Generalized Bloch Theory for Quasicrystals

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Diplom-Mathematiker

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Zusammenfassung in deutscher Sprache

Diese Diplomarbeit beschäftigt sich mit der speziellen Anordnung der Atome oder Moleküle in Quasikristallen. Unser Ziel ist es, eine geeignete Zerlegung für einen entsprechenden Schrödingeroperator anzugeben. Zur Beschreibung der Struktur eines Quasikristalls werden wir zunächst ein gut geeignetes Modell angeben. Als nächstes werden wir Konzepte einführen, die wir für eine Theorie benötigen, die unabhängig von der Lokalisation des Ursprungs ist. Dabei werden wir auch den Begriff des direkten Integrals kennen lernen. Diese Methode hat sich als nützlich herausgestellt, um komplizierte Operatoren zu untersuchen. Dann werden wir unseren weitere Überlegungen auf den $\mathbb{R}^d$ beschränken. Es wird um jedes Atom oder Molekül eine Zelle konstruiert, die den Einflussbereich der zugehörigen Potentiale enthält. Aufgrund der Beschaffenheit der Quasikristalle wird sich herausstellen, dass es nur eine endliche Anzahl an verschiedenen Zellen und damit an verschieden Potentialen gibt. Es wird sich zeigen, dass wir damit einen Hilbertraum konstruieren können und dieser unitär äquivalent zu $L^2(\mathbb{R}^d)$ ist. Die entsprechende unitäre Abbildung heißt Wannier-Transformation inspiriert durch die Arbeit von [BNM]. Abschließend werden wir in Anlehnung an diese Arbeit die Zerlegung des Schrödingeroperators bezüglich dieser Transformation angeben und beweisen.

Abstract

This diploma thesis engages with the special arrangement of atoms or molecules of quasicrystals. We would like to find an appropriate decomposition of the corresponding Schrödinger operator. Therefore, we will first specify a mathematical model which describes the structure of a quasicrystal in a good way. Next we introduce concepts that are necessary for a theory independently of the origin's position. Besides, we will get to know the notion of a direct integral. It is well-known that this method is useful to analyze difficult operators. Then, we restrict our further considerations to the space $\mathbb{R}^d$. We will construct a cell around every atom or molecule such that it contains the area of influence of the corresponding potentials. As a result of the configuration of quasicrystals it points out that there can be at most a finite number of different cells and potentials, respectively. According to this we will construct a Hilbert space which is unitary equivalent to $L^2(\mathbb{R}^d)$. The related unitary map is called Wannier transform inspired by [BNM]. Finally, we indicate a decomposition of the Schrödinger operator with respect to this transformation and prove it by following [BNM].

Contents

Chapter 1

Introduction

A crystal is a solid material consisting of periodically arranged atoms. Each such a material satisfies a translation invariance condition according to their lattice structure. In 1984 Dan Shechtman analyzed the structure of an aluminum-manganese composition which had been cooled down fast. He discovered an unusual structure of this material, which is called quasicrystal today. He was awarded the Nobel Prize in Chemistry for his work ([SBGC]) in 2011. In this paper it turns out that the symmetry of the material coincides with the icosahedral point group. This is incompatible with the lattice translations. Since such an alloy has a high degree of hardness and a good elasticity it is of special interest in physics as well as in other fields.

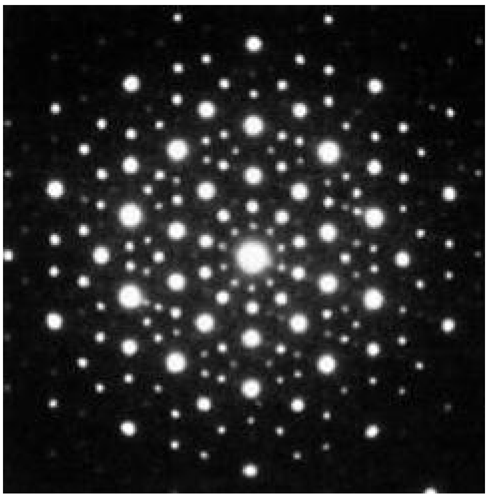

Figure 1 [1] :
The diffraction pattern of a quasicrystal

The configuration of the atoms and molecules in quasicrystals appears regular but not periodic. In Figure 1 we can see the diffraction pattern of the kind of quasicrystals

[1] Source: [SBGC]

observed by Shechtmann. One feature of quasicrystals is that there can be found at most a finite number of different sections. This property is called finite local complexity.

A mathematical model of such structures is typically embedded in a group. For instance, one can consider the group $(\mathbb{R}^d, +)$. This diploma thesis can be divided in two parts. In chapter 2., 3. and 4. our elaborations are dealing with locally compact groups in general. Chapters 5., 6. and 7. are restricted to $\mathbb{R}^d$, where we exploit special properties of the euclidean space. The second part is inspired by [BNM].

We will proceed in the following way: First, in chapter 2. the mathematical model which represents our quasicrystals is given. This part is mainly inspired by [BLM] and [BNM]. First it makes sense to take into account the fact that the atoms or molecules of a material maintain a minimal distance to each other. Further, we would like to exclude the case that there are gaps in the material meaning that we have a maximal distance bewteen the particles. As mentioned above one feature of quasicrystals is that they are of finite local complexity. A set holding these conditions is called D-set. Then, a D-set $\mathcal{D}$ displays the positions of the atoms or molecules of the material. In section 2.3 ("Hull and Transversal") we introduce the notion of the transversal $\mathcal{T}$. It can be imagined as the set of all translates of the origin to any atom of the material. This is necessary for a theory which is independent of the choice of the origin's position.

In chapter 3 we construct the Lagarias group and its dual group, which will be used to transform our space into another. Moreover, there is given an introduction to the theory of Haar measures and dual groups in general, see [DE], [LOO]. Further, in chapter 4 we will declare the main ideas of the concepts about groupoids, continuous fields of Hilbert spaces and representations. These are useful tools to decompose difficult operators to determine their spectrum easier. The underlying concept deals with direct integrals.

In the next chapter we introduce the concept of a tiling of $\mathbb{R}^d$. In detail, we define a cell, called Voronoi cell, for each element of our D-set $\mathcal{D}$ such that the set of all these cells tiles our space $\mathbb{R}^d$. For instance, we can imagine the Penrose tiling (Figure 2) where our aperiodic Delone set is the set of all barycenters of each of these cells. In our further considerations any cell will be endowed with a color (so called Collar) which depends on the adjacent cells of it. The colors in Figure 2 are for aesthetic effects only. They do not have any relation to the Collar of the cells. Then, we define a set $\mathcal{P}^{(d)}$ of equivalence classes of such cells with respect to their Collar. It is called the set of all collared Voronoi proto d-cells. According to the fact that quasicrystals are of finite local complexity it follows that $\mathcal{P}^{(d)}$ is finite. This is one essential property of quasicrystals.

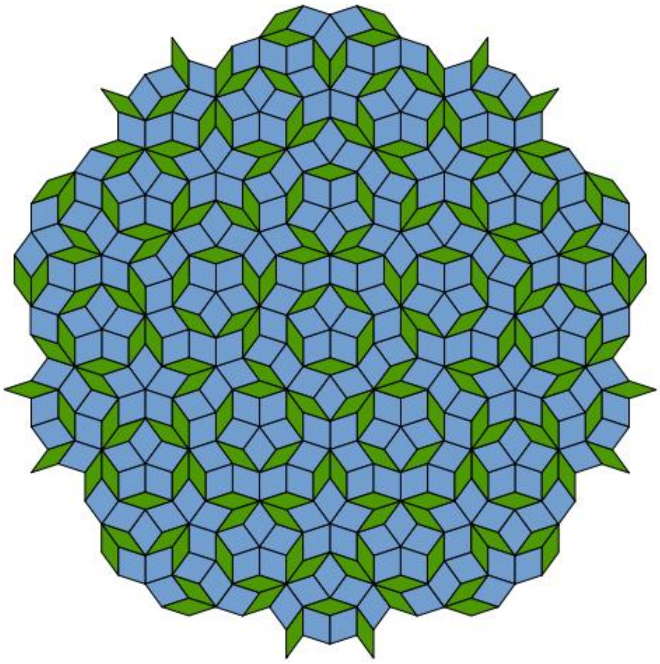

Figure 2 [2] :
The Penrose tiling

Our main aim in this work is to find a useful decomposition of a Schrödinger operator of a quasicrystal such that we can determine the spectrum of the operator easier. The knowledge of the eigenfunctions and their eigenvalues is of particular importance, because the eigenvalues specify the admissible energy states of the system. Note that the eigenfunctions of an operator are usually elements of the function space $L^2(\mathbb{R}^d)$. In order to do so it is necessary to construct a convenient Hilbert space which is unitary equivalent to this L^2-space. We suppose that the essential part of the potential of an atom or molecule lies in the corresponding Voronoi cell of it. According to our previous considerations there are only a finite set of different cells. Thus, it is sufficient for us to draw our attention to a finite number of different potentials. In detail, the procedure works as follows:

The property that $\mathcal{P}^{(d)}$ is finite will be used to define finitely many closed subspaces of the L^2-space of the dual Lagarias group. Using this, one can construct a family of Hilbert spaces $(\mathcal{H}_t)_{t\in\mathcal{T}}$ over the transversal $\mathcal{T}$. By defining a map $\mathcal{W}_t : \mathcal{C}_\mathrm{c}(\mathbb{R}^d) \to \mathcal{H}_t$, called Wannier transform, we will show that these transformations satisfy some useful features, as a covariance condition. It turns out that we can extend $\mathcal{W}_t$ to a unitary map between $L^2(\mathbb{R}^d)$ and $\mathcal{H}_t$.

Furthermore, we can endow the family $(\mathcal{H}_t)_{t\in\mathcal{T}}$ with a structure of a continuous field of Hilbert spaces. We will also specify a strongly continuous, unitary representation

[2] Source: http://en.wikipedia.org/wiki/File:Penrose_Tiling_(Rhombi).svg, January, 27th, 2012

with respect to the family $(\mathcal{H}_t)_{t\in\mathcal{T}}$. Thus, our theory is independent of the choice of the origin's position.

Finally, we adopt the Wannier transform to decompose the Schrödinger operator. We expect that this decomposition allows us to determine the spectrum of such operators in an easier way. Since we have a strongly continuous, unitary representation the origin can be replaced without changing the spectrum of the Schrödinger operator.

At this point I would like to thank my advisor Prof. Dr. Daniel Lenz for his caring support in the last year. Especially I thank him for giving me the opportunity to take a trip to Paris to visit Dr. Giuseppe de Nittis and Prof. Dr. Jean Bellissard. My special gratitude goes to Marcel Schmidt who read this work very carefully and who delivered many good ideas to me. I would like to express my thanks to Felix Pogorzelski, Dr. Matthias Keller and Sebastian Haeseler for their friendly aid and giving me a lot of inspirations. Moreover, I have always appreciated spending their time to read my work and helped me to understand some problems. My gratitude goes also to Dr. Giuseppe de Nittis who spent his time to meet me in France and helped me with some questions. I thank all the other readers of this diploma thesis taking their time. More precisely, I thank Juliane Neubauer, Peter Beckus, Petra Hofmann and Saskia Skeide. Further, I thank Prof. Dr. Jean Bellissard, Dr. Giuseppe de Nittis and Dr. Vida Milani for giving me the possibility to read their preliminary paper [BNM].

Chapter 2

Delone sets

Let G be a group with group composition "+". We denote by $-x$ the inverse element of $x \in G$ and by 0 the neutral element of the group G, which are unique. It is common to write $x - y$ for $x, y \in G$ instead of $x + (-y)$.

Further, we call G an *abelian group* if for each $x, y \in G$ the equation

$$x + y = y + x$$

holds.

In the second chapter we will use linear $\mathbb{Z}$-combinations of elements of the group G and of subsets of the group. Since we denote the composition of the group by "+" we will use the following notations. For $x_1, \ldots, x_k \in G$ we write $\sum_{j=1}^{k} x_j$ instead of $x_1 + x_2 + \ldots + x_k$, which is an element of G. Now let n be any element of $\mathbb{Z}$ and $x \in G$, then,

$$n \cdot x := \begin{cases} 0 & n = 0 \\ \sum_{j=1}^{n} x & n > 0 \\ \sum_{j=1}^{|n|} -x & n < 0 \end{cases}.$$

These notations are motivated by the form of linear $\mathbb{Z}$-combination of $\mathbb{R}^d$. Using this we define the set of all finite linear $\mathbb{Z}$-combinations of some subset $\mathcal{F} \subseteq G$ by

$$[\mathcal{F}] := \left\{ \sum_{j=1}^{N} n_j \cdot x_j \mid N \in \mathbb{N},\ x_j \in \mathcal{F} \text{ and } n_j \in \mathbb{Z} \right\}.$$

Moreover, we call a finite subset $\{x_1, \ldots, x_k\} \subseteq G$ *linear $\mathbb{Z}$-independent* if

$$\sum_{j=1}^{k} n_j \cdot x_j = 0 \quad \Rightarrow \quad n_1 = \ldots = n_k = 0.$$

Let A, B be subsets of G and $x \in G$, then,

$$A + B := \{y + z \mid y \in A, z \in B\},$$
$$-A := \{-y \mid y \in A\},$$
$$x + A := \{x + y \mid y \in A\}.$$

The pair $(G, \mathcal{O})$ is called a *topological group* if G is a group, $(G, \mathcal{O})$ is a topological space, where $\mathcal{O}$ is the topology on G and for each $x, y \in G$ the maps

$$(x, y) \overset{\rho}{\mapsto} x + y,$$
$$x \overset{i}{\mapsto} -x,$$

are continuous. Note that $G \times G$ is endowed with the product topology. An element $x \in G$ has a *compact neighborhood* if there is some open set $V \in \mathcal{O}$ and some compact subset K of G such that $x \in V \subseteq K$. We say that $(G, \mathcal{O})$ is a *locally compact group* if each $x \in G$ has a compact neighborhood.

Let A be some subset of G. An element $b \in A$ is called an *isolated point* if there is an open neighborhood $V \in \mathcal{O}$ of b such that $A \cap V = \{b\}$. We call the set $A \subseteq G$ *discrete* if any element of A is an isolated point. The *induced topology* of $B \subseteq G$ is given by $\mathcal{O}(B) := \{V \cap B \mid V \text{ open in } G\}$. Thus, for any discrete subset A of G the induced topology contains every single element of A. This topology is called *discrete topology.*

A topological group G is called *σ-compact* if there is a countable family of compact subsets $(K_i)_{i=1}^{\infty}$ of G such that $G = \bigcup_{i=1}^{\infty} K_i$. Furthermore, a topological group is *compactly generated* if every subset $A \subseteq G$ is closed, if and only if $A \cap K$ is closed for each compact subset $K \subseteq G$.

We consider here a compactly generated, locally compact, abelian group which is σ-compact. Further, we suppose that G is a *Hausdorff space*, which means that for any two distinct elements x and y of G there are two disjoint open neighborhoods U_x and U_y of x respectively y.

Note further, that any compact subset of a Hausdorff space is closed, see [QUE]. For a set $A \subseteq G$ we will denote by $\sharp(A)$ the number of elements of A.

In this work we consider a locally compact, abelian, Hausdorff group $(G, \mathcal{O})$ which is compactly generated and σ-compact only. We write G instead of $(G, \mathcal{O})$.

In the section "Hull and Transversal" we will need the following concept of a net. This is a more general concept than sequences. A pair $(I, \lhd)$ is called a *directed set* if " $\lhd$ " is a relation on the non-empty set I such that the following assertions hold.

(i) For any $\iota \in I$ the relation $\iota \lhd \iota$ holds.

(ii) For any $\iota_1, \iota_2, \iota_3 \in I$ such that $\iota_1 \lhd \iota_2$ and $\iota_2 \lhd \iota_3$ implies that $\iota_1 \lhd \iota_3$.

(iii) For all $\iota_1, \iota_2 \in I$ there exists an $\iota_3 \in I$ such that the relations $\iota_1 \lhd \iota_3$ and $\iota_2 \lhd \iota_3$ are true.

Let $(I, \lhd)$ be a directed set and consider some subset $X \subseteq Y$ of a topological space Y. A map $\tilde{x} : I \to X$ is called *net*. As in the case of sequences we write $(x_\iota)_{\iota \in I} \subseteq Y$ instead of $\tilde{x}$. We say a net *converges to x ($x_\iota \to x$)* if for any neighborhood U of x there is an $\iota_0 \in I$ such that for each index $\iota \in I$ greater than ι_0 ($\iota \rhd \iota_0$) we have $x_\iota \in U$.

Let $(x_\iota)_{\iota \in I}$ be some net in $\mathbb{R}$, the limes superior $\overline{\lim_\iota}$ and the limes inferior $\underline{\lim_\iota}$ are defined by

$$\begin{aligned}
\overline{\lim_\iota}\, x_\iota &:= \lim_\iota \left(\sup_{k \rhd \iota} x_k \right) = \inf_\iota \left(\sup_{k \rhd \iota} x_k \right), \\
\underline{\lim_\iota}\, x_\iota &:= \lim_\iota \left(\inf_{k \rhd \iota} x_k \right) = \sup_\iota \left(\inf_{k \rhd \iota} x_k \right).
\end{aligned}$$

Note that, if the topological space Y is a Hausdorff space, it can be shown that a net has at most one limit point in Y, see [QUE]. The topology of Y has a *countable base* or Y is *second countable* if there exists a countable collection of open sets such that each other open set can be expressed by the union of some of these open sets.

Let X and Y be topological spaces. We call a map $f : X \to Y$ *continuous*, if for any $x \in X$ and some open neighborhood $V \subseteq Y$ of $f(x)$ there is an open neighborhood $W \subseteq X$ of x such that $f(W) \subseteq V$. In a space with a countable base the concepts of continuity and closure are described by sequences, else the notion of a net is necessary, see [QUE].

Lemma 2.1 ([QUE])**.** *Let X and Y be topological spaces. Then, $f : X \to Y$ is continuous in $x \in X$, iff for any net $(x_\iota)_{\iota \in I} \subseteq X$ with $x_\iota \to x$, it follows that $f(x_\iota) \to f(x)$ in Y.*

2.1 Definition and properties

This section is inspired by [BLM] and [BNM]. A subset $\mathcal{D} \subseteq G$ is called *aperiodic* if for some $t \in G$ the equality $t + \mathcal{D} = \mathcal{D}$ implies that $t = 0$.

Furthermore, a subset $\mathcal{D} \subseteq G$ is *uniformly discrete* if there is an open neighborhood U of $0 \in G$ such that for any $x \in G$ the sets $x + U$ and $\mathcal{D}$ have at most one common point. Then, $\mathcal{D}$ is called *U-uniformly discrete*. We say for $x \in G$ that the set $x + U$ is a *translate* of U. The definition of a set $\mathcal{D}$ which is U-uniformly discrete implies

$$(\alpha + U) \cap \mathcal{D} = \{\alpha\}, \qquad \text{for all } \alpha \in \mathcal{D}.$$

Further, we call $\mathcal{D}$ *relatively dense* if there exists a compact set K such that

$$\mathcal{D} + K = \bigcup_{\alpha \in \mathcal{D}} (\alpha + K) = G.$$

In this situation we say $\mathcal{D}$ is *K-relatively dense.*

Imagine $\mathcal{D}$ as the set of positions of atoms in some material. The U-uniform discreteness means that these atoms have a minimal distance between each other whereas the definition of K-relatively dense does not allow too big gaps in the material. The property that $\mathcal{D}$ is aperiodic is an important feature of quasicrystals. In detail, the order of the atoms of a quasicrystal is not a cubic-lattice.

For instance, consider $G = \mathbb{R}$ and let $\mathcal{D} = \mathbb{Z}$. Then, $\mathcal{D}$ is U-uniformly discrete and K-relatively dense. More precisely, U can be chosen as the open interval $(-\frac{1}{2}, \frac{1}{2})$ and K can be chosen as the closed interval $[-\frac{1}{2}, \frac{1}{2}]$. Note that $\mathbb{Z}$ is not aperiodic. In fact, $\mathcal{D}$ satisfies that $t + \mathcal{D} = \mathcal{D}$ for any $t \in \mathbb{Z}$. At the end of this chapter we give a further example which is also aperiodic.

Lemma 2.2. *Let G be an abelian topological group and $\mathcal{D} \subseteq G$. Then, the following assertions are equivalent for a compact subset $K \subseteq G$.*

(i) The set $\mathcal{D}$ is K-relatively dense.

(ii) For any $x \in G$ the set $(x + (-K)) \cap \mathcal{D}$ is not empty .

Proof. Let $K \subseteq G$ be compact such that $\mathcal{D} + K = G$. Moreover, we know that a continuous function on a topological space maps compact sets onto compact sets. Since $i : G \to G$ is continuous we know that $(-K)$ is compact.

"$\Rightarrow$": Let x be any element of G. We will show that $(x + (-K)) \cap \mathcal{D} \neq \emptyset$. Since $\mathcal{D} + K = G$ there is a $y \in K$ and an $\alpha \in \mathcal{D}$ such that $\alpha + y = x$ and therefore,

$$\alpha = x + (-y).$$

It is $(-y) \in (-K)$ by definition implying that $(x + (-K)) \cap \mathcal{D} \neq \emptyset$.

"$\Leftarrow$": Showing that $G \subseteq \bigcup_{\alpha \in \mathcal{D}} (\alpha + K)$ suffices, since the converse inclusion is obvious. Consider $x \in G$ and as above there is some $\alpha \in \mathcal{D}$ and $y \in K$ such that $x = \alpha + y$.

□

Consider $\mathcal{D}$ which is a U-uniformly discrete and K-relatively dense subset of G. Then, we call $\mathcal{D}$ a *Delone set*, more precisely, a (U, K)-Delone set.

Let K be a compact subset of G. A *cluster* or *patch* in K is a finite set of the form $P_K := -x + P'$, where $P' \subseteq K \cap \mathcal{D}$ and $x \in P'$. In particular, you can imagine P_K as all connecting lines from a point x to any other element of P' if G is equal to $\mathbb{R}^d$.

Let $K \subseteq G$ be a compact set and $\mathcal{D} \subseteq G$ a U-uniformly discrete set. We denote by

$$Clu(\mathcal{D}, K) := \{(-\alpha + \mathcal{D}) \cap K \mid \alpha \in \mathcal{D}\}$$

the set of all clusters of $\mathcal{D}$ of the form K. Because the set $\mathcal{D}$ is U-uniformly discrete, the intersection $(-\alpha + \mathcal{D}) \cap K$ is finite, see the following Lemma 2.3. If $Clu(\mathcal{D}, K)$ is finite for every compact $K \subseteq G$ we call $\mathcal{D}$ of *finite local complexity (FLC)*. According to our definition of cluster adding the condition that $\mathcal{D}$ is of finite local complexity declares that there are only a finite number of different clusters. This property is really essential in order to define the Lagarias group. Moreover, it entails that we only need a finite number of different cells which tile our group G, see Proposition 5.4.

A Delone set $\mathcal{D}$ is called a *set of finite type* if

$$(\mathcal{D} - \mathcal{D}) := \{\alpha - \beta \mid \alpha, \beta \in \mathcal{D}\}$$

is discrete and closed.

A subset S of a locally compact, abelian topological group G is called *locally finite* if for any compact $K \subseteq G$ the set $S \cap K$ is finite or empty.

Lemma 2.3. *Let $\mathcal{D}$ be a subset of a locally compact abelian group G.*

(i) The set $\mathcal{D}$ is an aperiodic Delone set, iff for any $t \in G$ the set $t + \mathcal{D}$ is also an aperiodic Delone set.

(ii) The set $\mathcal{D}$ is of finite local complexity, iff for each $t \in G$ the set $t + \mathcal{D}$ is of finite local complexity.

(iii) If $\mathcal{D}$ is U-uniformly disrete then $\mathcal{D}$ is locally finite.

(iv) If $(\mathcal{D} - \mathcal{D})$ is locally finite, then, $\mathcal{D}$ is locally finite.

Proof. Note that in (i) and (ii) it is sufficient to show one direction, the converse follows immediately for $t = 0 \in G$.

(i) Let $t \in G$ be fixed but arbitrary. Suppose for some $x \in G$ that

$$x + (t + \mathcal{D}) = t + \mathcal{D}.$$

Because G is abelian it follows that $x + \mathcal{D} = \mathcal{D}$. Since $\mathcal{D}$ is aperiodic we get $x = 0$. Consequently, the set $t + \mathcal{D}$ is aperiodic.

Now we have to show that $t + \mathcal{D}$ is a (U, K)-Delone set. For any $x \in G$ there is a $y \in G$ such that $x = t + y$. Further, we get by an easy computation

$$\begin{aligned}(x + U) \cap (t + \mathcal{D}) &= t + ((\underbrace{-t + x}_{=y} + U) \cap (-t + t + \mathcal{D})) \\ &= t + ((y + U) \cap \mathcal{D}).\end{aligned}$$

Moreover, $\mathcal{D}$ is U-uniformly discrete implying that $(y + U) \cap \mathcal{D}$ contains at most one element and so any translate of it as well. Hence, $t + \mathcal{D}$ is U-uniformly discrete.

Since $\mathcal{D}$ is K-relatively dense it follows by Lemma 2.2 and for any $x \in G$ that

$$\begin{aligned}(x + (-K)) \cap (t + \mathcal{D}) &= t + ((\underbrace{-t + x}_{=:y \in G} + (-K)) \cap (-t + t + \mathcal{D}) \\ &= t + \underbrace{((y + (-K)) \cap \mathcal{D})}_{\neq \emptyset} \neq \emptyset.\end{aligned}$$

Consequently, $t + \mathcal{D}$ is K-relatively dense.

(ii) In the first part of the proof of (i) we have shown that for each $t \in \mathcal{D}$ the set $t + \mathcal{D}$ is again U-uniformly discrete. Thus, it suffices to show that $Clu(t + \mathcal{D}, K) = Clu(\mathcal{D}, K)$ for any compact $K \subseteq G$.

Let K be some compact subset of G, then,

$$\begin{aligned}Clu(t + \mathcal{D}, K) &= \{(-\alpha + (t + \mathcal{D})) \cap K \mid \alpha \in t + \mathcal{D}\} \\ &\overset{\alpha = t + \beta}{=} \{(-(\beta + t) + (t + \mathcal{D})) \cap K \mid \beta \in \mathcal{D}\} \\ &= \{(-\beta + \mathcal{D}) \cap K \mid \beta \in \mathcal{D}\} \\ &= Clu(\mathcal{D}, K)\,.\end{aligned}$$

(iii) Let $K \subseteq G$ be some compact set and assume the set $\mathcal{D} \cap K$ is not finite. We choose the following covering of K

$$K \subseteq \bigcup_{y \in K} (y + U).$$

Because K is compact we find $y_1, \ldots, y_N \in K$ such that

$$K \subseteq \bigcup_{j=1}^{N} (y_j + U).$$

Hence, for at least one $j \in \{1, \ldots, N\}$ there has to be an infinite number of elements of $\mathcal{D}$ in $y_j + U$. This contradicts the fact that $\mathcal{D}$ is U-uniformly discrete, which means that $x + U$ and $\mathcal{D}$ have at most one common element for any $x \in G$.

