous, then the map $\widetilde{f} : \mathcal{K}(X) \to \mathcal{K}(Y), K \mapsto f(K)$, is continuous in the Vietoris topology.

(b) The maps $\sup : \mathcal{K}(\mathbb{R}) \to \mathbb{R},\ K \mapsto \sup K$ and $\inf : \mathcal{K}(\mathbb{R}) \to \mathbb{R},\ K \mapsto \inf K$ are continuous in the Vietoris topology on $\mathcal{K}(X)$.

Exercise 3 (4 points)**.** Consider the dynamical system $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$ for $\mathcal{A} := \{a, b\}$ and $\omega \in \mathcal{A}^{\mathbb{Z}}$ is defined by

$$\omega(n) := \begin{cases} a, & n \neq 0, \\ b, & n = 0, \end{cases} \qquad n \in \mathbb{Z}.$$

Is the corresponding orbit closure $\Omega := \overline{Orb(\omega)}$ minimal? In order to check this, compute the set $\overline{Orb(\omega)}$. If Ω is not minimal, compute all its minimal components.

Exercise 4 (4 points)**.** Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces over $\mathbb{C}$ and $T : E \to F$ be linear (i.e. $T(\lambda x + y) = \lambda T(x) + T(y)$ for all $x, y \in E$ and $\lambda \in \mathbb{C}$). Prove that the following statements are equivalent.

(i) T is continuous,
(ii) T is continuous at 0,
(iii) there is an $C > 0$ such that $\|T(x)\| \leq C\|x\|$ holds for all $x \in E$,
(iv) T is uniformly continuous.

Bonus exercise 1 (1 point)**.** Let (X, G) be a dynamical system. Prove that if $Y, Z \in \mathcal{J}$ are minimal, then either $Y \cap Z = \emptyset$ or $Y = Z$.

Bonus exercise 2 (1 point)**.** Show that the Vietoris topology and Chabauty-Fell topology are different on $\mathcal{K}(\mathbb{R})$.
Hint: Consider the sequence $A_n := [0, 1] \cup \{n\} \in \mathcal{K}(\mathbb{R})$