(iv) For any $\alpha \in \mathcal{D}$ we know that $-\alpha + \mathcal{D} \subseteq (\mathcal{D} - \mathcal{D})$. Thus, for any compact $K \subseteq G$ we get $(-\alpha + \mathcal{D}) \cap K \subseteq (\mathcal{D} - \mathcal{D}) \cap K$ which is finite. On the other hand, since the group composition is continuous any translate of a compact set is again compact. Hence, $\mathcal{D}$ is locally finite.

$\square$

For some compact set $K \subseteq G$ the set

$$Rep(\mathcal{D}, K) := \{x \in G \mid (-x + \mathcal{D}) \cap K = \mathcal{D} \cap K\}$$

is called the *repetition of* $\mathcal{D}$ with respect to K.

Note that $(-x + \mathcal{D}) \cap K = -x + (\mathcal{D} \cap (x + K))$. We can imagine the intersection $\mathcal{D} \cap K$ as the image of $\mathcal{D}$ in the window K. Then, the repetition is the set of all positions where the translated images are equal to the image in the window K. A set $\mathcal{D} \subseteq G$ of finite local complexity is called *repetitive* if $Rep(\mathcal{D}, K)$ is a relatively dense subset of G for every compact set $K \subseteq G$.

In the following we call $\mathcal{D}$ a *D-set* if $\mathcal{D}$ is an aperiodic, repetitive Delone set of finite local complexity.

2.2 Delone sets of finite local complexity

In this section our aim is to relate the concepts of finite local complexity and finite type to each other.

Proposition 2.4. *Let $\mathcal{D}$ be a Delone set of a locally compact, abelian group G. The following statements are equivalent.*

(i) The set $\mathcal{D}$ is of finite local complexity, which means that $Clu(\mathcal{D}, K)$ is finite for any compact set $K \subseteq G$.

(ii) For any compact set $K \subseteq G$ the intersection $(\mathcal{D} - \mathcal{D}) \cap K$ is finite or empty.

(iii) The set $\mathcal{D}$ is a set of finite type, which means that $(\mathcal{D} - \mathcal{D})$ is discrete and closed.

Proof.

"$(i) \Rightarrow (ii)$":
Since $\mathcal{D}$ is of finite local complexity for any compact $K \subseteq G$ we can choose $\alpha_1, \ldots \alpha_N \in \mathcal{D}$ such that

$$Clu(\mathcal{D}, K) := \{(-\alpha_1 + \mathcal{D}) \cap K \ , \ \ldots \ , \ (-\alpha_N + \mathcal{D}) \cap K\}.$$

Then,

$$\begin{aligned}(\mathcal{D} - \mathcal{D}) \cap K &= \bigcup_{\alpha \in \mathcal{D}} (-\alpha + \mathcal{D}) \cap K = \bigcup_{((-\alpha+\mathcal{D})\cap K) \in Clu(\mathcal{D},K)} (-\alpha + \mathcal{D}) \cap K \\ &\overset{\text{FLC}}{=} \bigcup_{j=1}^{N} (-\alpha_j + \mathcal{D}) \cap K.\end{aligned}$$

According to Lemma 2.3 we know for each $j \in \{1, \ldots, N\}$ that $(-\alpha_j + \mathcal{D})$ is a Delone set. Thus, $(-\alpha_j + \mathcal{D}) \cap K$ is finite for any compact $K \subseteq G$. Moreover, the finite union of finite sets is again finite and so $(\mathcal{D} - \mathcal{D}) \cap K$ is finite for any compact set K.

"$(ii) \Rightarrow (iii)$":
We would like to show that $(\mathcal{D} - \mathcal{D})$ is a closed and discrete subset of G. To do so assume $(\mathcal{D} - \mathcal{D})$ is not discrete.

Then, there exists an $x \in (\mathcal{D} - \mathcal{D})$ for which any neighborhood $V \subseteq G$ of x satisfies $\sharp((\mathcal{D} - \mathcal{D}) \cap V) \geq 2$. Let K be a compact neighborhood of x and $x \in V \subseteq K$ open. We can find at least one element $y \in (\mathcal{D} - \mathcal{D}) \cap V$ different to x. Since

G is a Hausdorff space, there is another open neighborhood $W \subseteq V$ of x such that $y \notin W$. Because $\sharp((\mathcal{D}-\mathcal{D}) \cap W)$ is greater than or equal to two the number $\sharp((\mathcal{D}-\mathcal{D}) \cap V)$ is greater than or equal to three. Using this procedure inductively we obtain $\sharp((\mathcal{D}-\mathcal{D}) \cap V) = \infty$.

On the other hand, we supposed that $(\mathcal{D}-\mathcal{D}) \cap K$ is finite for K compact which leads to a contradiction. Hence, $(\mathcal{D}-\mathcal{D})$ has to be discrete.

To show that $(\mathcal{D}-\mathcal{D})$ is closed, take some convergent net $(x_\iota) \subseteq (\mathcal{D}-\mathcal{D})$ which tends to $x \in G$ with respect to the topology of G. By the definition of a locally compact set there is a compact neighborhood K of X. Thus, there is some $\iota_0 \in I$ such that for any $\iota \rhd \iota_0$ the element x_ι is in K and further, the intersection $(\mathcal{D}-\mathcal{D}) \cap K$ is finite. Thus, the set of values in $(\mathcal{D}-\mathcal{D})$ of the net x_ι is finite.

Assume x is not an element of $(\mathcal{D}-\mathcal{D})$. Thus, there is an open neighborhood $V \subseteq K$ of x such that $((\mathcal{D}-\mathcal{D}) \cap V) \subseteq ((\mathcal{D}-\mathcal{D}) \cap K)$ is empty. This is a contradiction to $(x_\iota)_{\iota \in I} \subseteq (\mathcal{D}-\mathcal{D})$ converging to x.

"$(iii) \Rightarrow (i)$":
Let K be some compact subset of G. The intersection $(\mathcal{D}-\mathcal{D}) \cap K$ is finite, because $(\mathcal{D}-\mathcal{D})$ is discrete. Hence, there is a finite number of different subsets of $(\mathcal{D}-\mathcal{D}) \cap K$ only. On the other hand, for each $\alpha \in \mathcal{D}$ the cluster $(-\alpha+\mathcal{D}) \cap K$ is a subset of $(\mathcal{D}-\mathcal{D}) \cap K$. Thus,

$$Clu(\mathcal{D}, K) = \bigcup_{\alpha \in \mathcal{D}} \{(-\alpha + D) \cap K\}$$

has to be finite for any compact K.

□

One should not confuse the property that $(\mathcal{D}-\mathcal{D}) \cap K$ is finite for any compact K as the definition of weakly uniformly discrete. We say $W \subseteq G$ is *weakly uniformly discrete* if $\sup_{x \in G} \sharp(W \cap (x+K))$ is finite for any compact set $K \subseteq G$.

For instance, consider $G = \mathbb{R}$ and the compact set $K := [-\frac{1}{3}, \frac{1}{3}]$ in $\mathbb{R}$. Define

$$W := \bigcup_{m \in \mathbb{N} \setminus \{0\}} \{x = m + \frac{k}{4m} \in \mathbb{R} \mid k \in \mathbb{N} \text{ with } -m \leq k \leq m\}.$$

In fact, $m + \frac{k}{4m}$ lies in the interval $m + [-\frac{1}{3}, \frac{1}{3}] =: K_m$, because $|\frac{k}{4m}| \leq \frac{1}{4}$. The blue points in Figure 3 illustrate the set W.

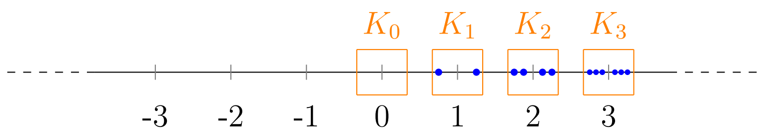

Figure 3 :
A set of finite local complexity which is not weakly uniformly discrete

Moreover, we can easily see

$$\sup_{x\in\mathbb{R}} (W \cap (x+K)) \geq \sup_{m\in\mathbb{N}} (W \cap (K_m)) = \infty$$

and so W is not weakly uniformly discrete. On the other hand, $W \cap K$ is finite or empty for any compact set $K \subseteq \mathbb{R}$.

In the paper of [BLM] it is shown for a compactly generated group G and a Delone set $\mathcal{D} \subseteq G$ that $(\mathcal{D} - \mathcal{D})$ is weakly uniformly discrete if and only if $(\mathcal{D} - \mathcal{D})$ is uniformly discrete.

2.3 Hull and transversal

In this section we define a topology on subsets of points of G. We denote by Λ_U the set of all U-uniformly discrete sets in G. First we will show that Λ_U is closed with respect to our topology. Moreover, we will characterize convergence in Λ_U. Then, we will consider a (U,K)-Delone set $\mathcal{D}$ of finite local complexity and construct two closed subsets $\Omega_{\mathcal{D}}$ and $\mathcal{P}_{\mathcal{T}}$ of Λ_U with $\mathcal{D}$. By using the results on Λ_U it will be proved that some properties of $\mathcal{D}$ will persist for the elements of $\Omega_{\mathcal{D}}$ and $\mathcal{P}_{\mathcal{T}}$, respectively.

In order to define a topology on Λ_U we need the concept of Radon measures. Here we give a short summary of some results and definitions of these concepts following [BAU] and [ELS]. The more interested reader may find further information there.

Let G be a locally compact abelian group which is Hausdorff and σ-compact. Let μ be a measure defined on the Borel σ-algebra $\mathfrak{B}(G)$ of G. We say μ is *locally finite* if for any $x \in G$ exists an open neighborhood V_x of x such that $\mu(V_x)$ is finite. Further, we call μ *inner regular* if

$$\mu(B) = \sup\{\mu(K) \mid K \subseteq B,\ K \text{ is compact}\}$$

for any Borel set $B \in \mathfrak{B}(G)$. Then, μ is a *Radon measure* if μ is locally finite and inner regular. Further, a measure μ is called *outer regular* if for any Borel set $B \in \mathfrak{B}(G)$

$$\mu(B) = \inf\{\mu(V) \mid B \subseteq V, V \text{ is open}\}.$$

Since the group G is σ-compact and Hausdorff this implies that any Radon measure on G is also outer regular, see [ELS]. We denote by $\mathcal{M}$ the set of all Radon measures on G. Furthermore, $\mathcal{M}$ is endowed on a natural way with the vague topology.

The *support* of a function f on G is defined by

$$supp(f) := \overline{\{x \in G \mid f(x) \neq 0\}}.$$

The set of all continuous functions f on G with compact support is denoted by $\mathrm{C}_c(\mathrm{G})$.

A net $(\mu_\iota)_{\iota \in I} \subseteq \mathcal{M}$ is called *convergent to* μ with respect to the vague topology if

$$\lim_\iota \int_{\mathrm{G}} f \, d\mu_\iota = \int_{\mathrm{G}} f \, d\mu$$

for any $f \in \mathrm{C}_c(\mathrm{G})$. Note that $\mathcal{M}$ is closed with respect to the vague topology. For our further considerations we need the following well known proposition.

Proposition 2.5 (Urysohn,[BAU])**.** *Consider a locally compact Hausdorff space G. Let $K \subseteq G$ be a compact set and $W \subseteq G$ some open subset with $K \subseteq W$. Then, there exists a function $f \in \mathrm{C}_c(\mathrm{G})$ with compact support in W such that $0 \leq f \leq 1$ and $f(x) = 1$ for any $x \in K$.*

Consider now for $x \in G$ the measure

$$\delta_x(A) := \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}, \qquad \text{for } A \in \mathfrak{B}(G).$$

Lemma 2.6. *The map $\mathcal{J} : \Lambda_U \to \mathcal{M}$ defined by*

$$\mathcal{J}(\mathcal{D}) := \delta_{\mathcal{D}} := \sum_{\alpha \in \mathcal{D}} \delta_\alpha, \qquad \textit{for } \mathcal{D} \in \Lambda_U,$$

is well-defined and injective.

Proof. We have to verify that $\mathcal{J}$ maps into $\mathcal{M}$. In particular, we have to show that for each $\mathcal{D} \in \Lambda_U$ the image $\mathcal{J}(\mathcal{D}) := \sum_{\alpha \in \mathcal{D}} \delta_\alpha$ is an inner regular measure which is locally finite.

Since the elements $\mathcal{D} \in \Lambda_U$ are U-uniformly discrete it follows immediately that $\mathcal{J}(\mathcal{D})$ is locally finite. In detail, $x + U$ is an open neighborhood of $x \in G$ and so

$$(\mathcal{J}(\mathcal{D}))(x + U) = \sharp(\mathcal{D} \cap (x + U)) \leq 1 < \infty \qquad \text{for any } x \in G.$$

Let $B \in \mathfrak{B}(G)$ be a Borel set. Consider first the case that $\sharp(\mathcal{D} \cap B)$ is infinite. Thus, the number $\mathcal{J}(\mathcal{D})(B)$ is equal to $\sharp(\mathcal{D} \cap B) = \infty$. On the other hand, there is a countable subset $\{x_1, x_2, \ldots\}$ of $(\mathcal{D} \cap B)$ and the set $K_n := \{x_1, \ldots, x_n\} \subseteq (\mathcal{D} \cap B)$ is compact. Hence,

$$\sup\{\mu(K) \mid K \subseteq B \text{ compact}\} \geq \sup_{n \in \mathbb{N}}\{\mu(K_n) \mid K_n := \{x_1, \ldots x_n\}\} = \infty.$$

Now let $\sharp(\mathcal{D} \cap B) =: N$ be finite. We have for the compact set $K_N := (\mathcal{D} \cap B)$ by the monotonicity of a measure

$$N = \sharp(\mathcal{D} \cap B) = \mu(B) \geq \sup\{\mu(K) \mid K \subseteq B \text{ compact}\} \underset{\text{compact}}{\overset{K_N \subseteq B}{\geq}} \mu(K_N) = N.$$

Altogether, the measure $\mathcal{J}(\mathcal{D})$ is inner regular and locally finite and so it is a Radon measure. That $\mathcal{J}$ is injective is a direct implication of the definition of $\mathcal{J}$. □

If we consider a Radon measure $\mu \in \mathcal{J}(\Lambda_U)$ the corresponding point set is given by

$$\mathcal{D}_\mu := \{x \in G \mid \mu(\{x\}) = 1\}.$$

That $\mathcal{D}_\mu$ is unique, is a consequence of the fact that $\mathcal{J}$ is injective.
A subset V of Λ_U is called open if there exists an open set $O \subseteq \mathcal{M}$ with respect to the vague topology such that $V = \mathcal{J}^{-1}(O)$. We call the induced topology on Λ_U *pointset topology* and denote it by $\mathcal{O}(\Lambda_U)$ (initial topology).

The support *supp*(μ) of a Radon measure μ is defined by

$$\{x \in G \mid x \in B \in \mathfrak{B}(G) \text{ implies } \mu(B) > 0\}.$$

Proposition 2.7. *For any open set U in G the set $\mathcal{J}(\Lambda_U)$ is closed with respect to the vague topology.*

Proof. Consider some convergent net $(\mu_\iota)_{\iota \in I} \subseteq \mathcal{J}(\Lambda_U)$ which tends to μ with respect to the vague topology. We would like to show that μ can be expressed by the sum over some δ measures and that the corresponding support of μ is U-uniformly discrete.

Assume for some $x \in G$ that there are two different $\alpha, \beta \in (x+U) \cap supp(\mu)$. Since G is Hausdorff we can find two open, disjoint neighborhoods $V_\alpha \subseteq (x+U)$ and $V_\beta \subseteq (x+U)$ of α and β, respectively. Since α and β are elements of the support any open neighborhood of them has positive measure.

According to Proposition 2.5 we can find a continuous function $f : G \to [0,1]$ with compact support in V_α such that $f(\alpha) = 1$. Similar there is a continuous function $g : G \to [0,1]$ with compact support in V_β such that $f(\beta) = 1$. Since f is continuous there is some open neighborhood $W_\alpha \subseteq V_\alpha$ of α such that $f|_{W_\alpha} \geq \frac{1}{2}$. Analogously, there is some open neighborhood $W_\beta \subseteq V_\beta$ of β with $g|_{W_\beta}$. The measures of W_α and W_β are larger than some constant c, because α and β are elements of the support of μ.

Hence,

$$\begin{aligned} c \leq & \mu(W_\alpha) \int\limits_{W_\alpha} 1 \, d\mu \leq 2 \cdot \int\limits_{W_\alpha} f \, d\mu \leq 2 \cdot \int\limits_{\mathrm{G}} f \, d\mu \\ = & 2 \cdot \lim_\iota \int\limits_{\mathrm{G}} f \, d\mu_\iota = 2 \cdot \lim_\iota \int\limits_{V_\alpha} \underbrace{f}_{\leq 1} \, d\mu_\iota = 2 \cdot \lim_\iota \mu_\iota(V_\alpha) \end{aligned}$$

and

$$c \underset{\text{above}}{\overset{\text{see}}{\leq}} 2 \cdot \lim_\iota \mu_\iota(V_\beta).$$

Consequently, there is an $\iota_0 \in I$ such that for $\iota \rhd \iota_0$ the intersections $\mathcal{D}_{\mu_\iota} \cap V_\alpha$ and $\mathcal{D}_{\mu_\iota} \cap V_\beta$ are not empty. Since V_α and V_β are disjoint and both are subsets of $x + U$ it follows that for $\iota \rhd \iota_0$ the intersection $\mathcal{D}_{\mu_\iota} \cap (x + U)$ contains at least two elements. This contradicts that $\mathcal{D}_{\mu_\iota}$ is U-uniformly discrete.

Thus, the support is U-uniformly discrete and μ has the form

$$\mu = \sum_{\alpha \in supp(\mu)} c_\alpha \cdot \delta_\alpha$$

where c_α is some positive real number.

Next we would like to verify that c_α is equal to one for each $\alpha \in supp(\mu)$. Fix some $\alpha \in supp(\mu)$. According to Proposition 2.5 there is a continuous function $f : G \to [0, 1]$ with compact support $\alpha + U$ such that $f(\alpha) = 1$ and

$$0 < c_\alpha = \mu(\{\alpha\}) \underset{\text{above}}{\overset{\text{like}}{=}} \lim_\iota \int\limits_{\alpha+U} f \, d\mu_\iota = \lim_\iota f(\alpha_\iota).$$

where some $\iota_0 \in I$ exists such that for $\iota \rhd \iota_0$ the equality $\{\alpha_\iota\} = (\alpha + U) \cap \mathcal{D}_\iota$ holds. If we now show that $\alpha_\iota \to \alpha$ we get $f(\alpha_\iota) \to f(\alpha) = 1$ by Lemma 2.1.

Let $V \subseteq (\alpha + U)$ be some open neighborhood of α. By Proposition 2.5 there exists a continuous function $f_V : G \to [0, 1]$ with compact support in V such that $f_V(\alpha) = 1$ and so

$$0 < c_\alpha = \mu(\{\alpha\}) = \lim_\iota f_V(\alpha_\iota^V).$$

This implies that for ι large enough

$$\{\alpha_\iota^V\} \in (\alpha + V) \cap \mathcal{D}_\iota \subseteq (\alpha + U) \cap \mathcal{D}_\iota = \{\alpha_\iota\}$$

and so $\alpha_\iota^V = \alpha_\iota$. Consequently, for any open neighborhood V we can find an $\iota_0 \in I$ such that for $\iota \rhd \iota_0$ the element α_ι lies in V. Hence, α_ι converges to α which concludes the proof. $\square$

Proposition 2.8. *Let $\mathcal{D}_\iota \subseteq \Lambda_U$ be a net. Then, $\mathcal{D}_\iota$ tends to $\mathcal{D}$ with respect to the pointset topology, if and only if the following two statements are true.*

(i) For any $\alpha \in \mathcal{D}$ there is a net $(\alpha_\iota)_{\iota\in I}$ with $\alpha_\iota \in \mathcal{D}_\iota$ and $\alpha_\iota \to \alpha$.

(ii) Let $\mathcal{D}_{\iota_j}$ be some subnet of $\mathcal{D}_\iota$ and $\beta_{\iota_j} \in \mathcal{D}_{\iota_j}$ such that the limit $\lim_j \beta_{\iota_j} =: \beta$ exists, then, $\beta \in \mathcal{D}$.

Proof. Take some net $\mathcal{D}_\iota \subseteq \Lambda_U$.

"$\Rightarrow$": Suppose D_ι converges to $\mathcal{D}$. Since Λ_U is closed the set $\mathcal{D}$ has to be an element of Λ_U, see Proposition 2.7. We denote by μ the corresponding measure of $\mathcal{D}$ and by μ_ι the corresponding measure of $\mathcal{D}_\iota$. Indeed, we know $\mu_\iota \to \mu$, which means that for each $f \in \mathrm{C}_c(\mathrm{G})$ the equation $\int_{\mathrm{G}} f\, d\mu = \lim_\iota \int_{\mathrm{G}} f\, d\mu_\iota$ is true.

"(i)": If $\mathcal{D}_\iota \to \mathcal{D}$ we have seen in the proof of Proposition 2.7 that for each $\alpha \in \mathcal{D}$ there is a net $(\alpha_\iota)_{\iota\in I}$ which tends to α.

"(ii)": Take a subnet $\mathcal{D}_{\iota_j}$ and let $\beta_{\iota_j} \in \mathcal{D}_{\iota_j}$ such that $\beta_{\iota_j} \to \beta \in G$. Note j is an element of a directed set J and for any $\iota_0 \in I$ there is a $j_0 \in J$ such that for all $j \rhd j_0$ we have $\iota_j \rhd \iota_0$. In particular, for any open set V_β which contains β there exists a $j_0 \in J$ such that for each $j \rhd j_0$ we know $\beta_{\iota_j} \in V_\beta$.

Assume now $\beta \notin \mathcal{D}$ which implies that $\mu(\{\beta\}) = 0$. Further, for some compact neighborhood K_β of β the intersection $K_\beta \cap \mathcal{D}$ is finite or empty, see Lemma 2.3 (iii). Thus, we can suppose that there is an open neighborhood $V_\beta \subseteq K_\beta$ of β such that $V_\beta \cap \mathcal{D} = \emptyset$, because G is a Hausdorff space. According to Proposition 2.5 we can find a continuous function $f : G \to [0,1]$ with compact support in V_β such that $f(\beta) = 1$. Since f is continuous there is an open neighborhood $W_\beta \subseteq V_\beta$ of β such that $f(x) \geq \frac{3}{4}$ for any $x \in W_\beta$. Altogether, since any subnet converges to the same limit

$$\begin{aligned} 0 = \mu(\{\beta\}) &\overset{supp(f)\subseteq V_\beta}{=} \sum_{\alpha\in(V_\beta\cap\mathcal{D})} f(\alpha) = \int_{\mathrm{G}} f\, d\mu = \lim_{\iota_j} \int_{\mathrm{G}} f\, d\mu_{\iota_j} \\ &\geq \lim_{\iota_j} \int_{W_\beta} \underbrace{f}_{\geq\frac{3}{4}}\, d\mu_{\iota_j} \geq \frac{3}{4} \lim_{\iota_j} \int_{W_\beta} d\mu_{\iota_j} = \frac{3}{4} \lim_{\iota_j} \mu_{\iota_j}(W_\beta). \end{aligned}$$

On the other hand, for ι_j large enough $\beta_{\iota_j} \in W_\beta$ by the convergence of β_{ι_j}. Thus, $\mu_{\iota_j}(W_\beta)$ is at least 1 which is a contradiction to our conclusion. Consequently, β has to be an element in $\mathcal{D}$.

"$\Leftarrow$": Suppose (i) and (ii) are true. We would like to show that the net $\mathcal{D}_\iota$ converges to $\mathcal{D}$. We denote by μ_ι the corresponding measure of $\mathcal{D}_\iota$. First of all, we have to check if $\mathcal{D}$ is U-uniformly discrete. To do so take $\alpha, \beta \in \mathcal{D}$, by (i) we know that there are two nets $\alpha_\iota, \beta_\iota \in \mathcal{D}_\iota$ such that $\alpha_\iota \to \alpha$ and $\beta_\iota \to \beta$, respectively.

Let $\tilde{U}$ be some translate of U such that $\alpha, \beta \in \tilde{U} \cap \mathcal{D}$. Then, there is an $\iota_0 \in I$ such that for any $\iota \rhd \iota_0$ we get that $\alpha_\iota, \beta_\iota \in \tilde{U}$, because $\tilde{U}$ is an open neighborhood of α and β, respectively. On the contrary, $\mathcal{D}_\iota$ is U-uniformly discrete. In fact, α and β can not be both an element of $\tilde{U}$, if they are different. Hence, $\mathcal{D}$ has to be an element of Λ_U. Thus, we can denote the corresponding measure of $\mathcal{D}$ by

$$\mu = \sum_{\alpha \in \mathcal{D}} \delta_\alpha.$$

We would like to show that for any $f \in \mathrm{C}_c(\mathrm{G})$ the equation

$$\lim_\iota \int_{\mathrm{G}} f \, d\mu_\iota = \int_{\mathrm{G}} f \, d\mu.$$

holds. By Proposition A.6 we can write any $g \in \mathrm{C}_c(\mathrm{G})$ as a finite sum of functions $\psi_j \in \mathrm{C}_c(\mathrm{G})$ with support contained in some translate $\tilde{U}$ of U. Thus, it is sufficient to consider a function f where the support K of f is contained in some translate $\tilde{U}$ of U. Take some $f \in \mathrm{C}_c(\mathrm{G})$ and let $K \subseteq \tilde{U}$ be the compact support of f.

"$\int_{\mathrm{G}} f \, d\mu \neq 0$":

Since $K \subseteq \tilde{U}$ and $\mathcal{D} \in \Lambda_U$ it follows that the number $\sharp(\mathcal{D} \cap K) \leq \sharp(\mathcal{D} \cap \tilde{U})$ is at most one. On the other hand, the integral

$$\int_{\mathrm{G}} f \, d\mu = \sum_{\beta \in \mathcal{D}} f(\beta) = \sum_{\beta \in (\mathcal{D} \cap K)} f(\beta)$$

is not zero and so the intersection $\mathcal{D} \cap K$ can not be empty. Hence, the intersection $\mathcal{D} \cap K$ contains exactly one element. Denote this element by α. Thus,

$$\int_{\mathrm{G}} f \, d\mu = \sum_{\beta \in (\mathcal{D} \cap K)} f(\beta) = f(\alpha) \neq 0.$$

By condition (i) there is a net $\alpha_\iota \in \mathcal{D}_\iota$ which tends to α. Consequently, there is some $\iota_0 \in I$ such that for each $\iota \rhd \iota_0$ is $\alpha_\iota \in \tilde{U}$. Since $\mathcal{D}_\iota$ is also U-uniformly discrete the intersection $\mathcal{D}_\iota \cap \tilde{U}$ contains the element α_ι only. Hence,

$$\int_{\mathrm{G}} f \, d\mu_\iota = \int_{\tilde{U}} f \, d\mu_\iota = f(\alpha_\iota) \xrightarrow[\text{Lemma 2.1}]{\iota} f(\alpha) = \int_{\mathrm{G}} f \, d\mu,$$

because f vanishes beyond $\tilde{U}$.

"$\int\limits_{G} f\, d\mu = 0$":

We would like to show that the integral $\int\limits_{G} f\, d\mu_\iota$ converges to zero. Since $\mathcal{D}_\iota \in \Lambda_U$ the intersection $\mathcal{D}_\iota \cap \tilde{U}$ contains at most one element, denoted by α_ι. Assume the contrary, which means that there is some subnet $\mathcal{D}_{\iota_j}$ of $\mathcal{D}_\iota$ such that for all $j \in J$ we have

$$|\int\limits_{G} f\, d\mu_{\iota_j}| = |\int\limits_{\tilde{U}} f\, d\mu_{\iota_j}| \overset{\mathcal{D}_{\iota_j} \in \Lambda_U}{=} |f(\alpha_{\iota_j})| \geq c > 0$$

Thus, α_{ι_j} is an element of the support K. Since K is compact there exists a convergent subnet $\alpha_{\iota_{j_k}}$ of α_{ι_j}. Hence,

$$|\int\limits_{G} f\, d\mu_{\iota_{j_k}}| = |f(\alpha_{\iota_{j_k}})| \underset{\text{Lemma 2.1}}{\overset{\iota}{\longrightarrow}} |f(\alpha)| \geq c > 0,$$

where $\lim\limits_{k \to \infty} \alpha_{\iota_{j_k}} =: \alpha \in K$. By condition (ii) α is an element of $\mathcal{D}$ and so

$$\int\limits_{G} f\, d\mu = f(\alpha) \neq 0$$

which leads to a contradiction.

$\square$

A good illustration of the convergence of a net $\mathcal{D}_\iota \subseteq \Lambda_U$ to $\mathcal{D}$ is that for $\iota_0 \in I$ large enough, any element of $\mathcal{D}_{\iota_0}$ is in a small open neighborhood of some element of $\mathcal{D}$ and its stays there for any larger $\iota \rhd \iota_0$. Because of the uniformly discreteness of $\mathcal{D}_\iota$ there can be at most one element of $\mathcal{D}_\iota$ in any of these open neighborhoods, see Figure 4.

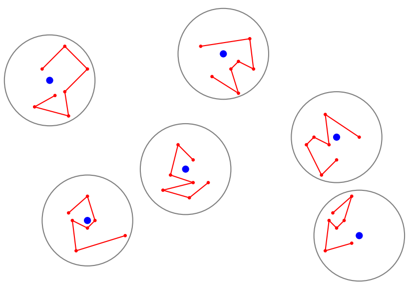

Figure 4:
An illustration of the convergence of a net of Delone sets

Imagine the blue points as the elements of $\mathcal{D}$ and the gray circles around them as the small open neighborhoods of the elements of $\mathcal{D}$. The red lines illustrate the jumps of the elements of $\mathcal{D}_\iota$ which converges to the corresponding element of $\mathcal{D}$.

The function which maps any subset $A \subseteq G$ to $t + A$ for some $t \in G$ is, by definition, a homeomorphism. Further, for each homeomorphism the image of a Borel set is again a Borel set. Thus, for $t \in G$ and some Borel set $B \subseteq G$ the translate $t + B$ is again a Borel set. Let μ be some Radon measure on G we denote the *associated shift* by

$$\tau^t \mu(B) := \mu(t + B)$$

for any $t \in G$ and $B \in \mathfrak{B}(G)$ some Borel set. Obviously, we have

$$\tau^{t_1}(\tau^{t_2}\mu(B)) = \tau^{(t_1+t_2)}\mu(B) \qquad \text{for any } t_1, t_2 \in G, B \in \mathfrak{B}(G)$$

and

$$\tau^0 \mu(B) = \mu(B).$$

Lemma 2.9. *Let $\mathcal{D}$ be any element of Λ_U. Then,*

$$\overline{\{\tau^t \delta_{\mathcal{D}} \mid t \in G\}}^V = \mathcal{J}(\overline{\{t + \mathcal{D} \mid t \in G\}}^P)$$

where on the left side of the equation we consider the closure with respect to the vague topology and on the right side with respect to the pointset topology.

Proof. At first we will show that

$$\{\tau^t \delta_{\mathcal{D}} \mid t \in G\} = \mathcal{J}(\{t + \mathcal{D} \mid t \in G\}).$$

According to Lemma 2.3 (i) the set $t + \mathcal{D}$ is an element of Λ_U and so $\mathcal{J}(\{t + \mathcal{D} \mid t \in G\})$ is well-defined.

"$\subseteq$": Let $t \in G$ be arbitrary. By an easy computation we get

$$\tau^t \delta_{\mathcal{D}} = \tau^t \sum_{\alpha \in \mathcal{D}} \delta_\alpha = \sum_{\alpha \in \mathcal{D}} \tau^t \delta_\alpha = \sum_{\alpha \in \mathcal{D}} \delta_{\alpha - t} = \sum_{\beta \in (-t+\mathcal{D})} \delta_\beta = \mathcal{J}(-t + \mathcal{D}).$$

In fact, we have that $\tau^t \delta_{\mathcal{D}} \in \mathcal{J}(\{t + \mathcal{D} \mid t \in G\})$.

"$\supseteq$": For any $t \in G$ we get

$$\mathcal{J}(t + \mathcal{D}) = \sum_{\beta \in (t+\mathcal{D})} \delta_\beta = \sum_{\alpha \in \mathcal{D}} \delta_{\alpha + t} = \sum_{\alpha \in \mathcal{D}} \tau^{-t} \delta_\alpha = \tau^{-t} \delta_{\mathcal{D}}.$$

Since $-t \in G$ it follows that $\mathcal{J}(t + \mathcal{D}) \in \{\tau^t \delta_{\mathcal{D}} \mid t \in G\}$.

By definition, the map $\mathcal{J}$ is continuous. Using the definition of the pointset topology and Proposition 2.7 the statement follows immediately. $\square$

We call the set $\Omega_{\mathcal{D}} := \overline{\{t+\mathcal{D} \mid t \in G\}}^{P}$ the *hull of* $\mathcal{D}$. According to Lemma 2.9 for any $\mu \in \overline{\{\tau^{-t}\delta_{\mathcal{D}} \mid t \in G\}}^{V}$ we get an element of $\Omega_{\mathcal{D}}$ defined by

$$\mathcal{D}_{\mu} := \{x \in G \mid \mu(\{x\}) = 1\}.$$

For our further considerations we require that G is a second-countable space. Then, it is sufficient to consider sequences to characterize the toplogy, see [QUE].

Let $\mathcal{D}$ be a U-uniformly discrete set and take some sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of the hull $\Omega_{\mathcal{D}}$. Hence, there is a sequence $(t_n)_{n\in\mathbb{N}} \subseteq G$ such that $\mathcal{D}_n$ has the form $t_n + \mathcal{D}$ for any $n \in \mathbb{N}$. The characterization of Proposition 2.8 says that $\lim\limits_{\mathrm{n}\to\infty} \mathcal{D}_n = \tilde{D} \in \Omega_D$, iff the following two assertions hold.

(i) For any $\tilde{\alpha} \in \tilde{\mathcal{D}}$ there is a sequence $(\alpha_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ such that $\tilde{\alpha} = \lim\limits_{n\to\infty} (t_n + \alpha_n)$.

(ii) Consider some sequence $(\beta_n)_{\in\mathbb{N}} \subseteq \mathcal{D}$. If for some subsequence the limit

$$\lim_{k\to\infty} (t_{n_k} + \beta_{n_k}) = \tilde{\beta}$$

exists, then, $\tilde{\beta} \in \tilde{\mathcal{D}}$.

Note that, if we write $\lim\limits_{n\to\infty} x_n$ for a sequence $(x_n)_{n\in\mathbb{N}} \subseteq G$ we mean the convergence with respect to the topology in G.

The following statement deals with the convergence of patterns by using the feature of finite local complexity. By this statement follows immediately that any element of the hull of a (U,K)-Delone set $\mathcal{D}$ of finite local complexity has the same pattern as $\mathcal{D}$, see Lemma 2.15.

Proposition 2.10. *Consider some (U,K)-Delone set of finite local complexity. Let $(\alpha_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ be chosen such that the limit $\tilde{\mathcal{D}} := \lim\limits_{\mathrm{n}\to\infty} (-\alpha_n + \mathcal{D})$ exists. For any compact subset $K \subseteq G$ exists an $n_0 \in \mathbb{N}$ such that*

$$(-\alpha_n + \mathcal{D}) \cap K = \tilde{\mathcal{D}} \cap K, \qquad n \geq n_0.$$

Proof. Define $\mathcal{D}_n$ by $(-\alpha_n + \mathcal{D})$ and suppose that the limit $\lim\limits_{\mathrm{n}\to\infty} D_n$ exists. Denote by μ_n the corresponding Radon measure of $\mathcal{D}_n$ defined by

$$\sum_{\beta\in\mathcal{D}_n} \delta_{\beta}.$$

Since the limit $\lim_{n\to\infty} D_n$ exists the limit $\lim_{n\to\infty} \mu_n$ exists as well.

Fix some compact subset $K \subseteq G$. Because $\mathcal{D}$ is of finite local complexity there are only a finite number of different clusters $P_1, \dots, P_l$ such that

$$Clu(\mathcal{D}, K) := \{P_1, \dots, P_l\}.$$

Thus, there are only a finite number of different possibilities for $\mathcal{D}_n \cap K$. Now assume the contrary of the statement which means that there are two subsequences $(\mathcal{D}_{N_k})_{k\in\mathbb{N}}$ and $(\mathcal{D}_{M_k})_{k\in\mathbb{N}}$ of $\mathcal{D}_n$ such that without loss of generality

$$\mathcal{D}_{N_k} \cap K = P_1, \qquad k \in \mathbb{N}$$

and

$$\mathcal{D}_{M_k} \cap K = P_2, \qquad k \in \mathbb{N}$$

where $P_1 \neq P_2$. Hence, there is without loss of generality an $x \in P_1$ such that $x \notin P_2$. According to Lemma 2.3 the set P_2 is finite. Thus, there is some open neighborhood $V_x \subseteq x + U$ of x such that the intersection $V_x \cap P_2$ is empty.

Denote by μ_{N_k} and μ_{M_k} the corresponding Radon measure of $\mathcal{D}_{N_k}$ and $\mathcal{D}_{M_k}$, respectively. By Proposition 2.5 there is a continuous function $f : G \to [0, 1]$ with compact support in V_x and $f(x) = 1$. Consequently, for any $k \in \mathbb{N}$

$$\int_G f \, d\mu_{N_k} = \int_{V_x} f \, d\mu_{N_k} = \sum_{\beta\in\mathcal{D}_{N_k}\cap V_x} f(\beta) = f(x) = 1$$

and

$$\int_G f \, d\mu_{M_k} = \int_{V_x} f \, d\mu_{M_k} = \sum_{\beta\in\mathcal{D}_{M_k}\cap V_x} f(\beta) = 0.$$

This contradicts the fact that the limit of $(\mu_n)_{n\in\mathbb{N}}$ exists and so follows the statement. □

The latter considerations pose the question if one can convey the properties that $\mathcal{D}$ is an aperiodic Delone set of finite local complexity to the elements of the hull $\Omega_{\mathcal{D}}$.

Lemma 2.11. *Let $\mathcal{D}$ be a (U, K)-Delone set of finite local complexity. Then, for any $\tilde{\mathcal{D}} \in \Omega_{\mathcal{D}}$ it is $(\tilde{\mathcal{D}} - \tilde{\mathcal{D}}) \subseteq (\mathcal{D} - \mathcal{D})$.*

Proof. According to Proposition 2.8 for any $\tilde{\alpha}, \tilde{\beta} \in \tilde{\mathcal{D}}$ there is a sequence $(t_n)_{n\in\mathbb{N}} \subseteq G$ and further, a sequence $(\alpha_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ respectively $(\beta_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ such that

$$\tilde{\alpha} = \lim_{n\to\infty} (t_n + \alpha_n)$$

and

$$\tilde{\beta} = \lim_{\mathrm{n}\to\infty} (t_n + \beta_n).$$

Hence, by the continuity of the group composition we have

$$\tilde{\alpha} - \tilde{\beta} = \lim_{n\to\infty} (\alpha_n - \beta_n).$$

Moreover, $\mathcal{D}$ is of finite local complexity, which means by Proposition 2.4 that $(\mathcal{D} - \mathcal{D})$ is discrete and closed. Thus, there exists an $n_0 \in \mathbb{N}$ such that

$$\tilde{\alpha} - \tilde{\beta} = \alpha_n - \beta_n, \qquad \forall n \geq n_0.$$

In fact, we get that $\tilde{\alpha} - \tilde{\beta}$ is an element of $(\mathcal{D} - \mathcal{D})$ which means that

$$(\tilde{D} - \tilde{D}) \subseteq (\mathcal{D} - \mathcal{D}).$$

□

Proposition 2.12. *Let $\mathcal{D}$ be a (U, K)-Delone set of finite local complexity, then, any $\tilde{\mathcal{D}} \in \Omega_{\mathcal{D}}$ is a (U, K)-Delone set of finite local complexity as well.*

Proof. First note that we know by Lemma 2.3 that any element of the set $\{t+\mathcal{D} \mid t \in G\}$ is an aperiodic (U, K)-Delone set of finite local complexity. Further, by definition $\Omega_{\mathcal{D}}$ is closed with respect to the pointset topology. Let $\tilde{\mathcal{D}}$ be an element of the hull of $\mathcal{D}$. Then, there is a sequence $(\mathcal{D}_n)_{n\in\mathbb{N}} \subseteq \{t + \mathcal{D} \mid t \in G\}$ holding $\mathcal{D}_n \overset{\mathcal{O}(\Lambda_U)}{\to} \tilde{\mathcal{D}}$. By our previous considerations there is a corresponding sequence $(t_n)_{n\in\mathbb{N}} \subseteq G$ such that any $\mathcal{D}_n$ has the form $t_n + \mathcal{D}$ for some $n \in \mathbb{N}$.

"Delone set":

First we will show that $\tilde{\mathcal{D}}$ is K-relatively dense which means that $\tilde{\mathcal{D}} + K = G$. It is sufficient to show that $G \subseteq \tilde{\mathcal{D}} + K$, because the converse conclusion is obvious.

Choose some $x \in G$, then, for any $\mathcal{D}_n = t_n + \mathcal{D}$ there exists an $\alpha_n \in \mathcal{D}$ and a $\kappa_n \in K$ such that $t_n + \alpha_n + \kappa_n = x$. Since K is compact there is a convergent subsequence κ_{n_k} such that $\lim_{k\to\infty} x - \kappa_{n_k} = \lim_{k\to\infty} t_{n_k} + \alpha_{n_k}$ exists. According to Proposition 2.8 the limit $\tilde{\alpha} := \lim_{k\to\infty} t_{n_k} + \alpha_{n_k}$ has to be an element of $\tilde{\mathcal{D}}$. Thus,

$$x = \lim_{k\to\infty} (t_{n_k} + \alpha_{n_k} + \kappa_{n_k}) = \lim_{k\to\infty} (t_{n_k} + \alpha_{n_k}) + \lim_{k\to\infty} \kappa_{n_k} \in \tilde{\alpha} + K$$

by using that the group composition is continuous.

Secondly, we would like to show that $\tilde{\mathcal{D}}$ is U-uniformly discrete which means that $\sharp((x+U)\cap\tilde{\mathcal{D}}) \leq 1$ for any $x \in G$.

Assume there are two distinct element $\tilde{\alpha}, \tilde{\beta} \in ((x+U)\cap\tilde{D})$ for some $x \in G$. Thus, by Proposition 2.8 we have $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ such that

$$\tilde{\alpha} = \lim_{\mathrm{n}\to\infty} (t_n + \alpha_n),$$
$$\tilde{\beta} = \lim_{\mathrm{n}\to\infty} (t_n + \beta_n).$$

Since G is a Hausdorff space the elements $\tilde{\alpha}$ and $\tilde{\beta}$ can be seperated by two open, disjoint sets $V_{\tilde{\alpha}}, V_{\tilde{\beta}} \subsetneq (x+U)$. Hence, by the convergence there is some $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have

$$t_n + \alpha_n \in V_{\tilde{\alpha}} \subsetneq (x+U),$$
$$t_n + \beta_n \in V_{\tilde{\beta}} \subsetneq (x+U).$$

On the other hand, the corresponding $t_n + \mathcal{D}$ is U-uniformly discrete by Lemma 2.3. This leads to a contradiction, because $\sharp((x+U)\cap(t_n+\mathcal{D}))$ is at least two for $n \geq n_0$.

"FLC":

By our previous considerations we know that $\tilde{\mathcal{D}}$ is a (U,K)-Delone set. According to Proposition 2.4 the set $\mathcal{D}$ is of finite local complexity if and only if for any compact subset K of G the intersection $(\mathcal{D}-\mathcal{D})\cap K$ is finite or empty.

Thus, by Lemma 2.11 for any compact subset $K \subseteq G$ the intersection

$$(\tilde{\mathcal{D}}-\tilde{\mathcal{D}})\cap K \subseteq (\mathcal{D}-\mathcal{D})\cap K$$

is finite or empty. Hence, by Proposition 2.4 the set $\tilde{\mathcal{D}}$ is of finite local complexity as well.

□

Proposition 2.13 ([SCH])**.** *Let G be a locally compact, Hausdorff, abelian group which is σ-compact. Consider a Delone set $\mathcal{D}$ of finite local complexity. Then, the following assertions are equivalent.*

(i) The set $\mathcal{D}$ is repetitive.

(ii) For any element $\tilde{\mathcal{D}}$ of the hull $\Omega_{\mathcal{D}}$ of $\mathcal{D}$ the equality $\Omega_{\mathcal{D}} = \Omega_{\tilde{\mathcal{D}}}$ holds.

Using this Proposition 2.13 we get a strengthen statement as in Lemma 2.11.

Proposition 2.14. *Let $\mathcal{D}$ be a repetitive (U, K)-Delone set of finite local complexity. Then, for any $\tilde{\mathcal{D}} \in \Omega_{\mathcal{D}}$ it is true that $(\tilde{\mathcal{D}} - \tilde{\mathcal{D}}) = (\mathcal{D} - \mathcal{D})$.*

Proof. According to Lemma 2.11 we know for $\tilde{D} \in \Omega_{\mathcal{D}}$ that $(\tilde{\mathcal{D}} - \tilde{\mathcal{D}}) \subseteq (\mathcal{D} - \mathcal{D})$. By Proposition 2.12 the set $\tilde{D}$ is a (U, K)-Delone set of finite local complexity and by Proposition 2.13 we have $\mathcal{D} \in \Omega_{\tilde{D}}$. Using Lemma 2.11 for the (U, K)-Delone set $\tilde{\mathcal{D}}$ of finite local complexity, then, the inclusion

$$(\mathcal{D} - \mathcal{D}) \subseteq (\tilde{\mathcal{D}} - \tilde{\mathcal{D}})$$

follows, which leads to the statement. □

Lemma 2.15. *Consider an aperiodic, repetitive (U, K)-Delone set of finite local complexity. Then, for any compact subset $K \subseteq G$ and any $\tilde{\mathcal{D}} \in \Omega_{\mathcal{D}}$ the equality*

$$Clu(\mathcal{D}, K) = Clu(\tilde{\mathcal{D}}, K)$$

holds.

Proof. Fix some compact subset $K \subseteq G$ and consider some $\tilde{\mathcal{D}} \in \Omega_{\mathcal{D}}$. Then, there exists a sequence $(t_n)_{n\in\mathbb{N}} \subseteq G$ such that $\tilde{\mathcal{D}} := \lim_{n\to\infty}(t_n + \mathcal{D})$. Let P be some element of $Clu(\tilde{\mathcal{D}}, K)$. Thus, there is some $\tilde{\alpha} \in \tilde{\mathcal{D}}$ with $P = (-\tilde{\alpha} + \tilde{\mathcal{D}}) \cap K$. According to Proposition 2.8 there is a sequence $(\alpha_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ such that

$$-\tilde{\alpha} + \tilde{\mathcal{D}} = \lim_{n\to\infty}(-(t_n + \alpha_n) + (t_n + \mathcal{D})) = \lim_{n\to\infty}(-\alpha_n + \mathcal{D})$$

where the limit exists. According to Proposition 2.10 there exists an $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ the equation

$$P = (-\tilde{\alpha} + \tilde{\mathcal{D}}) \cap K = (-\alpha_n + \mathcal{D}) \cap K$$

holds. Hence, the cluster P is an element of $Clu(\mathcal{D}, K)$, because $\alpha_n \in \mathcal{D}$.

The converse inclusion follows similar by using the fact that $\mathcal{D}$ is repetitive and the equivalence in Proposition 2.13. □

Moreover, we get a strengthen statement of Proposition 2.12 as well using Proposition 2.13.

Proposition 2.16. *Let $\mathcal{D}$ be an aperiodic, repetitive (U, K)-Delone set of finite local complexity, then, any $\tilde{\mathcal{D}} \in \Omega_{\mathcal{D}}$ is an aperiodic (U, K)-Delone set of finite local complexity as well.*

Proof. Consider some fixed $\tilde{\mathcal{D}} \in \Omega_{\mathcal{D}}$. According to Proposition 2.12 it follows that $\tilde{\mathcal{D}}$ is a (U, K)-Delone set of finite local complexity. Thus, we only have to verify that $\tilde{\mathcal{D}}$ is aperiodic.

It suffices to check if for some $x \in G$ the equation $x + \tilde{\mathcal{D}} = \tilde{\mathcal{D}}$ holds it leads to the equality $x + \mathcal{D} = \mathcal{D}$. Then, by the aperiodicity of $\mathcal{D}$ it follows that x has to be the neutral element 0.

To do so let $x \in G$ be chosen such that

$$x + \tilde{\mathcal{D}} = \tilde{\mathcal{D}} \Leftrightarrow -x + \tilde{\mathcal{D}} = \tilde{\mathcal{D}}. \qquad (\star)$$

The following consideration can be made for x and $-x$ and so let y denote the element x and $-x$, respectively. Since G is locally compact there is a compact set which contains x and $-x$. Condition $(\star)$ implies that for each $\tilde{\alpha} \in \tilde{\mathcal{D}}$

$$y \in (-\tilde{\alpha} + \tilde{\mathcal{D}}).$$

Thus, any cluster $\tilde{P} \in Clu(\tilde{\mathcal{D}}, K)$ contains y, because any cluster $\tilde{P}$ has the form $(-\tilde{\alpha} + \tilde{\mathcal{D}}) \cap K$ for some $\tilde{\alpha} \in \tilde{\mathcal{D}}$.

Hence, according to Lemma 2.15 any cluster $P \in Clu(\mathcal{D}, K)$ contains y. If now y is $-x$ it follows that for any $\alpha \in \mathcal{D}$ there is a $\beta \in \mathcal{D}$ such that $-x = -\alpha + \beta$. Consequently,

$$-x + \mathcal{D} \subseteq \mathcal{D}.$$

The converse inclusion follows similar for $y = x$ which leads to our statement. □

The set

$$\mathcal{P}_{\mathcal{T}} := \{\tilde{\mathcal{D}} \in \Omega_{\mathcal{D}} \mid 0 \in \tilde{\mathcal{D}}\} \subseteq \Omega_{\mathcal{D}} \subseteq \Lambda_U$$

is called *pointset of the transversal.* Since for any $\alpha \in \mathcal{D}$ the set $(-\alpha + \mathcal{D})$ is an element of the hull of $\mathcal{D}$ and $0 \in (-\alpha + \mathcal{D})$ the set $\mathcal{P}_{\mathcal{T}}$ is not empty.

According to that we define the *transversal* for a Delone set $\mathcal{D}$ by

$$\mathcal{T} := \{\mu \in \mathcal{J}(\Omega_{\mathcal{D}}) \mid \mathcal{D}_\mu \in \mathcal{P}_{\mathcal{T}}\}.$$

In words, the transversal $\mathcal{T}$ of a Delone set $\mathcal{D}$ is the set of all Radon measures μ, where the corresponding "translated" pointset $\mathcal{D}_\mu$ is U-uniformly discrete and contains the origin.

Proposition 2.17. *The transversal $\mathcal{T}$ of an aperiodic Delone set $\mathcal{D}$ of finite local complexity is closed with respect to the vague topology.*

Proof. Assume that $\mathcal{T}$ is not closed, more precisely, there is a sequence $(t_k)_{k\in\mathbb{N}} \subseteq \mathcal{T}$ which converges to $t \in \mathcal{J}(\Lambda_U)$ and $t \notin \mathcal{T}$. Let for $k \in \mathbb{N}$ the set $\mathcal{D}_{t_k} \subseteq \mathcal{P}_{\mathcal{T}}$ be the corresponding pointset of $(t_k)_{k\in\mathbb{N}}$ and $\mathcal{D}_t \subseteq \Lambda_U$ is the pointset of t. Then, by our assumption $0 \notin \mathcal{D}_t$ but 0 is an element of $\mathcal{D}_{t_k}$ for any $k \in \mathbb{N}$.

According to Proposition 2.16 the set $\mathcal{D}_t$ is an aperiodic Delone set of finite local complexity. Since G is a Hausdorff space there exists an open neighborhood V of 0 such that $\mathcal{D}_t \cap V$ is empty. Then, there is a continuous function $f : G \to [0,1]$ with compact support $supp(f) \subseteq V$ such that $f(0) = 1$. Thus,

$$0 = \int\limits_{\mathrm{G}} f\ dt = \lim_{\mathrm{n}\to\infty} \int\limits_{\mathrm{G}} f\ dt_n \overset{0\in\mathcal{D}_{t_n}}{\geq} \lim_{\mathrm{n}\to\infty} f(0) = 1$$

which is a contradiction. □

Lemma 2.18. *Let $\mathcal{D}$ be an aperiodic Delone set of finite local complexity. For any $\tilde{\mathcal{D}} \in \Omega_{\mathcal{D}}$ the following assertions are equivalent.*

(i) The set $(b + \tilde{\mathcal{D}})$ contains $\alpha + b$ for each $b \in G$.

(ii) The element α lies in $\tilde{\mathcal{D}}$.

Proof. Let μ be the corresponding Radon measure of $\tilde{\mathcal{D}}$ and ν the Radon measure of $b + \tilde{\mathcal{D}}$ for some $b \in G$. Then,

$$\mu = \sum_{\alpha\in\tilde{\mathcal{D}}} \delta_\alpha \quad \text{and} \quad \nu = \sum_{\beta\in(b+\tilde{\mathcal{D}})} \delta_\beta.$$

Thus, for any $x \in G$

$$\begin{aligned}\nu(\{x\}) = &\sum_{\beta\in(b+\tilde{\mathcal{D}})} \delta_\beta(\{x\}) = \sum_{\alpha\in\tilde{\mathcal{D}}} \delta_{\alpha+b}(\{x\}) = \sum_{\alpha\in\tilde{\mathcal{D}}} \delta_\alpha(\{x-b\}) \\ = &\tau^b \sum_{\alpha\in\tilde{\mathcal{D}}} \delta_\alpha(\{x\}) = \tau^b \mu(\{x\}).\end{aligned}$$

Hence, for some $b \in G$

$$\alpha + b \in (b + \tilde{\mathcal{D}}) \overset{\text{def.}}{\Leftrightarrow} 1 = \nu(\{\alpha + b\}) = \tau^b \mu(\{\alpha + b\}) = \mu(\{\alpha\}) \overset{\text{def.}}{\Leftrightarrow} \alpha \in \tilde{\mathcal{D}}.$$

□

2.4 Example

We give an example of an aperiodic Delone set $\mathcal{D} \subset \mathbb{R}$ of finite local complexity. It will be given the idea and no exact proof of its properties.

Consider some infinite word $\omega \in \{0,1\}^{\mathbb{Z}}$ such that any finite word over $\{0,1\}$ occurs in ω. We denote by W_n for $n \in \mathbb{N} \setminus \{0\}$ the set of all different words over $\{0,1\}$ with length n. For instance,

$$\begin{aligned} W_1 &= \{0;1\} \\ W_2 &= \{00;01;10;11\} \\ W_3 &= \{000;001;010;100;110;011;101;111\} \\ &\vdots \end{aligned}$$

Since $\{0,1\}$ is finite any W_n is finite. In detail, $\sharp(W_n)$ equals 2^n. Nevertheless, we can number consecutively our finite words by $\omega_1 = 0, \omega_2 = 1, \omega_3 = 00, \omega_4 = 01$ and so on. Then, we define an infinite word ω by

$$\ldots \omega_4 \, \omega_3 \, \omega_2 \, \omega_1 \mid \omega_1 \, \omega_2 \, \omega_3 \, \omega_4 \ldots$$

respectively

$$\ldots 11\ 10\ 01\ 00\ 1\ 0 \mid 0\ 1\ 00\ 01\ 10\ 11 \ldots$$

where $|$ denotes the origin of the word ω. By definition any finite word over $\{0,1\}$ is contained in ω. It is unimportant in which order we number consecutively our finite words, we only have to hold to our convention. A word is aperiodic if we cannot translate the origin of ω such that the translated word ω' is equal to ω component-by-component. Note that one component is only 1 or 0. For instance, if we liked to translate our origin by $t = 3 \in \mathbb{Z}$, then,

$$\ldots 11\ 10\ 01\ 00\ 1\ 0 \mid 0\ 1\ 0 \mid 0\ 01\ 10\ 11 \ldots$$

where "$|$" is the old origin. Similarly, we can translate the origin for any $t \in \mathbb{Z}$. Denote by $\tau^t \circ \omega$ the translated word $\omega \in \{0,1\}^{\mathbb{Z}}$ by $t \in \mathbb{Z}$. The map $\tau : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ is called *shift* of the words.

Take two intervals $I_0 = [0,1]$ and $I_1 = [0,q]$ where $q \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$ is an irrational number. Note that the choice of q being an element of the interval $[0,1]$ is a convention. Besides, we need q being irrational to get $(\mathcal{D} - \mathcal{D})$ to be not weakly uniformly discrete only. It is possible to construct an aperiodic Delone set with any two intervals with different lengths.

Now, we can use ω to tile our space $\mathbb{R}$. More precisely, we will replace 0 by the interval I_0 and 1 by the interval I_1, respectively. In Figure 5 the construction of the tiling of $\mathbb{R}$ is sketched. The blue points represent the points where we attach two intervals. Thus, the orange numbers represent the distance between the origin and these points. Further, we can see that the interval I_0 is marked by 0 and I_1 is marked by 1, respectively. However, the set of blue points will be denoted by $\mathcal{D}$.

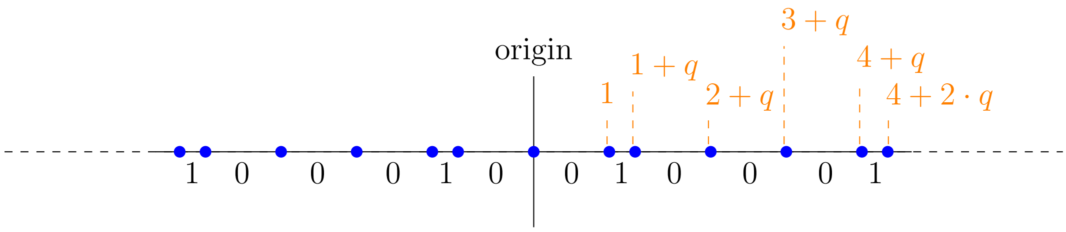

Figure 5:
An aperiodic Delone set of finite local complexity in $\mathbb{R}$ defined by an infinite word

We can indicate by an one-to-one correspondence the closed intervals with the elements of $\mathcal{D}$. In fact, two points determine a closed interval in $\mathbb{R}$ in a unique way. Consequently, the set $\mathcal{D} \subsetneq \mathbb{R}$ is also aperiodic, if the word ω is aperiodic. As mentioned above, we use here the fact that I_0 and I_1 have different lengths.

Assume the word ω is not aperiodic. Thus, there is a $t \in \mathbb{Z}\setminus\{0\}$ such that we can translate ω by t and the new infinite word ω' is equal to ω. Hence, by induction ω is t-periodic which means that $\tau^{z \cdot t} \circ \omega = \omega$ for $z \in \mathbb{Z}$. Consequently, we have a finite word ω_t of the length t attached to itself in both directions infinitely many times.

Without loss of generality, we can suppose that t is a positive integer. Then, the word ω has at most t different words of the length t. This contradicts the fact that $\sharp W_t$ has to be equal to 2^t, which is greater than t for $t \neq 0$. This implies that our word ω is aperiodic and so $\mathcal{D}$ is as well.

We can illustrate the sketch of the proof by a short example. Let $t = 3$ and our finite word $\omega_t = 010$. Then, ω has the form

$$\ldots 010\ 010\ 010\ 010\ 010 \mid 010\ 010\ 010\ 010\ 010 \ldots$$

We can immediately see that ω contains the words $010, 100, 001$ of length three only. But for example the word 110 never constists in ω.

Furthermore, the set $\mathcal{D}$ is U-uniformly discrete, where U is the open intervall $(0, q)$. Also $\mathcal{D}$ is K-relatively dense, where K is the closed interval $[0, 1]$. Note that we use here the fact that $q \in [0, 1]$.

An element of $\mathcal{D}$ is called 1-*point* if it is the boundary point on the right hand side of an interval of lenght one. Similar a q-*point* is the boundary point on the right hand side of an interval of lenght q.

Our next aim is to determine the set $(\mathcal{D}-\mathcal{D})$ and indicate some properties of it. The vectors in $(\mathcal{D}-\mathcal{D})$ are given by the distance of two points of $\mathcal{D}$. In other words, a vector $(\alpha-\beta)$ in $(\mathcal{D}-\mathcal{D})$ counts how many q-points and 1-points we have to go from β to reach α, see Figure 6.

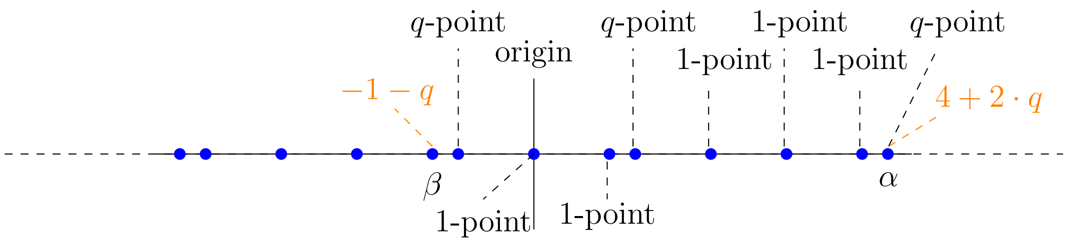

Figure 6:
An aperiodic Delone set of finite local complexity in $\mathbb{R}$ defined by an infinite word where the notation of the points is sketched

Thus, $(\alpha-\beta)=(4\cdot 1+2\cdot q)-(-1-q)=5\cdot 1+3\cdot q$ and so we have to go five times a 1-point step and three times a q-point step. This works for any $\alpha,\beta\in\mathcal{D}$ and so $(\alpha-\beta)$ can be written as $(m+n\cdot q)$ or $(-m-n\cdot q)$ for some $m,n\in\mathbb{N}$. If $(\alpha-\beta)$ is an element of $(\mathcal{D}-\mathcal{D})$, then, $(\beta-\alpha)$ as well. Indeed, there are no more other elements of $(\mathcal{D}-\mathcal{D})$.

The converse inclusion follows immediately by the fact that ω contains any finite word. To conclude our statements the equality

$$(\mathcal{D}-\mathcal{D})=\{m+n\cdot q \mid m,n\in\mathbb{N}\}\cup\{-m-n\cdot q \mid m,n\in\mathbb{N}\}.$$

holds.

Obviously, we have $(\mathcal{D}-\mathcal{D})\cap K$ is finite for any closed interval $K\subseteq\mathbb{R}$. Hence, by Proposition 2.4 the set $(\mathcal{D}-\mathcal{D})$ is discrete and closed or in other words $\mathcal{D}$ is of finite local complexity. As mentioned in the section 2.2 ("Delone sets of finite local complexity") if we consider a Delone set $\mathcal{D}\subsetneq G$ of a compactly generated group G the set $(\mathcal{D}-\mathcal{D})$ is weakly uniformly discrete, iff $(\mathcal{D}-\mathcal{D})$ is uniformly discrete. It can be shown that our set $(\mathcal{D}-\mathcal{D})$ is not uniformly discrete using that q is irrational. Since $\mathbb{R}^d$ is compactly generated it follows that $(\mathcal{D}-\mathcal{D})$ is not weakly uniformly discrete.

Chapter 3

Lagarias group

3.1 Definition and properties

Let $\mathcal{D}$ be a Delone set of G. Recall that we denote by $[\mathcal{F}]$ the set of all finite linear $\mathbb{Z}$-combinations of elements of $\mathcal{F} \subseteq G$. The set $[\mathcal{D} - \mathcal{D}]$ is called *finitely generated*, if there is a finite subset of $(\mathcal{D} - \mathcal{D})$ denoted by $\mathcal{B}_{(\mathcal{D}-\mathcal{D})}$ with $[\mathcal{D} - \mathcal{D}] = [\mathcal{B}_{(\mathcal{D}-\mathcal{D})}]$. We need the following short Lemma. Our first aim is to characterize when $[\mathcal{D} - \mathcal{D}]$ is finitely generated.

Lemma 3.1 ([BLM])**.** *Let G' be a topological space and $\mathcal{C}$ a compact space. Consider the product space $G = G' \times \mathcal{C}$ endowed with the product topology. Then, the projection $Pr : G \to G'$ maps locally finite subset of G to locally finite subsets of G'.*

Proof. Let $S \subseteq G$ be locally finite and so $S \cap K$ is finite or empty for any compact $K \subseteq G$. Assume $Pr(S)$ is not locally finite. Thus, there exists a compact $K' \subseteq G'$ such that $Pr(S) \cap K'$ is infinite. Hence, $S \cap (K' \times \mathcal{C})$ is infinite. On the other hand, if $\mathcal{C}$ is compact, $K' \times \mathcal{C}$ is compact with respect to the product topology. This contradicts the fact that S is locally finite. $\square$

In order to prove the next Theorem 3.3 we need the following statement of [DE], Theorem 4.2.2. Further information may be found there.

Proposition 3.2 ([DE])**.** *Let G be a locally compact, Hausdorff, compactly generated, abelian group. Then, for some $m, n \in \mathbb{N}$ and some compact, abelian group $\mathcal{C}$ the group G is topologically isomorphic to $\mathbb{R}^m \times \mathbb{Z}^n \times \mathcal{C}$.*

Theorem 3.3 ([BLM])**.** *Consider a locally compact, Hausdorff, abelian group G which is compactly generated. Let $\mathcal{D} \subseteq G$ be a (U, K)-Delone set and suppose $\mathcal{D}$ is of finite local complexity. Then, there exists a finite subset $\mathcal{F}$ of G such that $[\mathcal{F}] = [\mathcal{D}]$.*

Proof. By Proposition 3.2 there are some $n, m \in \mathbb{N}$ and a compact, abelian group $\mathcal{C}$ such that we can identify G by $\mathbb{R}^m \times \mathbb{Z}^n \times \mathcal{C}$. Thus, we can consider G as a subgroup of $\mathbb{R}^{m+n} \times \mathcal{C}$.

Since $\mathcal{D}$ is relatively dense we find a compact set $K \subseteq G$ such that $\mathcal{D}+K = G$. Moreover, the projection from G onto $\mathbb{R}^m \times \mathbb{Z}^n$ is continuous. Hence, the projection of K is again a compact set and so there is a radius $R > 0$ such that

$$\mathbb{R}^m \times \mathbb{Z}^n \times \mathcal{C} = \mathcal{D} + (Pr(K) \times \mathcal{C}) \subseteq \mathcal{D} + (\mathcal{B}_R(\overrightarrow{0}) \times \mathcal{C})$$

where $\mathcal{B}_R(\overrightarrow{0})$ is the open ball with radius R and center $\overrightarrow{0} \in \mathbb{R}^{m+n}$. Without loss of generality, the radius R is chosen such that

$$\mathbb{R}^{m+n} \times \mathcal{C} = \mathbb{R}^m \times \mathbb{R}^n \times \mathcal{C} = \mathcal{D} + (\mathcal{B}_R(\overrightarrow{0}) \times \mathcal{C}). \qquad (\star)$$

Since $\mathcal{D}$ is a U-uniformly discrete set the set $\mathcal{D}$ is locally finite, see Lemma 2.3. According to Proposition 2.4 the set $(\mathcal{D} - \mathcal{D})$ is also locally finite. Consequently, the set

$$\begin{aligned} \mathcal{F} :=& (\mathcal{D} \cup (\mathcal{D} - \mathcal{D})) \cap (\overline{\mathcal{B}}_{2R}(\overrightarrow{0}) \times \mathcal{C}) \\ =& (\mathcal{D} \cap (\overline{\mathcal{B}}_{2R}(\overrightarrow{0}) \times \mathcal{C})) \cup ((\mathcal{D} - \mathcal{D}) \cap (\overline{\mathcal{B}}_{2R}(\overrightarrow{0}) \times \mathcal{C})) \end{aligned}$$

is finite, because $\overline{\mathcal{B}}_{2R}(\overrightarrow{0}) \times \mathcal{C}$ is compact. Now consider the projection map

$$Pr : \mathbb{R}^{m+n} \times \mathcal{C} \to \mathbb{R}^{m+n}\,.$$

We will show that

$$[\mathcal{F}] := \{\sum_{j=1}^{N} n_j \cdot f_j \mid N \in \mathbb{N},\ n_j \in \mathbb{Z} \text{ and } f_j \in \mathcal{F}\} = [\mathcal{D}]$$

Note that it suffices to show that $\mathcal{F} \subseteq [\mathcal{D}]$ and $\mathcal{D} \subseteq [\mathcal{F}]$, because any finite linear $\mathbb{Z}$-combination of $\mathcal{F}$ is in $[\mathcal{D}]$ and of $\mathcal{D}$ is in $[\mathcal{F}]$, respectively.

"$\subseteq$": By definition of $\mathcal{F}$, any element $f \in \mathcal{F}$ has to be an element of $\mathcal{D}$ or $(\mathcal{D} - \mathcal{D})$. Thus, each $f \in \mathcal{F}$ lie in $[\mathcal{D}]$.

"$\supseteq$": Let α be some element of $\mathcal{D}$. If $\alpha \in (\overline{\mathcal{B}}_{2R}(\overrightarrow{0}) \times \mathcal{C})$, it follows immediately by construction of F that $\alpha \in F$. Suppose α is not an element of $(\overline{\mathcal{B}}_{2R}(\overrightarrow{0}) \times \mathcal{C})$. Now choose an $x' \in \mathbb{R}^{m+n}$ on the segment $[\overrightarrow{0}, Pr(\alpha)]$ such that

$$\|x' - Pr(\alpha)\| = R$$

where $\| \cdot \|$ is the standard Euclidean norm. We can choose such an $x' \in \mathbb{R}^{m+n}$ because $\alpha \notin (\overline{\mathcal{B}}_{2R}(\overrightarrow{0}) \times \mathcal{C})$ and so $\| Pr(\alpha) - \overrightarrow{0} \| > 2R$. Further, this x' is unique, see Figure 7.

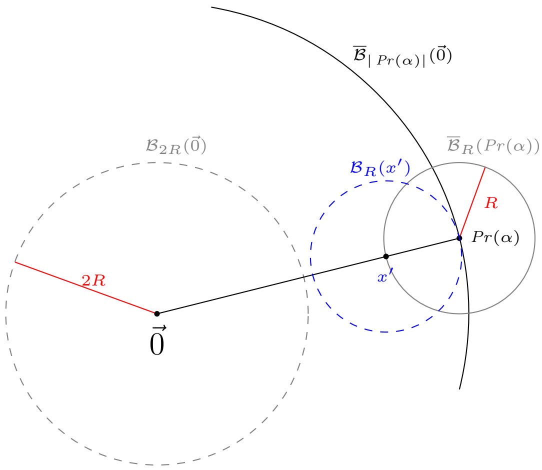

Figure 7:
The construction of the element x' for some given α

Indeed,

$$Pr(\alpha) \in \overline{\mathcal{B}}_R(x') \setminus \mathcal{B}_R(x') = \partial\,\mathcal{B}_R(x')$$

where $\partial\,\mathcal{B}_R(x')$ is the boundary of the open ball $\mathcal{B}_R(x')$. Set $x := (x', 0_{\mathcal{C}}) \in \mathbb{R}^{m+n} \times \mathcal{C}$ where $0_{\mathcal{C}}$ is the neutral element of the compact, abelian group $\mathcal{C}$. Consequently, $Pr(x)$ is equal to x'. Because of our choice of the radius R (see $(\star)$) there is a $\beta_1 \in \mathcal{D}$ and a $y \in \mathcal{B}_R(\overrightarrow{0}) \times \mathcal{C}$ such that $x = \beta_1 + y$.

Since the projection is linear and $Pr(y) \in \mathcal{B}_R(\overrightarrow{0})$ we get

$$\|\,Pr(x) - Pr(\beta_1)\| = \|x' - Pr(\beta_1)\| = \|\,Pr(y)\| < R$$

which implies that $Pr(\beta_1) \in \mathcal{B}_R(x')$. Moreover, by construction $Pr(\beta_1) \in \mathcal{B}_R(x') \subsetneq \overline{\mathcal{B}}_{\|\,Pr(\alpha)\|}(\overrightarrow{0})$, see Figure 8. Hence,

$$\|\,Pr(\beta_1)\| < \|\,Pr(\alpha)\|. \qquad (\star\star)$$

Since $Pr(\beta_1) \in \mathcal{B}_R(x')$ and $Pr(\alpha) \in \partial\,\mathcal{B}_R(x')$ the inequality

$$\|\,Pr(\alpha) - Pr(\beta_1)\| < 2R$$

holds and so $Pr(\alpha - \beta_1) \in \mathcal{B}_{2R}(\overrightarrow{0})$. In particular, this means that $\alpha - \beta_1$ is an element of $\mathcal{B}_{2R}(\overrightarrow{0}) \times \mathcal{C}$. On the other hand, α, β_1 are elements of $\mathcal{D}$ and thus,

$$\alpha - \beta_1 \in (\mathcal{D} - \mathcal{D}) \cap (\mathcal{B}_{2R}(\overrightarrow{0}) \times \mathcal{C}) \subseteq \mathcal{F}\,.$$

Thus, there exists some $f_1 \in \mathcal{F}$ such that $\alpha - \beta_1 = f_1$ respectively

$$\alpha = \beta_1 + f_1$$

for some $f_1 \in \mathcal{F}$, $\beta_1 \in \mathcal{D}$ with $\|\,Pr(\beta_1)\| < \|\,Pr(\alpha)\|$, see $(\star\star)$.

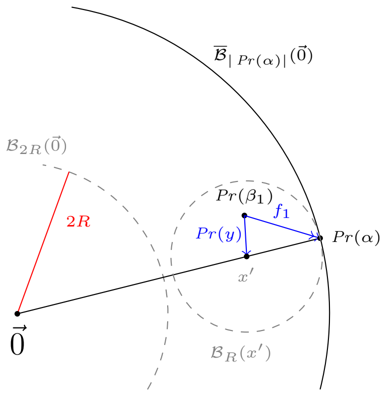

Figure 8:
The illustration of the first step of the iteration

Now we can follow the same steps for β_1 and construct a $\beta_2 \in \mathcal{D}$ such that $\|\, Pr(\beta_2)\| < \|\, Pr(\beta_1)\|$ and $\beta_1 = \beta_2 + f_2$ for some $f_2 \in \mathcal{F}$. In the following we will see that we can iterate this construction until we have a $\beta_K \in \mathcal{D}$ such that $\|\, Pr(\beta_K)\| < 2R$.

According to Lemma 3.1 the set $Pr(\mathcal{D})$ is locally finite and so $Pr(\mathcal{D}) \cap \overline{\mathcal{B}}_{|\,Pr(\alpha)|}(\overrightarrow{0})$ is finite or empty. It turns out that the intersection

$$Pr(\mathcal{D}) \cap \overline{\mathcal{B}}_{2R}(\overrightarrow{0}) \subseteq Pr(\mathcal{D}) \cap \overline{\mathcal{B}}_{|\,Pr(\alpha)|}(\overrightarrow{0})$$

is not empty:

We know that $(\overrightarrow{0}, 0_{\mathcal{C}})$ is an element of $\mathbb{R}^{m+n} \times \mathcal{C} = \mathcal{D} + (\mathcal{B}_R(\overrightarrow{0}) \times \mathcal{C})$. Hence, there is a $\gamma \in \mathcal{D}$ such that $(\overrightarrow{0}, 0_{\mathcal{C}}) \in \gamma + (B_R(\overrightarrow{0}) \times \mathcal{C})$. Consequently, $\overrightarrow{0} \in Pr(\gamma) + \mathcal{B}_R(\overrightarrow{0})$ which leads to

$$\|\, Pr(\gamma)\| = \|\, Pr(\gamma) - \overrightarrow{0}\,\| \leq R < 2R.$$

Thus, $Pr(\gamma) \in Pr(\mathcal{D})$ is an element of $\overline{\mathcal{B}}_{2R}(\overrightarrow{0})$ and so $Pr(\mathcal{D}) \cap \overline{\mathcal{B}}_{2R}(\overrightarrow{0})$ is not empty. Further, during one iteration the distance between $\overrightarrow{0}$ and the constructed β_j really gets smaller.

Altogether, for any $\alpha \in \mathcal{D}$ we find some $f_1, \ldots, f_K \in \mathcal{F}$ and a $\beta_K \in \mathcal{D} \cap (\mathcal{B}_{2R}(\overrightarrow{0}) \times \mathcal{C}) \subseteq \mathcal{F}$ such that

$$\alpha = \sum_{j=1}^{K} f_j + \beta_K.$$

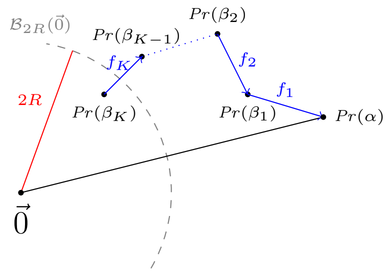

Figure 9:
An illustration of the idea of the proof

In particular, we can write α as a finite linear $\mathbb{Z}$-combination of elements of $\mathcal{F}$. Consequently, $\alpha \in \mathcal{D}$ is an element of $[\mathcal{F}]$.

We have shown that $[\mathcal{D}] = [\mathcal{F}]$ where $\mathcal{F}$ is a finite subset of G. $\square$

In Figure 9 there is sketched the idea of the proof, where f_j are elements of $\mathcal{F}$ and β_j are points of $\mathcal{D}$. One can imagine that for the construction of $[\mathcal{D}]$ it is sufficient to consider only the elements of $\mathcal{D}$ and the differences of them in some compact set containing the origin, denoted by $K_{\mathcal{D}} \simeq \overline{\mathcal{B}}_{2R}(\overrightarrow{0}) \times \mathcal{C}$.

Proposition 3.4. *Consider a locally compact, abelian, Hausdorff group G which is compactly generated. Let $\mathcal{D} \subseteq G$ be a (U, K)-Delone set of finite local complexity. If further, $\mathcal{D}$ contains the neutral element 0 we get that $[\mathcal{D} - \mathcal{D}]$ is finitely generated and moreover, the equation $[\mathcal{D} - \mathcal{D}] = [\mathcal{D}]$ holds.*

Proof. First we will show that $[\mathcal{D} - \mathcal{D}] = [\mathcal{D}]$. As in the proof above it suffices to show that $(\mathcal{D} - \mathcal{D}) \subseteq [\mathcal{D}]$ and $\mathcal{D} \subseteq [\mathcal{D} - \mathcal{D}]$, respectively.

"$\subseteq$": For any element of $(\mathcal{D} - \mathcal{D})$, we find an $\alpha, \beta \in \mathcal{D}$ such that we can represent this element by $\alpha - \beta$. This is a finite linear $\mathbb{Z}$-combination of elements of $\mathcal{D}$ and so $\alpha - \beta$ is in $[\mathcal{D}]$.

"$\supseteq$": Let $\alpha \in \mathcal{D}$ be arbitrary. Since $0 \in \mathcal{D}$ we know

$$\alpha = \alpha - 0 \in (\mathcal{D} - \mathcal{D}) \subseteq [\mathcal{D} - \mathcal{D}].$$

Thus, we have verified the equality $[\mathcal{D} - \mathcal{D}] = [\mathcal{D}]$.

In fact, we have shown that $\mathcal{D} \subseteq (\mathcal{D} - \mathcal{D})$. In the proof of Theorem 3.3 we have proven that for

$$\mathcal{F} := (\mathcal{D} \cup (\mathcal{D} - \mathcal{D})) \cap (\overline{\mathcal{B}}_{2R}(\overrightarrow{0}) \times \mathcal{C})$$

the equation $[\mathcal{F}] = [\mathcal{D}]$ holds. Hence, we have $[\mathcal{F}] = [\mathcal{D} - \mathcal{D}]$ and further, $\mathcal{F}$ is a subset of $(\mathcal{D} - \mathcal{D})$, since

$$\mathcal{F} \subseteq (\mathcal{D} - \mathcal{D}) \cup \mathcal{D} \; = (\mathcal{D} - \mathcal{D}).$$

Thus, we have a finite subset of $(\mathcal{D} - \mathcal{D})$ which generates $[\mathcal{D} - \mathcal{D}]$, or in other words, $[\mathcal{D} - \mathcal{D}]$ is finitely generated. $\square$

For any D-set $\mathcal{D}$ we call $[\mathcal{D} - \mathcal{D}]$ the *Lagarias group* of $\mathcal{D}$ and denote it by $\mathbb{L}_{\mathcal{D}}$. The name of this abelian group is motivated by the work of Lagarias [LAG1], [LAG2] and [LP].

Consider some finite set $\mathcal{B}_{(\mathcal{D}-\mathcal{D})}$ with $[\mathcal{B}_{(\mathcal{D}-\mathcal{D})}] = \mathbb{L}_{\mathcal{D}}$. This set is called a *minimal generator of the Lagarias group* if for any other finite set $\mathcal{B}'_{(\mathcal{D}-\mathcal{D})}$ with $[\mathcal{B}'_{(\mathcal{D}-\mathcal{D})}] = [\mathcal{D}-\mathcal{D}]$ the inequality

$$\sharp(\mathcal{B}'_{(\mathcal{D}-\mathcal{D})}) \geq \sharp(\mathcal{B}_{(\mathcal{D}-\mathcal{D})})$$

holds. Obviously, there is more than one minimal generator. For instance, the set $-\mathcal{B}_{(\mathcal{D}-\mathcal{D})}$ is also a minimal generator of $\mathbb{L}_{\mathcal{D}}$. An example is given at the end of this chapter.

The next proposition gives us a good characterization of the Lagarias group $\mathbb{L}_{\mathcal{D}}$.

Proposition 3.5. *Let $\mathcal{D} \subseteq G$ be a (U, K)-Delone set of finite local complexity and let $\mathcal{D}$ contain the neutral element 0. Consider a minimal generator $\mathcal{B}_{(\mathcal{D}-\mathcal{D})}$ of $\mathbb{L}_{\mathcal{D}}$. Then, there is a bijective, homomorphic map between $\mathbb{L}_{\mathcal{D}}$ and $\mathbb{Z}^M$ for $M := \sharp(\mathcal{B}_{(\mathcal{D}-\mathcal{D})})$.*

Proof. Let $\mathcal{B}_{(\mathcal{D}-\mathcal{D})} := \{e_1, \ldots, e_M \mid e_k \in (\mathcal{D}-\mathcal{D}) \text{ for } 1 \leq k \leq M\}$ be a minimal generator of $\mathbb{L}_{\mathcal{D}}$. Define for $M = \sharp(\mathcal{B}_{(\mathcal{D}-\mathcal{D})})$ the map $\Phi : \mathbb{L}_{\mathcal{D}} \to \mathbb{Z}^M$ by

$$\Phi(b) := \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_M \end{pmatrix}, \qquad \text{where } b = \sum_{j=1}^{M} n_j \cdot e_j \in \mathbb{L}_{\mathcal{D}}\,.$$

First, we have to check that Φ is well-defined: More precisely, that if

$$\sum_{j=1}^{M} n_j \cdot e_j = \sum_{j=1}^{M} n'_j \cdot e_j$$

it follows that for any $j \in \{1, \dots, M\}$ that $n_j = n'_j$. Assume the contrary, then, there is at least one $j_0 \in \{1, \dots, M\}$ such that

$$e_{j_0} = \sum_{\substack{j=1 \\ j \neq j_0}}^{M} (n'_j - n_j) \cdot e_j .$$

Thus, $\mathcal{B}_{(\mathcal{D}-\mathcal{D})} \setminus \{e_{j_0}\}$ generates $\mathbb{L}_{\mathcal{D}}$. This contradicts the minimality of $\mathcal{B}_{(\mathcal{D}-\mathcal{D})}$.

We have to verify that Φ is bijective and a homomorphism. Since the elements of $\mathcal{B}_{(\mathcal{D}-\mathcal{D})}$ are pairwise distinct the map is bijective. Further, we can easily see that Φ is a homomorphism. □

Define the map $\Psi : \mathbb{Z}^M \to \mathbb{L}_{\mathcal{D}}$ by

$$\begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_M \end{pmatrix} \longmapsto \sum_{j=1}^{M} n_j \cdot e_j .$$

As we have seen in the proof of Proposition 3.5 the fact that Φ is well-defined is equivalent to the property that Ψ is injective. Obviously this map is the inverse map of Φ.

Because of the last Proposition 3.5 we can identify $\mathbb{L}_{\mathcal{D}}$ with the group $\mathbb{Z}^M$. We endow $\mathbb{L}_{\mathcal{D}}$ with the discrete topology. Thus, the map $\Phi : \mathbb{L}_{\mathcal{D}} \to \mathbb{Z}^M$ is a homeomorphism which means that Φ and Ψ are continuous.

Lemma 3.6. *Consider $G = \mathbb{R}^d$ and an aperiodic (U, K)-Delone set $\mathcal{D} \subseteq \mathbb{R}^d$ of finite local complexity. Let $\mathcal{B}_{(\mathcal{D}-\mathcal{D})}$ be a minimal generator of $\mathbb{L}_{\mathcal{D}}$. Then, $M := \sharp \mathcal{B}_{(\mathcal{D}-\mathcal{D})}$ has to be greater than or equal to d.*

Proof. First, we will show that $\mathcal{D}$ is not contained in a hyperplane of $\mathbb{R}^d$. Assume the contrary, so, $\mathcal{D}$ is contained in a hyperplane $hyp(\mathcal{D})$ of $\mathbb{R}^d$. More precisely, there is a $0 \neq x \in \mathbb{R}^d$ such that for all $\alpha \in \mathcal{D}$ the equation

$$\langle \alpha \mid x \rangle_{\mathbb{R}^d} = 0$$

is true. Further, $\mathcal{D}$ is K-relatively dense, which means that $\mathcal{D} + K = \mathbb{R}^d$ for some compact $K \subseteq \mathbb{R}^d$. Let

$$diam_K := \sup_{k_1, k_2 \in K} \|k_1 - k_2\|_d < \infty$$

where $\|\cdot\|_d$ is the Euclidean norm in $\mathbb{R}^d$. Without loss of generality, we can suppose that the distance between x and $hyp(\mathcal{D})$ is greater than $diam_K$. Hence,

$$x \notin \mathcal{D} + K = \mathbb{R}^d$$

which leads to a contradiction.

Assume that $M < d$ then, there is an element $0 \neq x \in \mathbb{R}^d$ such that for any $b \in \mathbb{L}_{\mathcal{D}}$ the equation

$$\langle b \mid x \rangle = 0$$

holds. Thus, the Lagarias group is contained in a hyperplane in $\mathbb{R}^d$ which leads to a contradiction. □

Conclusion 3.7. *Let $\mathcal{D}$ be an D-set. For any $\tilde{\mathcal{D}} \in \Omega_{\mathcal{D}}$ the induced Lagarias group $\mathbb{L}_{\tilde{\mathcal{D}}}$ is equal to $\mathbb{L}_{\mathcal{D}}$.*

Proof. First recall the definition of a D-set. A D-set is an aperiodic, repetitive (U, K)-Delone set of finite local complexity. By Proposition 2.16 $\tilde{\mathcal{D}}$ is an aperiodic (U, K)-Delone set of finite local complexity. According to Proposition 2.14 we know that $(\tilde{\mathcal{D}} - \tilde{\mathcal{D}})$=$(\mathcal{D} - \mathcal{D})$ and so $[\tilde{\mathcal{D}} - \tilde{\mathcal{D}}] = [\mathcal{D} - \mathcal{D}]$. □

3.2 The dual group and $L^2(\widehat{\mathbb{L}_{\mathcal{D}}})$

In this section we need the concept of the dual group and the Haar measure. We start with a short introduction on these topics. Further information may be found in [DE] and [LOO].

Let $\mathcal{A}$ be a locally compact, abelian group. We call a continuous group homomorphism $\eta : \mathcal{A} \to \mathcal{S}_1 := \{z \in \mathbb{C} \mid |z| = 1\}$ a *character of* $\mathcal{A}$. Note that $\mathcal{S}_1$ is a compact, abelian group with respect to multiplication. Moreover, we denote by $\widehat{\mathcal{A}}$ the set of all characters of $\mathcal{A}$. The set $\widehat{\mathcal{A}}$ is again an abelian group with respect to the multiplication defined by

$$(\eta \cdot \psi)(x) = \eta(x) \cdot \psi(x), \qquad \eta, \psi \in \widehat{\mathcal{A}}, x \in \mathcal{A}.$$

The composition $\eta \cdot \psi$ is again a group homomorphism. This becomes clear by a short computation for $x, y \in A$

$$\begin{aligned}(\eta \cdot \psi)(x + y) =& \eta(x + y) \cdot \xi(x + y) = (\eta(x) \cdot \eta(y)) \cdot (\psi(x) \cdot \psi(y)) \\ =& (\eta(x) \cdot \psi(x)) \cdot (\eta(y) \cdot \psi(y)) = (\eta \cdot \psi)(x) \cdot (\eta \cdot \psi)(y).\end{aligned}$$

Further, the inverse element for $\eta \in \widehat{\mathcal{A}}$ is given by

$$\eta^{-1}(x) = \frac{1}{\eta(x)} = \overline{\eta(x)} = \eta(-x), \qquad\qquad x \in \mathcal{A}\,.$$

The neutral element of $\widehat{\mathcal{A}}$ is determined by the continuous group homomorphism ϕ which maps any element of $\mathcal{A}$ to 1. Consequently,

$$(\eta \cdot \phi)(x) = \eta(x) \cdot \underbrace{\phi(x)}_{=1} = \eta(x) = \underbrace{\phi(x)}_{=1} \cdot \eta(x) = (\phi \cdot \eta)(x).$$

By an easy computation we get that any character $\psi \in \widehat{\mathcal{A}}$ maps 0 to $1 \in \mathcal{S}_1$.
Consider some topological group $\mathcal{A}$ and the set of all contiunous functions $\mathcal{C}(\mathcal{A})$. The topology generated by the sets

$$O(V, K) := \{f \in \mathcal{C}(\mathcal{A}) \mid f(K) \subseteq V\}$$

for any compact set $K \subseteq \mathcal{A}$ and each open set $V \subseteq \mathbb{C}$ is called the *compact-open topology.* It turns out that $\mathcal{C}(\mathcal{A})$ is a topological Hausdorff space with the compact-open topology and $\widehat{\mathcal{A}}$ is a closed subset of $\mathcal{C}(\mathcal{A})$ with respect to this topology ([DE]).

Lemma 3.8 ([DE])**.** *The group $\widehat{\mathcal{A}}$ gets a topological Hausdorff group with the compact-open topology.*

Proposition 3.9 (Pontryagin Duality, [DE])**.** *The locally compact, abelian groups $\mathcal{A}$ and $\widehat{\widehat{\mathcal{A}}}$ are isomorphic.*

We call $\widehat{\mathcal{A}}$ the dual group of $\mathcal{A}$. The last Proposition 3.9 explains the notation dual group.

Lemma 3.10 ([DE])**.** *Let $\mathcal{A}$ be a locally compact, abelian group. Then, the following assertions are true.*

(i) The dual group $\widehat{\mathcal{A}}$ is a locally compact, abelian group and it is a Hausdorff space.

(ii) If $\mathcal{A}$ is discrete, then, $\widehat{\mathcal{A}}$ is compact.

(iii) For any $0 \neq x \in \mathcal{A}$ there exists a character $\psi \in \widehat{\mathcal{A}}$ such that $\psi(x) \neq 1$. In fact, the dual group $\widehat{\mathcal{A}}$ seperates the points of $\mathcal{A}$.

(iv) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are both locally compact abelian groups, then, $\widehat{\mathcal{A}_1 \times \mathcal{A}_2}$ is equal to $\widehat{\mathcal{A}_1} \times \widehat{\mathcal{A}_2}$.

The following part is inspired by [LOO]. A measure ϱ on a locally compact, abelian group $\mathcal{A}$ is called *left-invariant*, if for all $a \in \mathcal{A}$ and each Borel set $S \subseteq \mathcal{A}$ the equation

$$\varrho(a+S) = \varrho(S)$$

holds. In fact, it is shown that for any locally compact, abelian group $\mathcal{A}$ there exists a unique (up to a positive multiple), left-invariant measure ϱ.

Lemma 3.11 ([LOO])**.** *If $\mathcal{A}$ is a locally compact, abelian group. Then, $\mathcal{A}$ is compact, iff there is a left-invariant measure ϱ on $\mathcal{A}$ such that $\varrho(\mathcal{A}) < \infty$.*

By using the last Propostion 3.11 for any compact, abelian group $\mathcal{A}$ there exists a left-invariant measure ϱ such that $\varrho(\mathcal{A}) = 1$. This measure ϱ is called *Haar measure* of the compact, abelian group $\mathcal{A}$.

Lemma 3.12 ([LOO])**.** *Let $\mathcal{A}$ be a compact, abelian group and ϱ its corresponding Haar measure. Then,*

$$\int\limits_{\mathcal{A}} f(\tilde{k} \cdot k)\, d\varrho(k) = \int\limits_{\mathcal{A}} f(k)\, d\varrho(k)$$

for any $f \in L^1(\mathcal{A}, \varrho)$ and $\tilde{k} \in \mathcal{A}$.

Since we have endowed the abelian group $\mathbb{L}_{\mathcal{D}}$ with the discrete topology, it is locally compact. Thus, $\widehat{\mathbb{L}_{\mathcal{D}}}$ is well-defined. According to Lemma 3.11 $\widehat{\mathbb{L}_{\mathcal{D}}}$ is a Hausdorff, compact, abelian group. Moreover, $\widehat{\mathbb{L}_{\mathcal{D}}}$ is a locally compact, abelian group and it is a Hausdorff space. The measure ϱ denotes the corresponding Haar measure on $\widehat{\mathbb{L}_{\mathcal{D}}}$.

Lemma 3.13. *For any $0 \neq c \in \mathbb{L}_{\mathcal{D}}$ the equation*

$$\int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} f_c(k)\, d\varrho(k) = 0,$$

holds, where $f_c(k) := k(c)$ for any character $k \in \widehat{\mathbb{L}_{\mathcal{D}}}$.

Proof. According to Lemma 3.12 we get that for $\tilde{k} \in \widehat{\mathbb{L}_{\mathcal{D}}}$

$$\begin{aligned}
\int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} f_c(k)\, d\varrho(k) &= \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} k(c)\, d\varrho(k) \\
&\overset{\text{L. 3.12}}{=} \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} (\tilde{k} \circ k)(c)\, d\varrho(k) \\
&= \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} \tilde{k}(c) \cdot k(c)\, d\varrho(k) \\
&= \tilde{k}(c) \cdot \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} \underbrace{k(c)}_{=f_c(k)}\, d\varrho(k).
\end{aligned}$$

Hence,

$$\int_{\widehat{\mathbb{L}_{\mathcal{D}}}} f_c(k)\ d\varrho(k) \cdot (1 - \tilde{k}(c)) = 0.$$

This equation holds if and only if one of the factors is zero. Further, c is not the neutral element. Thus, by Lemma 3.10 we can find a character $\tilde{k}$ such that $\tilde{k}(c) \neq 1$. In this case, the term $(1 - \tilde{k}(c))$ is not equal to zero. Consequently,

$$\int_{\widehat{\mathbb{L}_{\mathcal{D}}}} f_c(k)\ d\varrho(k) = 0.$$

□

Note that we did not use a property of $\widehat{\mathbb{L}_{\mathcal{D}}}$. Indeed, this result is true for general locally compact, discrete, abelian groups.

Let $\langle u \mid v\rangle_{\mathbb{C}} = \bar{u} \cdot v$ be the scalar product in $\mathbb{C}$ for $u, v \in \mathbb{C}$. Further, the absolut value of a complex number is given by $|u|^2 = \langle u \mid u\rangle_{\mathbb{C}}$. Consider now the function space

$$\mathcal{L}^2(\widehat{\mathbb{L}_{\mathcal{D}}}) := \{f : \widehat{\mathbb{L}_{\mathcal{D}}} \to \mathbb{C} : f \text{ measurable}, \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} |f(k)|^2\ d\varrho(k) < \infty\}$$

endowed with the semiscalar product

$$\langle f \mid g\rangle := \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} \overline{f(k)} \cdot g(k)\ d\varrho(k).$$

Set $N := \{f \in \mathcal{L}^2(\widehat{\mathbb{L}_{\mathcal{D}}}) \mid \langle f \mid f\rangle = 0\}$. Then, the quotient space

$$L^2(\widehat{\mathbb{L}_{\mathcal{D}}}) := \mathcal{L}^2(\widehat{\mathbb{L}_{\mathcal{D}}}) / N$$

is a Hilbert space with scalar product

$$\langle f \mid g\rangle_{\widehat{\mathbb{L}_{\mathcal{D}}}} := \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} \overline{f(k)} \cdot g(k)\ d\varrho(k).$$

and the induced norm

$$\|f\|_{L^2(\widehat{\mathbb{L}_{\mathcal{D}}})} := \left(\int_{\widehat{\mathbb{L}_{\mathcal{D}}}} \overline{f(k)} \cdot f(k)\ d\varrho(k) \right)^{\frac{1}{2}}.$$

In order to prove the next proposition, we need the following Theorem of Stone-Weierstrass.

Proposition 3.14 (Theorem of Stone-Weierstrass,[QUE]). *Let X be a compact Hausdorff-space and S a subset of $\mathcal{C}(X)$ with the following conditions*

(SW1) For every $x \in X$ there is an $f \in S$ such that $f(x) \neq 0$.

(SW2) For each $x, y \in X$ with $x \neq y$ exists an $f \in S$ such that $f(x) \neq f(y)$. (S seperates points)

(SW3) For any $f \in S$ the complex conjugated element $\overline{f}$ is an element of S as well.

Then, the generated algebra of S is a dense set in $\mathcal{C}(X)$.

The following Proposition 3.15 says that the set of characters of a group forms an orthonormal basis of the L^2-space of the group.

Proposition 3.15 ([LOO]). *The family $\left(f_b : \widehat{\mathbb{L}_{\mathcal{D}}} \to \mathbb{C}\right)_{b \in \mathbb{L}_{\mathcal{D}}}$ defined by*

$$f_b(k) := k(b), \qquad\qquad k \in \widehat{\mathbb{L}_{\mathcal{D}}}$$

is an orthonormal basis of $L^2(\widehat{\mathbb{L}_{\mathcal{D}}})$.

Proof. Note that any map $f_b : \widehat{\mathbb{L}_{\mathcal{D}}} \to \mathbb{C}$ is by definition a continuous map. We will prove that the family $(f_b)_{b \in \mathbb{L}_{\mathcal{D}}}$ is an orthonormal system. We sketch the rest of the proof and omit the details. The interested reader may refer to [LOO].

"$b = c$": By using the fact that any character maps $0 \in \mathbb{L}_{\mathcal{D}}$ to $1 \in \mathcal{S}_1$ we get

$$\begin{aligned}
\langle f_b \mid f_c \rangle_{L^2(\widehat{\mathbb{L}_{\mathcal{D}}})} &= \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} \overline{f_b(k)} \cdot f_c(k) \, d\varrho(k) \\
&= \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} \overline{k(b)} \cdot k(c) \, d\varrho(k) \\
&= \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} k(-b) \cdot k(c) \, d\varrho(k) \\
&\overset{k \text{ homomorphism}}{=} \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} k(\underbrace{c - b}_{=0}) \, d\varrho(k) \\
&= \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} 1 \, d\varrho(k) = 1.
\end{aligned}$$

"$b \neq c$": According to Lemma 3.10 (iii) there exists a character $\tilde{k} \in \widehat{\mathbb{L}_{\mathcal{D}}}$ such that $\tilde{k}(c-b)$ is not equal to $1 \in \mathcal{S}_1$ and so

$$
\begin{aligned}
\langle f_b \mid f_c \rangle_{L^2(\widehat{\mathbb{L}_{\mathcal{D}}})} &\overset{\text{see above}}{=} \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} k(c-b)\, d\varrho(k) \\
&\overset{\varrho \text{ left-invariant}}{=} \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} (\tilde{k} \circ k)(c-b)\, d\varrho(k) \\
&= \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} \tilde{k}(c-b) \cdot k(c-b)\, d\varrho(k) \\
&= \tilde{k}(c-b) \cdot \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} \underbrace{k(c-b)}_{=\overline{f_b(k)} \cdot f_c(k)}\, d\varrho(k) \\
&= \tilde{k}(c-b) \cdot \langle f_b \mid f_c \rangle_{L^2(\widehat{\mathbb{L}_{\mathcal{D}}})}.
\end{aligned}
$$

Hence,

$$\langle f_b \mid f_c \rangle_{L^2(\widehat{\mathbb{L}_{\mathcal{D}}})} \cdot (1 - \tilde{k}(c-b)) = 0.$$

As in Lemma 3.13 it follows that $\langle f_b \mid f_c \rangle_{L^2(\widehat{\mathbb{L}_{\mathcal{D}}})}$ is equal to zero, since $\tilde{k}(c-b) \neq 1$.

To sum up, for any $b, c \in \mathbb{L}_{\mathcal{D}}$ the family $(f_b)_{b \in \mathbb{L}_{\mathcal{D}}}$ satisfies

$$\langle f_b \mid f_c \rangle_{L^2(\widehat{\mathbb{L}_{\mathcal{D}}})} = \begin{cases} 1, & b = c \\ 0, & b \neq c \end{cases},$$

which means that the system $(f_b)_{b \in \mathbb{L}_{\mathcal{D}}}$ is orthonormal.

Note that $\widehat{\mathbb{L}_{\mathcal{D}}}$ is a compact Hausdorff space, see Lemma 3.10 (i) and (ii). We would like to use the Theorem of Stone-Weierstrass (Proposition 3.14) for the subset

$$S := \{ f_b : \widehat{\mathbb{L}_{\mathcal{D}}} \to \mathbb{C} \mid b \in \mathbb{L}_{\mathcal{D}} \}.$$

In order to do so choose the function $f_0 \in S$. For any character $k \in \widehat{\mathbb{L}_{\mathcal{D}}}$ the equations

$$1 = k(0) = f_0(k)$$

hold. Consequently, the family $(f_b)_{b \in \mathbb{L}_{\mathcal{D}}}$ satisfies condition (SW1).

Let k_1, k_2 be two distinct elements of the dual Lagarias group $\widehat{\mathbb{L}_{\mathcal{D}}}$. In fact, there is at least one $b \in \mathbb{L}_{\mathcal{D}}$ such that $k_1(b) \neq k_2(b)$, because else they are equal. Consequently, for the continuous function $f_b \in S$ is true that

$$f_b(k_1) \neq f_b(k_2).$$

Thus, the set S seperates the points (SW2).

Consider some $b \in \mathbb{L}_{\mathcal{D}}$ and the corresponding function $f_b \in S$. Since $\mathbb{L}_{\mathcal{D}}$ is a group the element $-b$ is in $\mathbb{L}_{\mathcal{D}}$ and so $f_{-b} \in S$. Furthermore,

$$\overline{f_b(k)} = \overline{k(b)} = k(-b) = f_{-b}(k), \qquad k \in \widehat{\mathbb{L}_{\mathcal{D}}}\,.$$

Hence, the family S satisfies also the property (SW3), which allows us to use Proposition 3.14. $\square$

Lemma 3.16. *For $b, c \in \mathbb{L}_{\mathcal{D}}$ and each $k \in \widehat{\mathbb{L}_{\mathcal{D}}}$ the following two equalities*

(i) $f_{b-c}(k) = f_b(k) \cdot \overline{f_c(k)}$,

(ii) $\overline{f_b(k)} \cdot f_b(k) = 1$

are true.

Proof. As mentioned in the beginning of this section the equation $k(-c) = \overline{k(c)}$ holds. Thus,

$$f_{b-c}(k) = k(b + (-c)) \overset{\text{homomorphism}}{=} k(b) \cdot k(-c) = f_b(k) \cdot \overline{f_c(k)}.$$

Further,

$$\overline{f_b(k)} \cdot f_b(k) \overset{(i)}{=} f_{b-b}(k) = k(0) \overset{\text{homomorphism}}{=} 1.$$

$\square$

Our last aim of this chapter is to characterize the dual Lagarias group $\widehat{\mathbb{L}_{\mathcal{D}}}$. In particular, we know by Proposition 3.5 that $\mathbb{L}_{\mathcal{D}}$ is isomorphic to to $\mathbb{Z}^M$. Hence, we can identify our dual Lagarias group $\widehat{\mathbb{L}_{\mathcal{D}}}$ with the dual group of $\mathbb{Z}^M$.

The dual group $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$ is equal to

$$\{e^{-i \cdot \langle h \mid \cdot \rangle_{\mathbb{R}}} : \mathbb{Z} \to \mathcal{S}_1 \mid h \in [0, 2\pi)\}.$$

Further, by Lemma 3.10 (iv) we know that $\widehat{\mathbb{Z}^M}$ is equal to $\bigtimes_{n=1}^{M} \widehat{\mathbb{Z}}$.

Lemma 3.17. *Let $\mathbb{L}_{\mathcal{D}}$ be the Lagarias group of an aperiodic, Delone set $\mathcal{D}$ of finite local complexity. Then, there exists an isomorphism between $\widehat{\mathbb{L}_{\mathcal{D}}}$ and*

$$\{e^{-i \cdot \langle h \mid \cdot \rangle_{\mathbb{R}^M}} : \mathbb{Z}^M \to \mathcal{S}_1 \mid h \in [0, 2\pi)^M\}$$

where $\langle \cdot \mid \cdot \rangle_{\mathbb{R}^M}$ is the standard scalar product on $\mathbb{R}^M$ defined by

$$\langle x \mid y \rangle_{\mathbb{R}^M} = \sum_{j=1}^{M} x_j \cdot y_j$$

for any $x, y \in \mathbb{R}^M$.

Proof. The statement follows immediately by the previous considerations and Proposition 3.5. $\square$

3.3 Example

Recall the example of the last chapter. We have the pointset $\mathcal{D} \subsetneq \mathbb{R}$ defined by an infinite word ω which contains any finite word over $\{0, 1\}$. Then, $\mathcal{D}$ is an aperiodic Delone set of finite local complexity and

$$(\mathcal{D} - \mathcal{D}) = \{m \cdot 1 + n \cdot q \mid m, n \in \mathbb{N}\} \cup \{-m - n \cdot q \mid m, n \in \mathbb{N}\} \qquad (\star)$$

where $q \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]$ is a fixed irrational number. Recall the definition of the Lagarias group of $\mathcal{D}$, which is the group of all linear $\mathbb{Z}$-combinations of elements of $(\mathcal{D} - \mathcal{D})$. Since $\mathcal{D}$ is an aperiodic Delone set of finite local complexity, which contains the neutral element 0, the Lagarias group has to be finitely generated, see Proposition 3.4. It follows immediately that $\mathbb{L}_{\mathcal{D}}$ is generated by the set $\mathcal{B}_{(\mathcal{D}-\mathcal{D})} = \{1, q\}$, see $(\star)$. Indeed, the set $\mathcal{B}_{(\mathcal{D}-\mathcal{D})}$ is a minimal generator. Note that for example the set $\{1, -q\}$ is also a minimal generator.

We can identify the Lagarias group with $\mathbb{Z}^2$, see Proposition 3.5. Besides, we can see that $\mathcal{D}$ satisfies the dimension inequality of Lemma 3.6, since

$$M = 2 > 1 = \dim(\mathbb{R}).$$

Further, our dual Lagarias group is given by

$$\widehat{\mathbb{L}_{\mathcal{D}}} \cong \{e^{i \cdot \langle h \mid \cdot \rangle_{\mathbb{R}^2}} : \mathbb{Z}^2 \cong \mathbb{L}_{\mathcal{D}} \to \mathcal{S}_1 \mid h \in [0, 2\pi)^2\}.$$

Chapter 4

Groupoid

4.1 Groupoid on the tiling space $\mathcal{T}$

Let Γ^0 and Γ^1 be some sets and suppose we are given maps $r : \Gamma^1 \to \Gamma^0$ and $s : \Gamma^1 \to \Gamma^0$, called the *range map* and the *source map*, respectively. Define

$$\Gamma^2 := \{(\gamma_1, \gamma_2) \in \Gamma^1 \times \Gamma^1 \mid s(\gamma_1) = r(\gamma_2)\}.$$

We then say that $\Gamma = (\Gamma^0, \Gamma^1)$ with the composition map $\circ : \Gamma^2 \to \Gamma^1$ is a *groupoid*, if the following assertions hold.

(G1) For any $(\gamma_1, \gamma_2), (\gamma_2, \gamma_3) \in \Gamma^2$ the equation $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$ holds.

(G2) For each $x \in \Gamma^0$ there is an $e_x \in \Gamma^1$ with $r(e_x) = s(e_x) = x$ such that

$$e_x \circ \gamma = \gamma \text{ for each } \gamma \in \Gamma^1, \text{ where } r(\gamma) = x$$

and

$$\gamma \circ e_x = \gamma \text{ for each } \gamma \in \Gamma^1, \text{ where } s(\gamma) = x.$$

(G3) For all $\gamma \in \Gamma^1$ exists an $\eta \in \Gamma^1$ such that the following is true.

- ·) We have the equalities $r(\gamma) = s(\eta)$ and $s(\gamma) = r(\eta)$.
- ·) For any $\xi \in \Gamma^1_{\mathcal{T}}$ with $r(\gamma) = r(\xi)$ it satisfies $(\gamma \circ \eta) \circ \xi = \xi$.
- ·) For each $\xi \in \Gamma^1_{\mathcal{T}}$ with $s(\gamma) = r(\xi)$ it is true that $\xi \circ (\gamma \circ \eta) = \xi$.

The set Γ^0 can be thought of as a set of points and Γ^1 as a set of arrows between two points of Γ^0. The elements of Γ^1 are endowed with a direction, characterized by the source and the range map.

Lemma 4.1. *Let Γ be a groupoid. Then, the following statements are true.*

(i) For $x \in \Gamma^0$ the arrow e_x in (G2) is unique.

(ii) For $\gamma \in \Gamma^1$ the arrow η in (G3) is unique.

Proof.

(i) Let x be an arbitrary element of Γ^0. Assume we have two arrows $e_x^{(1)}, e_x^{(2)} \in \Gamma^1$ satisfying (G2). Thus, $(e_x^{(1)}, e_x^{(2)})$ and $(e_x^{(2)}, e_x^{(1)})$ are both elements of Γ^2 and so $e_x^{(1)} = e_x^{(1)} \circ e_x^{(2)} = e_x^{(2)}$.

(ii) Let γ be some element of Γ^1 and $\eta^{(1)}, \eta^{(2)} \in \Gamma^1$ satisfying (G3). More precisely, we have $\gamma \circ \eta^{(1)} = e_{r(\gamma)}$ and $\eta^{(2)} \circ \gamma = e_{s(\gamma)}$ where $e_{r(\gamma)}$ and $e_{s(\gamma)}$ are both unique by (i). According to condition (G3) we know that $r(\eta^{(1)}) = s(\gamma)$ and $r(\gamma) = s(\eta^{(2)})$. Hence,

$$\begin{aligned}\eta^{(1)} &\overset{(G2)}{=} e_{r(\eta^{(1)})} \circ \eta^{(1)} = e_{s(\gamma)} \circ \eta^{(1)} = (\eta^{(2)} \circ \gamma) \circ \eta^{(1)} \\ &\overset{(G1)}{=} \eta^{(2)} \circ (\gamma \circ \eta^{(1)}) = \eta^{(2)} \circ e_{r(\gamma)} = \eta^{(2)} \circ e_{s(\eta^{(2)})} \overset{(G2)}{=} \eta^{(2)}.\end{aligned}$$

□

Because of the uniqueness in (G2) and (G3) we call the arrow $e_x \in \Gamma^1$ the *unit* of $x \in \Gamma^0$ and the arrow $\eta \in \Gamma^1$ in (G3) the *inverse element* of $\gamma \in \Gamma^1$, respectively. We denote the inverse element of $\gamma \in \Gamma^1$ by $\gamma^{-1} \in \Gamma^1$.

Moreover, because of Lemma 4.1 (i) we can consider Γ^0 as a subset of Γ^1, represented by the set of the units. Hence, we can identify the groupoid Γ with the set Γ^1.

Consider a groupoid $\Gamma = (\Gamma^0, \Gamma^1)$ where Γ^0 and Γ^1 are topological spaces. Then, the groupoid Γ is called *topological* if the range map 'r', source map 's', the composition map '$\circ$' and the map which maps γ to its inverse element γ^{-1} are continuous.

Lemma 4.2. *Let $\mathcal{D} \subseteq G$ be an aperiodic Delone set of finite local complexity and $\mathcal{T}$ the corresponding transversal. We set $\Gamma^0_{\mathcal{T}} := \mathcal{T}$ and*

$$\Gamma^1_{\mathcal{T}} := \{(t, b) \in \mathcal{T} \times G \mid \tau^{-b} t \in \mathcal{T}\}$$

where τ^{-b} is the continuous associated shift of a measure, see section 2.3. Moreover, the range map is given by $r((t, b)) := t$ and the source map is given by $s((t, b)) := \tau^{-b} t$ for all $(t, b) \in \Gamma^1_{\mathcal{T}}$. Then,

$$\Gamma^2_{\mathcal{T}} = \{((t_1, b_1), (t_2, b_2)) \in \Gamma^1_{\mathcal{T}} \times \Gamma^1_{\mathcal{T}} \mid \tau^{-b_1} t_1 = t_2\}.$$

Furthermore, the composition map $\circ : \Gamma^2_{\mathcal{T}} \to \Gamma^1_{\mathcal{T}}$ *is defined by*

$$(t_1, b_1) \circ (t_2, b_2) := (t_1, b_1 + b_2),$$

for each $((t_1, b_1), (t_2, b_2)) \in \Gamma^2_{\mathcal{T}}$. *The inverse element of* $(t, b) \in \Gamma^1_{\mathcal{T}}$ *is given by*

$$(t, b)^{-1} = (\tau^{-b} t, -b).$$

Then, $\Gamma_{\mathcal{T}} = (\Gamma^0_{\mathcal{T}}, \Gamma^1_{\mathcal{T}})$ *is a continuous groupoid.*

Proof. By definition of $\Gamma^1_{\mathcal{T}}$ the image of the range and source map is $\mathcal{T}$ and so they are well-defined. The characterization of $\Gamma^2_{\mathcal{T}}$ follows immediately by the definition of the range and source map.

"(G1)" Let $((t_1, b_1), (t_2, b_2))$ and $((t_2, b_2), (t_3, b_3))$ be some elements of $\Gamma^2_{\mathcal{T}}$. Hence, by the characterization of $\Gamma^2_{\mathcal{T}}$ we have that $\tau^{-b_1} t_1 = t_2$ and $\tau^{-(b_1+b_2)} t_1 = \tau^{-b_2} t_2 = t_3$. Consequently,

$$\begin{aligned}
[(t_1, b_1) \circ ((t_2, b_2)] \circ (t_3, b_3) &= \left[(t_1, b_1) \circ ((\tau^{-b_1} t_1, b_2)\right] \circ (\tau^{-(b_1+b_2)} t_1, b_3) \\
= (t_1, b_1 + b_2) \circ (\tau^{-(b_1+b_2)} t_1, b_3) &= (t_1, b_1 + b_2 + b_3) \\
= (t_1, b_1) \circ (\tau^{-b_1} t_1, b_2 + b_3) &= (t_1, b_1) \circ \left[(\tau^{-b_1} t_1, b_2) \circ (\tau^{-(b_1+b_2)} t_1, b_3)\right] \\
&= (t_1, b_1) \circ [(t_2, b_2) \circ (t_3, b_3)] .
\end{aligned}$$

"(G2)" Let t be some element of $\Gamma^0_{\mathcal{T}}$. Then, $e_t := (t, 0)$ is an element of $\Gamma^1_{\mathcal{T}}$. Further,

$$r(e_t) = r((t, 0)) = t = \tau^{-0} t = s((t, 0)) = s(e_t).$$

Any $(t, b) \in \Gamma^1_{\mathcal{T}}$ has the range t and any $(\tau^b t, b) \in \Gamma^1_{\mathcal{T}}$ has the source t. Moreover, the following equations are true

$$\begin{aligned}
e_t \circ (t, b) = (t, 0) \circ (\tau^{-0} t, b) = (t, 0 + b) &= (t, b), && \text{for all } (t, b) \in \Gamma^1_{\mathcal{T}}, \\
(\tau^b t, b) \circ e_t = (\tau^b t, b) \circ (t, 0) = (\tau^b t, b + 0) &= (\tau^b t, b), && \text{for all } (\tau^b t, b) \in \Gamma^1_{\mathcal{T}} .
\end{aligned}$$

Thus, $e_t := (t, 0) \in \Gamma^1_{\mathcal{T}}$ is the unit of $t \in \mathcal{T} = \Gamma^0_{\mathcal{T}}$.

"(G3)" Let (t, b) be some element of $\Gamma^1_{\mathcal{T}}$. Obviously, the pair $(\tau^{-b} t, -b)$ is an element of $\Gamma^1_{\mathcal{T}}$. By definition

$$\begin{aligned}
s((t, b)) &= \tau^{-b} t = r((\tau^{-b} t, -b)), \\
r((t, b)) &= t = \tau^b (\tau^{-b} t) = s((\tau^{-b} t, -b)).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
(\tau^{-b}t, -b) \circ (t, b) = \ & (\tau^{-b}t, b - b) = \ (\tau^{-b}t, 0) \overset{\text{(G2)}}{=} \ e_{s((t,b))}, \\
(t, b) \circ (\tau^{-b}t, -b) = \ & (t, b - b) \quad = \ (t, 0) \quad \overset{\text{(G2)}}{=} \ e_{r((t,b))}.
\end{aligned}$$

Hence, $(\tau^{-b}t, -b)$ is the inverse element of (t, b).

Now we still have to show that the groupoid $\Gamma_{\mathcal{T}}$ is topological. Consider the set

$$X = \{x \in G \mid \exists t \in \mathcal{T} \text{ such that } \tau^{-x}t \in \mathcal{T}\} \subseteq G.$$

Assume X is not discrete, which means for some $b \in X$ that any open neighborhood V of b contains a different element $c \neq b$ of X. Choose c such that $c - b$ is an element of U. By definition of X there are a $t_1 \in \mathcal{T}$ and $t_2 \in \mathcal{T}$ such that $\tau^{-b}t_1 \in \mathcal{T}$ and $\tau^{-c}t_2 \in \mathcal{T}$. Thus, $\tau^{-b}t_1(\{0\})$ and $\tau^{-c}t_2(\{0\})$ are equal to one which means that both are equal. Since $\mathcal{D}_{t_1}$ is U-uniformly discrete and $c - b \neq 0$ it follows that $t_1(\{c - b\}) = 0$, see Proposition 2.16. Hence, we get

$$1 \overset{t_2 \in \mathcal{T}}{=} t_2(\{0\}) = \tau^c(\tau^{-c}t_2(\{0\})) = \tau^c(\tau^{-b}t_1(\{0\})) = t_1(\{c - b\}) = 0,$$

which is a contradiction. Thus, the set X is discrete and so its induced topology is the discrete topology.

The set $\Gamma^1_{\mathcal{T}}$ is in a natural way endowed with the product topology and also $\Gamma^2_{\mathcal{T}}$. Hence, $\Gamma^0_{\mathcal{T}}$ and $\Gamma^1_{\mathcal{T}}$ are both topological spaces. Now we have to check if r,s,$\circ$ and the inverse map $^{-1}$ are continuous. We will prove that $\circ$ is continuous. The other proofs work in a similar way.

Take some $((t_1, b_1), (t_2, b_2)) \in \Gamma^2_{\mathcal{T}}$ and an open neighborhood V of $(t_1, b_1) \circ (t_2, b_2) = (t_1, b_1 + b_2)$. By definition of the topology on $\Gamma^1_{\mathcal{T}}$, there are open sets $V_1, V_2 \subseteq \Gamma^1_{\mathcal{T}}$ such that $V_1 \times V_2 \subseteq V$ and $(t_1, b_1) \circ (t_2, b_2) \in V_1 \times V_2$. Since the shift map τ^b and its inverse map τ^{-b} are continuous, the set $\tau^{-b}(V_1)$ is an open neighborhood of t_2 and $\tau^b(\tau^{-b}(V_1)) = V_1$. Furthermore, $\{b_1\}$ and $\{b_2\}$ are open sets with respect to the induced topology on X. Altogether, the set $(V_1 \times \{b_1\}) \times ((\tau^{-b}V_1) \times \{b_2\}) \subseteq \Gamma^2_{\mathcal{T}}$ is an open neighborhood of $((t_1, b_1), (t_2, b_2)) \in \Gamma^2_{\mathcal{T}}$ and

$$(V_1 \times \{b_1\}) \circ ((\tau^{-b}V_1) \times \{b_2\}) = V_1 \times \{b_1 + b_2\} \subseteq V_1 \times V_2 = V,$$

which implies that the composition map $\circ$ is continuous. $\square$

4.2 Continuous fields of Hilbert spaces

The following notions are motivated by [DIXC] and [BOS]. Consider a family $(A_i)_{i\in\mathcal{I}}$ of subsets of G. Then, the product of these sets is defined by

$$\prod_{i\in\mathcal{I}} A_i := \{a : \mathcal{I} \to \bigsqcup_{i\in\mathcal{I}} A_i \mid a_i \in A_i \text{ for all } i \in \mathcal{I}\}$$

where $\bigsqcup$ denotes the disjoint union of these sets. Actually the disjoint union $\bigsqcup\limits_{i\in\mathcal{I}} A_i$ is equal to $\{(i,a) \mid i \in I, a \in A_i\}$. For the sake of convenience we denote such an element by a_i.

Let Σ be a locally compact space and consider the family $(\mathcal{H}(\sigma))_{\sigma\in\Sigma}$ of complex separable Hilbert spaces with index set Σ. For $\sigma \in \Sigma$ the scalar product in $\mathcal{H}(\sigma)$ is denoted by $\langle\cdot \mid \cdot\rangle_\sigma$. We write $\|\cdot\|_\sigma$ for the induced norm of the scalar product on $\mathcal{H}(\sigma)$. The elements of the set

$$\mathcal{V}(\Sigma, \mathcal{H}(.)) := \prod_{\sigma\in\Sigma} \mathcal{H}(\sigma)$$

are called *vector fields*. Furthermore, the set

$$\mathfrak{O}(\Sigma, B(\mathcal{H}(.))) := \prod_{\sigma\in\Sigma} B(\mathcal{H}(\sigma))$$

is called the set of *operator fields* on $\mathcal{H}(\sigma)$, where for any $\sigma \in \Sigma$ the set $B(\mathcal{H}(\sigma))$ denotes the set of all bounded and linear operators on the Hilbert space $\mathcal{H}(\sigma)$.

Consider a linear subspace $\mathcal{S} \subseteq \mathcal{V}(\Sigma, \mathcal{H}(.))$. The pair $((\mathcal{H}(\sigma))_{\sigma\in\Sigma}, \mathcal{S})$ is called a *continuous field of Hilbert spaces* if the following conditions are hold.

(H1) For any $\sigma \in \Sigma$ the set $\mathcal{S}_\sigma := \{v_\sigma \mid v \in \mathcal{S}\}$ is a dense subset in $\mathcal{H}(\sigma)$.

(H2) For all $v \in \mathcal{S}$ the map $\Phi_v : \Sigma \to \mathbb{R}$ defined by

$$\sigma \mapsto \|v_\sigma\|_\sigma$$

is continuous.

(H3) Let $v \in \mathcal{V}(\Sigma, \mathcal{H}(.))$ be a vector field. If for any $\sigma \in \Sigma$ and $\varepsilon > 0$ there is a $w \in \mathcal{S}$ and a neighborhood of σ such that $\|v_{\sigma'} - w_{\sigma'}\|_{\sigma'} \leq \varepsilon$ is true, then, v is an element of $\mathcal{S}$.

The set $\mathcal{S}$ is called a *generator* and we say any element of $\mathcal{S}$ is a *continuous vector field* or *section*.

Proposition 4.3 ([DIXC]). *Consider a locally compact space Σ and a family $(\mathcal{H}(\sigma))_{\sigma\in\Sigma}$ of complex seperable Hilbert spaces with index set Σ. If there is a linear subset Λ of $\mathcal{V}(\Sigma, \mathcal{H}(.))$ satisfying (H1) and (H2), then, the family $(\mathcal{H}(\sigma))_{\sigma\in\Sigma}$ can be uniquely endowed with a structure of a continuous field of Hilbert spaces.*

A subset of a Hilbert space is called *total* if the subspace of all finite linear combinations is a dense subset of the Hilbert space. According to that we call a subset $\Lambda \subseteq \mathcal{V}(\Sigma, \mathcal{H}(.))$ *total* if for each $\sigma \in \Sigma$ the set

$$\Lambda_\sigma := \{v_\sigma \mid v \in \Lambda\}$$

is total in $\mathcal{H}(\sigma)$.

Proposition 4.4 ([DIXC]). *Consider a continuous field of Hilbert spaces $((\mathcal{H}(\sigma))_{\sigma\in\Sigma}, \mathcal{S})$ over a locally compact space Σ and Λ a total subset of $\mathcal{S}$. For any $v \in \mathcal{S}$, $K \subseteq \Sigma$ compact and $\varepsilon > 0$ there are functions $\varphi_1, \ldots, \varphi_n \in \mathcal{C}_c(\Sigma)$ and $w^{(1)}, \ldots w^{(n)} \in \Lambda$ such that*

$$\|v_\sigma - (\sum_{j=1}^{n} \varphi_j(\sigma) \cdot w_\sigma^{(j)})\| \leq \varepsilon$$

holds for all $\sigma \in K$.

Denote the disjoint union $\bigsqcup_{\sigma\in\Sigma} \mathcal{H}(\sigma)$ by $\mathcal{H}$. As mentioned above any element h of $\mathcal{H}$ has the form (σ, h') where h' is an element of the Hilbert space $\mathcal{H}(\sigma)$. Now consider the projection $pr : \mathcal{H} \to \Sigma$ defined by

$$pr(h) := pr((\sigma, h')) := \sigma \in \Sigma.$$

For $\varepsilon > 0$, an open set $W \subseteq \Sigma$ and $v \in \mathcal{S}$ we define the set

$$U(\varepsilon, v, W) := \{h := (\sigma, h') \in \mathcal{H} \mid pr(h) \in W \text{ and } \|h' - v_{pr(h)}\|_\sigma < \varepsilon\}.$$

These sets forms a basis for a topology $\mathcal{O}(\mathcal{H})$ on $\mathcal{H}$.

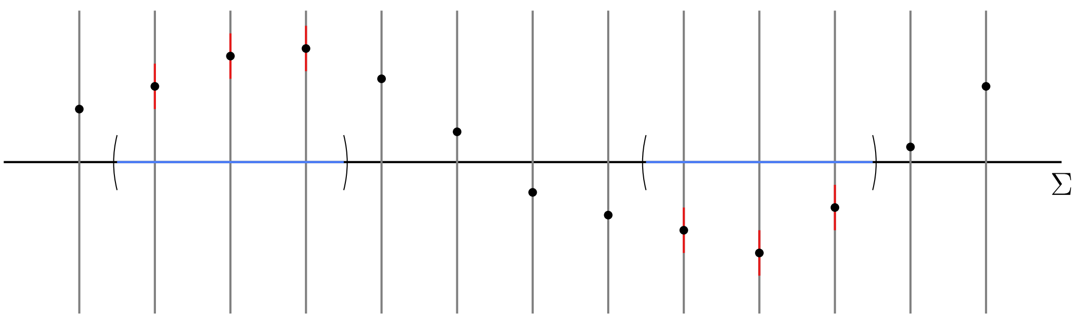

Figure 10:
An open set in $\mathcal{H}$

An open set in $\mathcal{H}$ is sketched in Figure 10. The blue lines describes an open set $W \subseteq \Sigma$ and any gray line represents a Hilbert space $\mathcal{H}(\sigma)$ located at $\sigma \in \Sigma$. The section $v \in \mathcal{S}$ is represented by the black points and the open set $U(\varepsilon, v, W)$ is drawn with the red lines.

Then, the set $\mathcal{C}(\Sigma, \mathcal{H})$ denotes the set of all *continuous functions in* $\prod_{\sigma\in\Sigma} \mathcal{H}(\sigma)$.

Lemma 4.5 ([DIXC]). *Consider a continuous field of Hilbert spaces* $((\mathcal{H}(\sigma))_{\sigma\in\Sigma}, \mathcal{S})$ *over* Σ. *For each* $\sigma \in \Sigma$ *and each* $x \in \mathcal{H}(\sigma)$ *there is a section* $v \in \mathcal{S}$ *such that*

$$v_\sigma = x.$$

4.3 Representation of groupoids

Consider a locally compact space Σ and a topological groupoid $\Gamma_\Sigma = (\Sigma, \Gamma^1_\Sigma)$ with unit space Σ. Let $(\mathcal{H}(\sigma))_{\sigma\in\Sigma}$ be a continuous field of Hilbert spaces and $\mathcal{S}$ be its generator. We denote $\bigsqcup_{\sigma\in\Sigma} \mathcal{H}(\sigma)$ by $\mathcal{H}$ endowed with the topology $\mathcal{O}(\mathcal{H})$ as above. In this section we follow [BOS].

Let $\big(A(\gamma) : \mathcal{H}_{s(\gamma)} \to \mathcal{H}_{r(\gamma)}\big)_{\gamma\in\Gamma_\Sigma}$ be a family of unitary invertible bounded operators. The triple $((\mathcal{H}(\sigma))_{\sigma\in\Sigma}, \mathcal{S}, (A(\gamma))_{\gamma\in\Gamma_\Sigma})$ is called a *unitary representation* if the following assertions hold.

(R1) For any $\sigma \in \Sigma$ and its corresponding unit $e_\sigma \in \Gamma^1_\Sigma$ the equation $A(e_\sigma) = id_{\mathcal{H}(\sigma)}$ is true.

(R2) For all $(\gamma_1, \gamma_2) \in \Gamma^2_\Sigma$ the equation $A(\gamma_1 \circ \gamma_2) = A(\gamma_1) \circ A(\gamma_2)$ holds.

(R3) For each $\gamma \in \Gamma_\Sigma$ it is true that $A(\gamma^{-1}) = A(\gamma)^{-1}$.

Note that for any unitary representation the map $\gamma \mapsto \|A(\gamma)\|$ is locally bounded, because for $\gamma \in \Gamma_\Sigma$ the operator $A(\gamma)$ is unitary and hence, it has norm one. A unitary representation $((\mathcal{H}(\sigma))_{\sigma\in\Sigma}, \mathcal{S}, (A(\gamma))_{\gamma\in\Gamma_\Sigma})$ is called

·) *weakly continuous* if the map $\Psi : \Gamma^1_\Sigma \to \mathbb{C}$ defined by

$$\gamma \overset{\Psi}{\mapsto} \langle u_{r(\gamma)} \mid A(\gamma) v_{s(\gamma)} \rangle$$

is continuous for all $u, v \in \mathcal{S}$.

·) *strongly continuous* if the map $\Phi : \Gamma^1_\Sigma \to \bigsqcup_{\sigma\in\Sigma} \mathcal{H}(\sigma)$ defined by

$$\gamma \overset{\Phi}{\mapsto} A(\gamma) v_{s(\gamma)}$$

is continuous for each $v \in \mathcal{S}$.

·) *continuous* if the map $\Upsilon : \Gamma^1_\Sigma \times \bigsqcup_{\sigma\in\Sigma} \mathcal{H}(\sigma) \to \bigsqcup_{\sigma\in\Sigma} \mathcal{H}(\sigma)$ defined by

$$(\gamma, h) \overset{\Upsilon}{\mapsto} A(\gamma)h$$

is continuous.

Proposition 4.6 ([BOS]). *Let $((\mathcal{H}(\sigma))_{\sigma\in\Sigma}, \mathcal{S}, (A(\gamma))_{\gamma\in\Gamma_\Sigma})$ be a weakly continuous, unitary representation. Then, this representation is strongly continuous.*

Proof. Let $v \in \mathcal{S}$ be arbitrary. Consider for some $\varepsilon > 0$ and an open set $W \subseteq \Sigma$ the neighborhood $\mathcal{N}(\varepsilon, v, W) \subseteq \mathcal{H}$ of $A(\gamma)v_{s(\gamma)}$ for a given $\gamma \in \Gamma^1_\Sigma$ such that for $u \in \mathcal{N}(\varepsilon, v, W)$ the equation

$$u_{r(\gamma)} = A(\gamma)v_{s(\gamma)} \qquad (\star)$$

holds. Because the representation is weakly continuous there is a neighborhood $\mathcal{K}_\gamma \subset \Gamma^1_\Sigma$ of γ such that for all $\eta \in \mathcal{K}_\gamma$ it is true that

$$\left|\langle u_{r(\eta)} \mid A(\eta)v_{s(\eta)}\rangle_{r(\eta)} - \langle u_{r(\gamma)} \mid A(\gamma)v_{s(\gamma)}\rangle_{r(\gamma)}\right| < \frac{\varepsilon}{4}. \qquad (\star\star)$$

Furthermore, $r(.)$ is a continuous map and so for $u \in \mathcal{N}(\varepsilon, v, W)$ we can find an open neighborhood $\mathcal{K}'_\gamma \subset \mathcal{K}_\gamma$ of γ such that for any $\gamma' \in \mathcal{K}'_\gamma$ the inequality

$$\left|\langle u_{r(\gamma')} \mid u_{r(\gamma')}\rangle_{r(\gamma')} - \langle u_{r(\gamma)} \mid u_{r(\gamma)}\rangle_{r(\gamma)}\right| < \frac{\varepsilon}{4} \qquad (\star\star\star)$$

holds. Since $A(\gamma)$ is unitary for all $\gamma \in \Gamma^1_\Sigma$ we have

$$\langle A(\gamma)v_{s(\gamma)} \mid A(\gamma)v_{s(\gamma)}\rangle_{r(\gamma)} = \langle v_{s(\gamma)} \mid v_{s(\gamma)}\rangle_{s(\gamma)}.$$

By using the triangle inequality we get for all $\gamma' \in \mathcal{K}'_\gamma$ that

$$\begin{aligned}
&\|u_{r(\gamma')} - A(\gamma')v_{s(\gamma')}\|^2_{r(\gamma')} \\
= &\left|\langle u_{r(\gamma')} - A(\gamma')v_{s(\gamma')} \mid u_{r(\gamma')} - A(\gamma')v_{s(\gamma')}\rangle_{r(\gamma')}\right| \\
\le &\left|\langle u_{r(\gamma')} \mid u_{r(\gamma')}\rangle_{r(\gamma')} - \langle u_{r(\gamma')} \mid A(\gamma')v_{s(\gamma')}\rangle_{r(\gamma')}\right| \\
&+ \left|\underbrace{\langle A(\gamma')v_{s(\gamma')} \mid A(\gamma')v_{s(\gamma')}\rangle_{r(\gamma')}}_{=\langle v_{s(\gamma')} \mid v_{s(\gamma')}\rangle_{s(\gamma')}} - \langle A(\gamma')v_{s(\gamma')} \mid u_{r(\gamma')}\rangle_{r(\gamma')}\right| \\
= &\left|\langle u_{r(\gamma')} \mid u_{r(\gamma')}\rangle_{r(\gamma')} - \langle u_{r(\gamma')} \mid A(\gamma')v_{s(\gamma')}\rangle_{r(\gamma')}\right| + \left|\langle v_{s(\gamma')} \mid v_{s(\gamma')}\rangle_{s(\gamma')} - \langle A(\gamma')v_{s(\gamma')} \mid u_{r(\gamma')}\rangle_{r(\gamma')}\right| \\
< &\varepsilon.
\end{aligned}$$

The last inequality follows by the following two computations

$$
\begin{aligned}
&\left|\langle u_{r(\gamma')} \mid u_{r(\gamma')}\rangle_{r(\gamma')} - \langle u_{r(\gamma')} \mid A(\gamma')v_{s(\gamma')}\rangle_{r(\gamma')}\right| \\
\leq\ &\underbrace{\left|\langle u_{r(\gamma')} \mid u_{r(\gamma')}\rangle_{r(\gamma')} - \langle u_{r(\gamma)} \mid u_{r(\gamma)}\rangle_{r(\gamma)}\right|}_{<\frac{\varepsilon}{4} \text{ because of } (\star\star\star)} + \left|\langle u_{r(\gamma)} \mid u_{r(\gamma)}\rangle_{r(\gamma)} - \langle u_{r(\gamma')} \mid A(\gamma')v_{s(\gamma')}\rangle_{r(\gamma')}\right| \\
\overset{(\star)}{<}\ &\frac{\varepsilon}{4} + \underbrace{\left|\langle u_{r(\gamma)} \mid A(\gamma)v_{s(\gamma)}\rangle_{r(\gamma)} - \langle u_{r(\gamma')} \mid A(\gamma')v_{s(\gamma')}\rangle_{r(\gamma')}\right|}_{<\frac{\varepsilon}{4} \text{ because of } (\star\star)} \\
<\ &\frac{\varepsilon}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\langle v_{s(\gamma')} \mid v_{s(\gamma')}\rangle_{s(\gamma')} - \langle A(\gamma')v_{s(\gamma')} \mid u_{r(\gamma')}\rangle_{r(\gamma')}\right| \\
\leq\ &\left|\langle v_{s(\gamma')} \mid v_{s(\gamma')}\rangle_{s(\gamma')} - \langle A(\gamma)v_{s(\gamma)} \mid u_{r(\gamma)}\rangle_{r(\gamma)}\right| + \left|\langle A(\gamma)v_{s(\gamma)} \mid u_{r(\gamma)}\rangle_{r(\gamma)} - \langle A(\gamma')v_{s(\gamma')} \mid u_{r(\gamma')}\rangle_{r(\gamma')}\right| \\
\overset{(\star)}{=}\ &\left|\langle v_{s(\gamma')} \mid v_{s(\gamma')}\rangle_{s(\gamma')} - \underbrace{\langle A(\gamma)v_{s(\gamma)} \mid A(\gamma)v_{s(\gamma)}\rangle_{r(\gamma)}}_{=\langle v_{s(\gamma)} \mid v_{s(\gamma)}\rangle_{s(\gamma)}}\right| \\
&+ \underbrace{\left|\langle A(\gamma)v_{s(\gamma)} \mid u_{r(\gamma)}\rangle_{r(\gamma)} - \langle A(\gamma')v_{s(\gamma')} \mid u_{r(\gamma')}\rangle_{r(\gamma')}\right|}_{<\frac{\varepsilon}{4} \text{ because of } (\star\star)} \\
<\ &\underbrace{\left|\langle v_{s(\gamma')} \mid v_{s(\gamma')}\rangle_{s(\gamma')} - \langle v_{s(\gamma)} \mid v_{s(\gamma)}\rangle_{s(\gamma)}\right|}_{<\frac{\varepsilon}{4} \text{ because of } (\star\star\star)} + \frac{\varepsilon}{4} \\
<\ &\frac{\varepsilon}{2},
\end{aligned}
$$

which leads to the statement. □

Proposition 4.7 ([BOS]). *If* $((\mathcal{H}(\sigma))_{\sigma\in\Sigma}, \mathcal{S}, (A(\gamma))_{\gamma\in\Gamma_\Sigma})$ *is a strongly continuous, unitary representation, then, this representation is continuous.*

Proof. Let $(\gamma, h) \in \Gamma^1_\Sigma \times \mathcal{H}$ be arbitrary. Consider some open neighborhood $\mathcal{N}(\varepsilon, v, W)$ in $\mathcal{H}$ of $A(\gamma)h$ such that the equation

$$v_{r(\gamma)} = A(\gamma)h$$

holds. Choose $w \in \mathcal{S}$ such that $w_{s(\gamma)}$ is equal to h. Since the representation is strongly continuous there is an open neighborhood $W_\gamma \subseteq \Gamma^1_\Sigma$ of γ such that for any $\gamma' \in W_\gamma$ we have the inequality

$$\|v_{r(\gamma')} - A(\gamma')w_{s(\gamma')}\|_{r(\gamma')} < \frac{\varepsilon}{2}. \qquad (\star)$$

The set

$$U_{\gamma,h} := \{(\gamma', h') \in \Gamma^1_\Sigma \times \mathcal{H} \mid \|h' - w_{s(\gamma')}\|_{s(\gamma')} < \frac{\varepsilon}{2},\ \gamma' \in W_\gamma\}$$

is obviously an open neighborhood of (γ, h), since for any $\varepsilon > 0$ we have

$$\|h - w_{s(\gamma)}\|_{s(\gamma)} = \|h - h\|_{s(\gamma)} = 0 < \frac{\varepsilon}{2}.$$

Thus, for any $(\gamma', h') \in U_{\gamma,h}$

$$\begin{aligned}
&\|v_{r(\gamma')} - A(\gamma')h'\|_{r(\gamma')} \\
\leq\ & \underbrace{\|v_{r(\gamma')} - A(\gamma')w_{s(\gamma')}\|_{r(\gamma')}}_{<\frac{\varepsilon}{2} \text{ because of } (\star)} + \|A(\gamma')w_{s(\gamma')} - A(\gamma')h'\|_{r(\gamma')} \\
<\ & \frac{\varepsilon}{2} + \underbrace{\|A(\gamma')\|}_{=1 \text{ since } A \text{ is unitary}} \cdot \underbrace{\|w_{s(\gamma')} - h'\|_{s(\gamma')}}_{<\frac{\varepsilon}{2} \text{ since } (\gamma',h') \in U_{\gamma,h}} \\
<\ & \varepsilon.
\end{aligned}$$

□

The last two propositions imply that a weakly continuous, unitary representation $((\mathcal{H}(\sigma))_{\sigma\in\Sigma}, \mathcal{S}, (A(\gamma))_{\gamma\in\Gamma_\Sigma})$ is continuous.

Chapter 5

Proto cells

From now on we consider the locally compact abelian group $(\mathbb{R}^d, +)$ endowed with the standard topology. The space $\mathbb{R}^d$ is σ-compact, because it can be written as the union of the family $([-n, n]^d)_{n\in\mathbb{N}}$ of compact subsets. Moreover, $\mathbb{R}^d$ is Hausdorff and second-countable. Thus, we can use all our previous results. Let $\mathcal{D}$ denote our D-set of $\mathbb{R}^d$.

Consider some real, positive numbers R_0 and R_1. We say $\mathcal{D}$ is R_0-*uniformly discrete*, if for each $x \in \mathbb{R}^d$ the interesection $\mathcal{D} \cap \mathcal{B}_{R_0}(x)$ contains at most one point. Further, the set $\mathcal{D}$ is called R_1-*relatively dense*, if the equation $\mathcal{D} + \overline{\mathcal{B}}_{R_1}(0) = \mathbb{R}^d$ holds. We immediately see that these definitions are special cases of the definitions of U-uniformly discrete and K-relatively dense. We call $\mathcal{D}$ an (R_0, R_1)-Delone set, if $\mathcal{D}$ is R_0-uniformly discrete and R_1-relatively dense. If we speak in the following about Delone sets, we actually mean (R_0, R_1)-Delone sets. Note that we suppose that R_0 is chosen to be maximal, i.e. as large as possible, and R_1 is chosen to be minimal. They satisfy the inequality $R_0 \leq R_1$.

In this chapter our main aim is to tile the space $\mathbb{R}^d$ by a finite number of different cells. In order to do so we will define the Voronoi cells of the set $\mathcal{D}$. By using them, we will construct the set of all collared Voronoi proto cells $\mathcal{P}$. Then, we will show that this set is finite and that the cells of dimension d tile our space. The following considerations are inspired by [BNM].

First, we give a short introduction to the concept of hyperplanes and polytopes. A *hyperplane* with respect to $u \in \mathbb{R}^d$ and $c \in \mathbb{R}$ is defined by

$$H := \{x \in \mathbb{R}^d \mid \langle u \mid x\rangle_{\mathbb{R}^d} = c\}.$$

Such a hyperplane splits the space $\mathbb{R}^d$ in two half-spaces

$$H_+ := \{x \in \mathbb{R}^d\} \mid \langle u \mid x\rangle_{\mathbb{R}^d} \geq c\}$$

and

$$H_- := \{x \in \mathbb{R}^d\} \mid \langle u \mid x\rangle_{\mathbb{R}^d} \leq c\}.$$

Let C be a convex and closed subset of $\mathbb{R}^d$ and $H \subseteq \mathbb{R}^d$ be some hyperplane. If further, the intersection $C \cap H$ is not empty and C is a subset of H_-, we call H a *supporting hyperplane* of the set C. The intersection $C \cap H$ is called *face* of $\mathcal{C}$, see Figure 11.

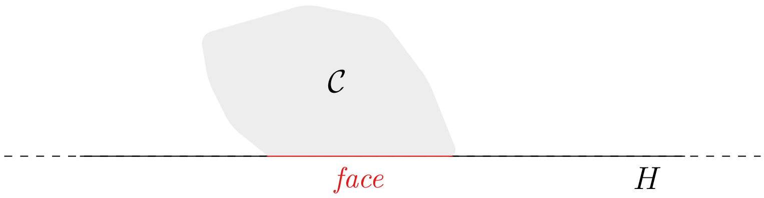

Figure 11:
An illustration of the face and a corresponding supporting hyperplane of a convex, closed subset of $\mathbb{R}^d$

Lemma 5.1. *Consider a closed and convex set $C \subseteq \mathbb{R}^d$ and a supporting hyperplane H of it. The face $F := C \cap H$ is convex and closed.*

Proof. By definition there are a $c \in \mathbb{R}$ and a $u \in \mathbb{R}^d$ such that

$$H = \{x \in \mathbb{R}^d \mid \langle u \mid x\rangle_{\mathbb{R}^d} = c\}.$$

Note that a set F is called convex, if for any $f_1, f_2 \in F$ and some $\lambda \in [0, 1]$ the linear combination $\lambda \cdot f_1 + (1 - \lambda) \cdot f_2$ is still an element of F. Let f_1, f_2 be some elements of F and $\lambda \in [0, 1]$ be arbitrary but fixed. Since $f_1, f_2 \in H$ the equations

$$\langle u \mid f_1\rangle_{\mathbb{R}^d} = c = \langle u \mid f_2\rangle_{\mathbb{R}^d}$$

hold. Thus,

$$\langle u \mid \lambda \cdot f_1 + (1 - \lambda) \cdot f_2\rangle_{\mathbb{R}^d} = \lambda \cdot \langle u \mid f_1\rangle_{\mathbb{R}^d} + (1 - \lambda) \cdot \langle u \mid f_2\rangle_{\mathbb{R}^d} = c.$$

Consequently, $\lambda \cdot f_1 + (1 - \lambda) \cdot f_2$ is an element of H. On the other hand, it is also an element of C, because C is convex. Altogether, the linear combination is an element of the intersection $C \cap H$. Obviously, the intersection $C \cap H$ is also closed. □

We call a convex and closed set C a *finite polytope*, if there is a finite number $N \in \mathbb{N}$ of hyperplanes $(H^j)_{j=1}^N$ such that $C = \bigcap_{j=1}^N H_-^j$. If, further, the set C is compact we call C a *bounded, finite polytope*. Here we will only consider bounded, finite polytopes. Thus, if we talk in the following about polytopes, we mean a bounded, finite polytope.

Any polytope C is contained in the subspace E_C of $\mathbb{R}^d$ which is defined by

$$E_C := span\{x \in \mathbb{R}^d \mid \exists \alpha, \beta \in C \text{ such that } x = \alpha - \beta\}.$$

We denote by $span(X)$ the set of all finite linear combinations of the elements of $X \subseteq \mathbb{R}^d$. The dimension of E_C is called the *dimension of the corresponding polytope C*.

For instance, consider the bounded, finite polytope

$$C := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_1, x_2 \in [0,1], x_3 = 0 \right\}.$$

Then, the space E_C determined by

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_3 = 0 \right\}.$$

Hence, the dimension of the bounded finite polytope is equal to two.

Recall that the *interior* of a convex subset X of $\mathbb{R}^d$ is the set

$$int(X) := \{x \in X \mid \exists B \text{ open s.t. } x \in B \subseteq X\}.$$

Now we give a short summary of the concept of a tiling of $\mathbb{R}^d$. This part is inspired by [SEN].

Let $(Z_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^d$ be a countable family of compact, convex sets with non-empty interior. We call $Z := \{Z_1, Z_2, \ldots\}$ a *tiling*, if for two different $k_1, k_2 \in \mathbb{N}$ the intersection $Z_{k_1} \cap int(Z_{k_2})$ is empty and $\bigcup\limits_{k=1}^{\infty} Z_k = \mathbb{R}^d$. The subset Z_k is called *tile*. Consider a tiling Z of $\mathbb{R}^d$. The non-empty intersection of $d-n+1$ pairwise different tiles $Z_{k_1}, \ldots, Z_{k_{d-n+1}}$ with $dim(\bigcap\limits_{j=1}^{d-n+1} Z_{k_j}) = n$ is called *n-face*. For example, Figure 12 shows a tiling of $\mathbb{R}^2$. The set F_1 is a 2-face, F_2 is a 1-face and F_3 is a 0-face.

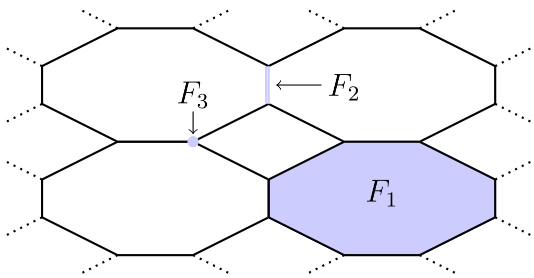

Figure 12:
The illustration of some k-faces for a special tiling of the space $\mathbb{R}^2$

A convex and compact set K specifies an affine space $\mathcal{A}_K := x + E_C$ where x is some element of K. Let ℓ be the Lebesgue measure on $\mathcal{A}_K$, then, the *barycenter* of a compact and convex set K is defined component-by-component by

$$b_K := \frac{1}{\int_K 1 \, d\ell(x)} \cdot \int_K x \, d\ell(x).$$

Later on, we will use the barycenter of a face of a Voronoi tile to puncture this face.

5.1 Voronoi cells

Now consider a D-set $\mathcal{D} \subseteq \mathbb{R}^d$. We define the *Voronoi cell* of an element $\alpha \in \mathcal{D}$ by

$$\begin{aligned} \mathcal{V}_{\mathcal{D}}(\alpha) :=& \{x \in \mathbb{R}^d \mid \forall \beta \in \mathcal{D} \setminus \{\alpha\} \; : \; \|x - \alpha\|_{\mathbb{R}^d} < \|x - \beta\|_{\mathbb{R}^d}\} \\ =& \bigcap_{\beta \in \mathcal{D} \setminus \{\alpha\}} \{x \in \mathbb{R}^d \mid \|x - \alpha\|_{\mathbb{R}^d} < \|x - \beta\|_{\mathbb{R}^d}\}. \end{aligned}$$

The sets $H_-^{(\beta)} := \{x \in \mathbb{R}^d \mid \|x - \alpha\|_{\mathbb{R}^d} < \|x - \beta\|_{\mathbb{R}^d}\}$ are halfspaces of the hyperplanes $H^{(\beta)} := \{x \in \mathbb{R}^d \mid \|x - \alpha\|_{\mathbb{R}^d} = \|x - \beta\|_{\mathbb{R}^d}\}$. Namely, such a hyperplane bisects perpendicularly the segment $-\alpha + \beta \in (-\alpha + \mathcal{D})$. Then, the Voronoi cell is the interior of all these half-spaces H_-, see Figure 13.

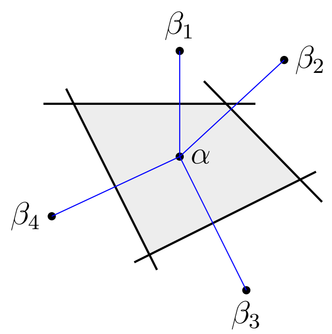

Figure 13:
The illustration how to construct a Voronoi cell of some given α

Figure 13 suggests that only a finite number of hyperplanes are essential for the construction of a Voronoi cell. This statement will be proven in the following lemma.

Lemma 5.2. *Let $\mathcal{D} \subseteq \mathbb{R}^d$ be a D-set, then, the following assertions hold.*

(i) For any $\alpha \in \mathcal{D}$ the closure of the Voronoi cell $\overline{\mathcal{V}_{\mathcal{D}}}(\alpha)$ is a bounded, finite polytope. Indeed, the closure of a Voronoi cell $\overline{\mathcal{V}_{\mathcal{D}}}(\alpha)$ is contained in $\overline{\mathcal{B}}_{R_1}(\alpha)$.

(ii) For each different $\alpha, \beta \in \mathcal{D}$ it follows that $\overline{\mathcal{V}_{\mathcal{D}}}(\alpha) \cap \mathcal{V}_{\mathcal{D}}(\beta)$ is empty.

(iii) The union $\bigcup_{\alpha \in \mathcal{D}} \overline{\mathcal{V}_{\mathcal{D}}}(\alpha)$ is equal to $\mathbb{R}^d$.

In fact, the family of the closure of the Voronoi cells with respect to $\mathcal{D}$ is a tiling of $\mathbb{R}^d$.

Proof. First note that for some $\alpha \in \mathcal{D}$ the closure of the Voronoi cell $\mathcal{V}_{\mathcal{D}}(\alpha)$ is equal to

$$\{x \in \mathbb{R}^d \mid \forall \beta \in \mathcal{D} \setminus \{\alpha\} \ : \ \|x - \alpha\|_{\mathbb{R}^d} \leq \|x - \beta\|_{\mathbb{R}^d}\}.$$

(i) Since $\mathcal{D}$ is R_1-relatively dense we have $\mathcal{D} + \overline{\mathcal{B}}_{R_1}(0) = \mathbb{R}^d$. Hence, for each $x \in \mathbb{R}^d \setminus \overline{\mathcal{B}}_{R_1}(\alpha)$ there is some $\beta \in \mathcal{D}$ such that $\|x - \beta\|_{\mathbb{R}^d} \leq R_1$. This implies that $\overline{\mathcal{V}_{\mathcal{D}}}(\alpha) \subseteq \overline{\mathcal{B}}_{R_1}(\alpha)$. Consequently, $\overline{\mathcal{V}_{\mathcal{D}}}(\alpha)$ is bounded and closed making it compact.

Further, for any element β of $\mathcal{D} \cap (\mathbb{R}^d \setminus \overline{\mathcal{B}}_{2R_1}(\alpha))$ the distance $\|\alpha - \beta\|_{\mathbb{R}^d}$ is greater than $2R_1$. Thus, the segments $(-\alpha + (\mathcal{D} \cap (\mathbb{R}^d \setminus \overline{\mathcal{B}}_{2R_1}(\alpha))))$ do not play a role for $\mathcal{V}_{\mathcal{D}}(\alpha)$, because $\mathcal{V}_{\mathcal{D}}(\alpha) \subseteq \overline{\mathcal{B}}_{R_1}(\alpha)$. On the other hand, the intersection $\mathcal{D} \cap \overline{\mathcal{B}}_{2R_1}(\alpha)$ contains at most a finite number of elements, because $\mathcal{D}$ is R_0-uniformly discrete (see Lemma 2.3 (iii)). Hence, only a finite number of segments $(-\alpha + \mathcal{D})$ contribute to the construction of $\mathcal{V}_{\mathcal{D}}(\alpha)$. Altogether, $\overline{\mathcal{V}_{\mathcal{D}}}(\alpha)$ is a polytope.

(ii) Let α and β be two different elements of $\mathcal{D}$. Assume $\overline{\mathcal{V}_{\mathcal{D}}}(\alpha) \cap \mathcal{V}_{\mathcal{D}}(\beta)$ is not empty and $x \in \overline{\mathcal{V}_{\mathcal{D}}}(\alpha) \cap \mathcal{V}_{\mathcal{D}}(\beta)$. Then,

$$\|x - \alpha\|_{\mathbb{R}^d} \overset{x \in \overline{\mathcal{V}_{\mathcal{D}}}(\alpha)}{\leq} \|x - \beta\|_{\mathbb{R}^d} \overset{x \in \mathcal{V}_{\mathcal{D}}(\beta)}{<} \|x - \alpha\|_{\mathbb{R}^d}$$

which is a contradiction.

(iii) The union $\bigcup_{\alpha \in \mathcal{D}} \overline{\mathcal{V}_{\mathcal{D}}}(\alpha)$ is obviously a subset of $\mathbb{R}^d$. Thus, we only have to verify the converse inclusion $\mathbb{R}^d \subseteq \bigcup_{\alpha \in \mathcal{D}} \overline{\mathcal{V}_{\mathcal{D}}}(\alpha)$.

Consider some $x \in \mathbb{R}^d$. Since $\mathcal{D}$ is uniformly discrete the intersection $\mathcal{D} \cap \overline{\mathcal{B}}_{R_1}(x)$ is finite or empty (Lemma 2.3 (iii)). But the intersection is non-empty, see Lemma 2.2. Consequently, we can take the minimum over the distances between x and the elements of $\mathcal{D} \cap \overline{\mathcal{B}}_{R_1}(x)$. Denote a minimum by $\beta \in \mathcal{D} \cap \overline{\mathcal{B}}_{R_1}(x)$. Hence, x is an element of $\overline{\mathcal{V}_{\mathcal{D}}}(\beta)$, because any other element of $\mathcal{D} \cap (\mathbb{R}^d \setminus \overline{\mathcal{B}}_{R_1}(x))$ has a distance to x greater than R_1, see (i).

Furthermore, for $\alpha \in \mathcal{D}$ the Voronoi cell $\mathcal{V}_{\mathcal{D}}(\alpha)$ is not empty, since $\mathcal{D}$ is R_0-uniformly discrete. More precisely, the open ball $\mathcal{B}_{\frac{R_0}{2}}(\alpha)$ is contained in $\mathcal{V}_{\mathcal{D}}(\alpha)$. Thus, the family $\left(\overline{\mathcal{V}_{\mathcal{D}}}(\alpha)\right)_{\alpha\in\mathcal{D}}$ is a tiling of $\mathbb{R}^d$. □

The closure of a Voronoi cell $\overline{\mathcal{V}_{\mathcal{D}}}(\alpha)$ is called *Voronoi tile* and it is punctured by $\alpha \in \mathcal{D}$. We call two elements $\alpha, \beta \in \mathcal{D}$ *nearest neighbors (n.n.)*, if the intersection $\overline{\mathcal{V}_{\mathcal{D}}}(\alpha) \cap \overline{\mathcal{V}_{\mathcal{D}}}(\beta)$ is not empty and has dimension $d-1$.

To get a better imagination of the definition of a nearest neighbor consider the group $\mathbb{R}^2$. Suppose $\mathcal{D} \subseteq \mathbb{R}^2$ is a D-set and we have given its Voronoi cells. By definition two elements of $\mathcal{D}$ are nearest neighbors, if the intersection of its Voronoi cells are not empty and has the dimension 1. In Figure 14 it can be seen that $\beta_1, \beta_3, \beta_4, \beta_5$ and β_7 are nearest neighbors of α. But β_2, β_6 and β_8 are not nearest neighbors of α, because the intersection of the Voronoi cells have the dimension zero.

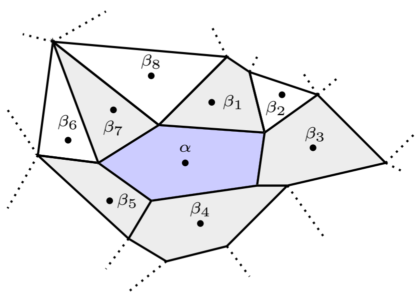

Figure 14:
An example which Voronoi cells are nearest neighbors of some given Voronoi cell $\mathcal{V}_{\mathcal{D}}(\alpha)$

We define the *relative interior* of a convex set X by

$$relint(X) := \{x \in X \mid \forall y \in X,\ \exists z \in X,\ \exists \lambda \in (0,1) \text{ such that } x = \lambda \cdot y + (1-\lambda) \cdot z\}.$$

A set $\mathcal{K}$ of convex polytopes is called *cell complex* of $\mathbb{R}^d$ if the following two assertions are true.

·) Any face of a convex polytope of $\mathcal{K}$ lies in $\mathcal{K}$.

·) Let $P_1, P_2 \in \mathcal{K}$ with non-empty intersection, then, their intersection is a common face of both.

The elements of $\mathcal{K}$ are called *cells*. The meaning of the second condition is sketched in Figure 15. There can be seen which cases we would like to exclude. The red line illustrates the intersection of these two convex polytopes. This red line is obviously not a face of either of the two polytopes.

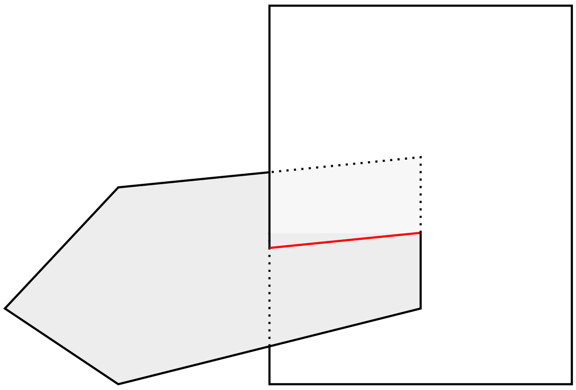

Figure 15:
The intersection of two polytopes

For $\alpha \in \mathcal{D}$ the pair $(\overline{\mathcal{V}_{\mathcal{D}}}(\alpha), \alpha)$ is called a *(Voronoi) d-cell.* For each $k \in \{0, \dots, d-1\}$ we say the pair $(\overline{F}, b_F)$ is a *(Voronoi) k-cell* if F is the relative interior of a face of dimension k of a Voronoi cell. The point $b_F \in \mathbb{R}^d$ is the puncture of this cell and it is defined by the barycenter of $\overline{F}$.

It is possible to choose a different puncture of the Voronoi k-cells. Necessary are the following. First, the puncture must lie in the relative interior of the face and secondly the choice of the puncture for any k-cell must be coherent.

For any $k \in \{0, \dots, d\}$ the translated support $(-b_F + F)$ of a k-cell $(\overline{F}, b_F)$ is called the *geometric support* of the k-cell. Then, $\mathcal{K}_{\mathcal{V}}$ is the set of all k-cells of the appertaining set $\mathcal{D}$. Obviously, the set $\mathcal{K}_{\mathcal{V}}$ is an infinite cell complex and we use the following notations

$$\begin{aligned} \mathcal{K}_{\mathcal{V}}^{(k)} &:= \{\sigma \in \mathcal{K}_{\mathcal{V}} \mid \sigma \text{ is a k-cell}\}, \\ \mathcal{D}^{(k)} &:= \{b \in \mathbb{R}^d \mid \exists F \subseteq \mathbb{R}^d \text{ such that } (\overline{F}, b) \in \mathcal{K}_{\mathcal{V}}^{(k)}\}. \end{aligned}$$

5.2 Collection of all proto cells

For any $\alpha \in \mathcal{D}$ we define the *collar of the Voronoi d-cell* $\sigma = (\overline{\mathcal{V}_{\mathcal{D}}}(\alpha), \alpha)$ by

$$Col_{\mathcal{D}}(\sigma) := Col_{\mathcal{D}}(\alpha) := \{\beta - \alpha \in (\mathcal{D} - \mathcal{D}) \mid \beta \text{ is nearest neighbor of } \alpha\}.$$

Similarly, the *collar of a Voronoi k-cell* $\sigma = (F, b_F) \in \mathcal{K}_{\mathcal{V}}^{(k)}$ is defined by

$$Col_{\mathcal{D}}(\sigma) := \{-b_F + \alpha \in (-b_F + \mathcal{D}) \mid \overline{\mathcal{V}_{\mathcal{D}}}(\alpha) \cap F \neq \emptyset\}.$$

Note that the definition of nearest neighbors requires that the dimension of the intersection of the Voronoi tiles is $d-1$ (dimension condition). But the definition of the collar of a Voronoi k-cell ($k \neq d$) asks for the intersection of the closure of their geometric supports being not empty. Reason for establishing the dimension condition is the uprising of problems with the construction of the boundary condition due to the admission of Voronoi tiles whose intersection has dimension less than $d-1$.

Lemma 5.3. *Let $\mathcal{D}$ be a D-set. Then, for two different $\alpha, \beta \in \mathcal{D}$ the following two statements are equivalent.*

(i) The collars of α and β are equal ($Col_{\mathcal{D}}(\alpha) = Col_{\mathcal{D}}(\beta)$).

(ii) The Voronoi cells $\mathcal{V}_{\mathcal{D}}(\alpha)$ and $\mathcal{V}_{\mathcal{D}}(\beta)$ coincide, which means

$$-\alpha + \mathcal{V}_{\mathcal{D}}(\alpha) = -\beta + \mathcal{V}_{\mathcal{D}}(\beta).$$

Proof. Let $\alpha, \beta \in \mathcal{D}$ be chosen such that $\alpha \neq \beta$.

$(i) \Rightarrow (ii)$ According to Lemma 5.2 (i) the sets $\overline{\mathcal{V}_{\mathcal{D}}}(\alpha) - \alpha$ and $\overline{\mathcal{V}_{\mathcal{D}}}(\beta) - \beta$ are polytopes. Furthermore, they have the same number of hyperplanes which determine them, because $\sharp\, Col_{\mathcal{D}}(\alpha) = \sharp\, Col_{\mathcal{D}}(\beta)$. Since the Voronoi cells are defined by the same segments the statement follows.

$(ii) \Rightarrow (i)$ Since the Voronoi cells $\mathcal{V}_{\mathcal{D}}(\alpha)$ and $\mathcal{V}_{\mathcal{D}}(\beta)$ be equal we have

$$\overline{\mathcal{V}_{\mathcal{D}}}(\alpha) = \alpha - \beta + \overline{\mathcal{V}_{\mathcal{D}}}(\beta). \qquad (\star)$$

Let $(\gamma - \alpha)$ be some element of $Col_{\mathcal{D}}(\alpha)$. Then, the intersection $\overline{\mathcal{V}_{\mathcal{D}}}(\gamma) \cap \overline{\mathcal{V}_{\mathcal{D}}}(\alpha)$ is not empty and the dimension of $\overline{\mathcal{V}_{\mathcal{D}}}(\gamma) \cap \overline{\mathcal{V}_{\mathcal{D}}}(\alpha)$ is equal to $d-1$. Moreover,

$$\overline{\mathcal{V}_{\mathcal{D}}}(\gamma) \cap \overline{\mathcal{V}_{\mathcal{D}}}(\alpha) \overset{(\star)}{=} \overline{\mathcal{V}_{\mathcal{D}}}(\gamma) \cap (-\beta + \alpha + \overline{\mathcal{V}_{\mathcal{D}}}(\beta)) = (\beta - \alpha + \overline{\mathcal{V}_{\mathcal{D}}}(\gamma)) \cap \overline{\mathcal{V}_{\mathcal{D}}}(\beta).$$

Hence, by definition $(\gamma - \alpha) = (\beta - \alpha + \gamma) - \beta$ is an element of $Col_{\mathcal{D}}(\beta)$.

□

Let $\sigma = (\overline{F}, b_F)$ and $\sigma' = (\overline{F'}, b'_F)$ be both elements of $\mathcal{K}_{\mathcal{V}}^{(k)}$ for some $k \in \{0, \dots, d\}$. We call them *translation equivalent ($\sigma \sim \sigma'$)* if the following statements are true.

(T1) The geometric supports coincide ($F - b_F = F' - b'_F$).

(T2) They have the same collar ($Col_{\mathcal{D}}(\sigma) = Col_{\mathcal{D}}(\sigma')$).

Note that in this definition also the Voronoi d-cells are included. By definition it is obvious that for two Voronoi d-cells which are translation equivalent the differences of their puncture is contained in $(\mathcal{D}-\mathcal{D})$. The relation $\sim$ defines an equivalence class $[\sigma]$ on $\mathcal{K}_{\mathcal{V}}$. The representator is given by $(\overline{F}, 0, Col_{\mathcal{D}}(\sigma))$ which is located at the origin.

Such a $[\sigma]$ of dimension k is called *collared Voronoi proto k-cell.* We define the *set of all collared Voronoi proto k-cells* by

$$\mathcal{P}^{(k)} := \mathcal{K}_{\mathcal{V}}^{(k)} / \sim \; = \{[\sigma] \mid \sigma \in \mathcal{K}_{\mathcal{V}}^{(k)}\}.$$

Proposition 5.4. *For every D-set $\mathcal{D}$ the set of all collared Voronoi proto k-cells $\mathcal{P}^{(k)}$ is finite for each $k \in \{0, \dots, d\}$.*

Proof. Consider at first the set of all collared Voronoi proto d-cells $\mathcal{P}^{(d)}$. First of all we will prove that it is sufficient to consider the elements $\beta \in (\mathcal{D} \cap \overline{\mathcal{B}}_{2R_1}(\alpha))$ to determine the collar $Col_{\mathcal{D}}(\alpha)$ of some $\alpha \in \mathcal{D}$.

Assume there is some $\gamma \in \mathcal{D}$ with $\|\alpha - \gamma\|_{\mathbb{R}^d} > 2R_1$ such that $\alpha - \gamma \in Col_{\mathcal{D}}(\alpha)$. By definition of $Col_{\mathcal{D}}(\alpha)$ there is some $x \in \mathbb{R}^d$ such that the intersection $\overline{\mathcal{V}_{\mathcal{D}}}(\gamma) \cap \overline{\mathcal{V}_{\mathcal{D}}}(\alpha)$ contains x. Thus,

$$2R_1 < \|\alpha - \gamma\|_{\mathbb{R}^d} \overset{\triangle\text{-inequality}}{\leq} \|\alpha - x\|_{\mathbb{R}^d} + \|x - \gamma\|_{\mathbb{R}^d} \overset{\text{L. 5.2 (i)}}{\leq} R_1 + R_1 = 2R_1$$

which is a contradiction. Consequently,

$$Col_{\mathcal{D}}(\alpha) \subseteq (-\alpha + \mathcal{D}) \cap \overline{\mathcal{B}}_{2R_1}(\alpha) \in Clu(\mathcal{D}, \overline{\mathcal{B}}_{2R_1}(\alpha)),$$

where $\overline{\mathcal{B}}_{2R_1}(\alpha)$ is compact. Since $\mathcal{D}$ is of finite local complexity there can be only a finite number of different collars. Thus, by Lemma 5.3 the set $\mathcal{P}^{(d)}$ is finite.

Consider now for $k \in \{0, \dots, d\}$ the collared Voronoi proto k-cells $\mathcal{P}^{(k)}$. Any element $[\sigma] \in \mathcal{P}^{(k)}$ has only a finite set of k-1 dimensional faces, because the k-cells are bounded, finite polytopes. Hence, there is a finite set of Voronoi (k-1)-cells which matches to one Voronoi k-cell $[\sigma]$ only.

Moreover, we have proven that $\mathcal{P}^{(d)}$ is finite. Consequently by our last consideration the set $\mathcal{P}^{(d-1)}$ has to be finite. Iterating this step it follows that for any $k \in \{0, \dots, d\}$ the set $\mathcal{P}^{(k)}$ is finite. $\square$

According to the last Proposition 5.4 there is for any set of all collared Voronoi proto k-cells $\mathcal{P}^{(k)}$ an $N_k = \sharp(\mathcal{P}^{(k)})$ such that

$$\begin{aligned}
\mathcal{P}^{(d)} &= \{[\sigma_1], \ldots, [\sigma]_{N_d}\}, && \text{(proto cells)} \\
\mathcal{P}^{(d-1)} &= \{[\sigma_1], \ldots, [\sigma]_{N_{d-1}}\}, && \text{(proto faces)} \\
&\vdots \\
\mathcal{P}^{(k)} &= \{[\sigma_1], \ldots, [\sigma]_{N_k}\}, && \text{(proto k-cells)} \\
&\vdots \\
\mathcal{P}^{(0)} &= \{[\sigma_1], \ldots, [\sigma]_{N_0}\}. && \text{(proto vertices)}
\end{aligned}$$

The disjoint union

$$\mathcal{P} := \bigsqcup_{k=1}^{d} \mathcal{P}^{(k)}$$

is called the *collection of all collared Voronoi proto cells* which is finite, see Proposition 5.4.

A relation between the elements of $\mathcal{P}^{(k)}$ and $\mathcal{P}^{(k-1)}$, called incidence number

$$[\cdot;\cdot]_\sim : \mathcal{P}^{(k)} \times \mathcal{P}^{(k-1)} \to \{-1, 0, 1\},$$

can be established, see [BNM]. It turns out that $\mathcal{P}$ is a CW-complex with homology induced by the incidence number $[\cdot;\cdot]_\sim$. This leads to a cell complex structure on $\mathcal{P}$ and it is called Anderson-Putnam Complex. A CW-complex gives an instruction how we have to glue the cells together.

Our main aim is to decompose the Schrödinger operator in a useful way. To do so we will use the structure of the D-set. According to Lemma 5.2 we can describe the whole space $\mathbb{R}^d$ by the elements of $\mathcal{P}^{(d)}$. If we suppose that the main potential of an atome or molecule lies in such a cell it suffices to draw our attention to a finitie number of cells.

Let $\mathcal{D}_\mu$ be some pointset of the Hull $\Omega_\mathcal{D}$ of $\mathcal{D}$. We define for any $p \in \mathcal{P}^{(d)}$ the set

$$\mathcal{D}_\mu(p) := \{\alpha \in \mathcal{D}_\mu \mid (\overline{\mathcal{V}_{D_\mu}}(\alpha), \alpha, Col_{\mathcal{D}_\mu}(\alpha)) \sim p\}$$

of all points of the set $\mathcal{D}_\mu$ such that the corresponding collared Voronoi proto d-cell is translation equivalent to p.

Lemma 5.5. *Let $\mathcal{D}$ be a D-set of $\mathbb{R}^d$ and $p \in \mathcal{P}^{(d)}$ be some collared Voronoi proto d-cell. Then, for each $t \in \mathcal{T}$ we have*

$$\mathcal{D}_t(p) \subseteq (\mathcal{D} - \mathcal{D}) \subseteq \mathbb{L}_\mathcal{D}\,.$$

Proof. First note that for any element $t \in \mathcal{T}$ the set $\mathcal{D}_t$ is by definition an element of the Hull $\Omega_{\mathcal{D}}$ and so $\mathcal{D}_t(p)$ is well-defined. According Proposition 2.14 the equation

$$\mathcal{D}_t - \mathcal{D}_t = \mathcal{D} - \mathcal{D} \subseteq \mathbb{L}_{\mathcal{D}}$$

holds. Since $\mathcal{D}_t$ is an element of $\mathcal{P}_{\mathcal{T}}$ the origin 0 lies in $\mathcal{D}_t$. Thus,

$$\mathcal{D}_t(p) \subseteq \mathcal{D}_t \overset{0 \in \mathcal{D}_t}{\subseteq} \mathcal{D}_t - \mathcal{D}_t = \mathcal{D} - \mathcal{D} \subseteq \mathbb{L}_{\mathcal{D}}\,.$$

□

Lemma 5.6. *Let $t \in \mathcal{T}$ and $\beta \in \mathcal{D}_t(p)$ be arbitrary and fixed. The following two statements are equivalent for any $p \in \mathcal{P}^{(d)}$.*

(*i*) *The set $\mathcal{D}_t(p)$ contains $\gamma - \beta$.*

(*ii*) *The set $\mathcal{D}_{\tau^{-\beta}t}(p)$ contains γ.*

Moreover, for each $\beta \in \mathcal{D}_t \in \mathcal{P}_{\mathcal{T}}$ the pointset $\mathcal{D}_{\tau^\beta t}$ is an element of $\mathcal{P}_{\mathcal{T}}$.

Proof. The origin is an element of $\mathcal{D}_{\tau^\beta t}(p)$ for $\beta \in \mathcal{D}_t(p)$, because

$$\tau^\beta t(\{0\}) = t(\{\beta\}) \overset{\beta \in \mathcal{D}_t(p)}{=} 1.$$

Thus, the set $\mathcal{D}_{\tau^\beta t} \in \Omega_{\mathcal{D}}$ is an element of the Transversal $\mathcal{T}$. Since the collar does not change by shifting and

$$t(\{\gamma - \beta\}) = \tau^{-\beta} t(\{\gamma\})$$

the equivalence follows. □

For some $p \in \mathcal{P}^{(d)}$ we denote by $Col(p)$ the corresponding collar of this Voronoi cell. We define

$$\mathcal{T}_p := \{t \in \mathcal{T} \mid Col_{\mathcal{D}_t}(0) = Col(p)\} \subseteq \mathcal{T}\,.$$

In detail, $\mathcal{T}_p$ is the set of all elements of the transversal where the collar of the Voronoi cell located at the origin 0 coincides with the collar of p. Note that by definition $0 \in \mathcal{D}_t$ and so the collar $Col_{\mathcal{D}_t}(0)$ is well-defined.

As mentiones above we number consecutively the set $\mathcal{P}^{(d)} = \{p_1, \ldots, p_{N_d}\}$. Thus, we define the *characteristic function* $\chi_p : \mathcal{T} \to \{0,1\}$ *of* $\mathcal{T}_p$ by

$$\chi_p(t) = \begin{cases} 1 & , \ t \in \mathcal{T}_p \\ 0 & , \ \text{else} \end{cases}.$$

5.3 Example

Recall the example of chapter 2 and chapter 3. Let $\mathcal{D} \subseteq \mathbb{R}$ be the aperiodic Delone set of finite local complexity as before. In chapter 2 we have seen that our minimal generator of the Lagarias group

$$\widehat{\mathbb{L}_{\mathcal{D}}} \cong \{e^{i \cdot \langle h \mid \cdot \rangle_{\mathbb{R}^2}} : \mathbb{Z}^2 \cong \mathbb{L}_{\mathcal{D}} \to \mathcal{S}_1 \mid h \in [0, 2\pi)^2\}$$

is given by $\{1, q\}$. We would like to characterize the set of all collared Voronoi proto d-cells $\mathcal{P}^{(d)}$ of $\mathcal{D}$. Obviously, we have only four possibilities of nearest neighbors for some $\alpha \in \mathcal{D}$, see Figure 16.

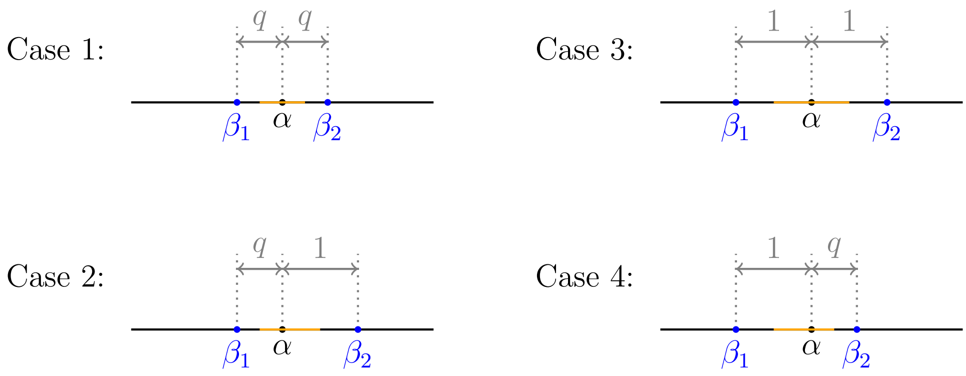

Figure 16:
The different possibilities of nearest neighbors in our setting

The orange lines represent the corresponding closed Voronoi cell of α. The blue points label the nearest neighbors of α. More precisely, we have

$$\begin{aligned}\mathcal{P}^{(d)} = \{(&\left[-\frac{q}{2}, \frac{q}{2}\right], 0, \{-q, q\}) \ ,\\ (&\left[-\frac{q}{2}, \frac{1}{2}\right], 0, \{-q, 1\}) \ ,\\ (&\left[-\frac{1}{2}, \frac{1}{2}\right], 0, \{-1, 1\}) \ ,\\ (&\left[-\frac{1}{2}, \frac{q}{2}\right], 0, \{-1, q\})\}.\end{aligned}$$

The elements of $\mathcal{P}^{(d)}$ have the form $(\overline{F}, 0, Col_{\mathcal{D}}(\alpha))$ where F is the geometric support and $Col_{\mathcal{D}}(\alpha)$ is the corresponding collar. That all these cells lie in $\mathcal{P}^{(d)}$ is clear, because ω contains any finite word.

Chapter 6

The Wannier transform

In this chapter we use the concepts of a tensor product and a direct sum, see [KR] and [WEI]. The main ideas of this chapter are based on [BNM]. We will add those proofs which are missing and carry out the corresponding arguments in a detailed and rigorous way.

First recall the definition of the Hilbert space

$$L^2(\widehat{\mathbb{L}_\mathcal{D}}) := \{f : \widehat{\mathbb{L}_\mathcal{D}} \to \mathbb{C} : f \text{ measurable}, \int\limits_{\widehat{\mathbb{L}_\mathcal{D}}} |f(k)|^2 \, d\varrho(k) < \infty\}/\sim$$

endowed with the scalar product

$$\langle f \mid g \rangle_{L^2(\widehat{\mathbb{L}_\mathcal{D}})} := \int\limits_{\widehat{\mathbb{L}_\mathcal{D}}} \overline{f(k)} \cdot g(k) \, d\varrho(k)$$

which we have discussed in chapter 3 about the Lagarias group. According to Proposition 3.15 the family $\left(f_b : \widehat{\mathbb{L}_\mathcal{D}} \to \mathbb{C}\right)_{b \in \mathbb{L}_\mathcal{D}}$ defined by

$$f_b(k) := k(b), \qquad\qquad k \in \widehat{\mathbb{L}_\mathcal{D}}$$

is an orthonormal basis of $L^2(\widehat{\mathbb{L}_\mathcal{D}})$.

As mentioned above we will now consider the group $G = \mathbb{R}^d$. Thus, by Proposition 3.5 we can uniquely identify our Lagarias group $\mathbb{L}_\mathcal{D}$ with $\mathbb{Z}^M$ and so by Lemma 3.17

$$\widehat{\mathbb{L}_\mathcal{D}} := \{k : \mathbb{L}_\mathcal{D} \to \mathcal{S}_1 \mid k \in \mathcal{C}(\mathbb{L}_\mathcal{D}) \text{ homomorphism}\} \cong \{e^{-i \cdot \langle h \,|\, \cdot \,\rangle_{\mathbb{R}^M}} : \mathbb{Z}^M \to \mathcal{S}_1 \mid h \in [0, 2\pi)^M\}.$$

In the last chapter we have seen that the set of all collared Voronoi proto d-cells $\mathcal{P}^{(d)}$ is finite. According to Lemma 5.2 the equality

$$\bigcup_{j=1}^{N_d} \left(\bigcup_{\alpha \in \mathcal{D}_t(p_j)} \alpha + \overline{\mathrm{V}_\mathrm{j}} \right) = \mathbb{R}^d$$

holds for each $t \in \mathcal{T}$, where V_j is the geometric support of $p_j \in \mathcal{P}^{(d)}$.

Lemma 6.1. *Consider two compact sets K_1 and K_2 of $\mathbb{R}^d$ and a U-uniformly discrete set $\mathcal{D} \subseteq \mathbb{R}^d$. Then, there is an $\mathcal{R} > 0$ such that*

$$\{\alpha \in \mathcal{D} \mid (\alpha + K_1) \cap K_2 \neq \emptyset\} \subseteq \mathcal{D} \cap \overline{\mathcal{B}}_{\mathcal{R}}(0)$$

and this set is finite.

Proof. Since $K_1, K_2 \subseteq \mathbb{R}^d$ are compact there exists an $\mathcal{R} > 0$ such that

$$K_1, K_2 \subseteq \overline{\mathcal{B}}_{\frac{\mathcal{R}}{2}}(0).$$

Let $\alpha \in \mathcal{D}$ be chosen such that $(\alpha + K_1) \cap K_2 \neq \emptyset$. Thus, there are $x_1 \in K_1$ and $x_2 \in K_2$ with

$$\alpha + x_1 = x_2 \;\Leftrightarrow\; \alpha = x_2 - x_1.$$

Hence,

$$\|\alpha\|_{\mathbb{R}^d} = \|x_2 - x_1\|_{\mathbb{R}^d} \leq \|x_2\|_{\mathbb{R}^d} + \|x_1\|_{\mathbb{R}^d} \leq \frac{\mathcal{R}}{2} + \frac{\mathcal{R}}{2} = \mathcal{R}$$

which implies

$$\{\alpha \in \mathcal{D} \mid (\alpha + K_1) \cap K_2 \neq \emptyset\} \subseteq \mathcal{D} \cap \overline{\mathcal{B}}_{\mathcal{R}}(0).$$

According to Lemma 2.3 (iii) the set $\mathcal{D} \cap \overline{\mathcal{B}}_{\mathcal{R}}(0)$ is finite and so

$$\{\alpha \in \mathcal{D} \mid (\alpha + K_1) \cap K_2 \neq \emptyset\}$$

is finite. □

Lemma 6.2. *Consider a D-set $\mathcal{D}$ and a collared Voronoi proto d-cell $p_j \in \mathcal{P}^{(d)}$ with geometric support V_j. Then, for any $\psi \in \mathcal{C}_c(\mathbb{R}^d)$ there exists an $\mathcal{R} > 0$ such that the equation*

$$\sum_{\alpha \in \mathcal{D}_t(p_j)} \psi(\alpha + r) = \sum_{\alpha \in \mathcal{D}_t(p_j) \cap \overline{\mathcal{B}}_{\mathcal{R}}(0)} \psi(\alpha + r)$$

holds uniformly in $r \in \overline{V_j}$. In particular, this sum is finite.

Proof. Note that if we have verified that there is such a closed ball $\overline{\mathcal{B}}_{\mathcal{R}}(0)$ independent of $r \in \overline{V_j}$ the statement follows by Lemma 2.3 (iii).

According to Lemma 5.2 the closure of the geometric support $K_1 := \overline{V_j}$ is compact. Let K_2 be the compact support of ψ. Note that $\mathcal{D}_t(p_j) \subseteq \mathcal{D}_t$ is U-uniformly discrete. Hence, by Lemma 6.1 there is an $\mathcal{R} > 0$ such that

$$\{\alpha \in \mathcal{D}_t(p_j) \mid (\alpha + K_1) \cap K_2 \neq \emptyset\} \subseteq \mathcal{D}_t(p_j) \cap \overline{\mathcal{B}}_{\mathcal{R}}(0).$$

Since ψ vanishes on $\mathbb{R}^d \setminus K_2$ the equation

$$\sum_{\alpha \in \mathcal{D}_t(p_j)} \psi(\alpha + r) = \sum_{\alpha \in \mathcal{D}_t(p_j) \cap \overline{\mathcal{B}}_{\mathcal{R}}(0)} \psi(\alpha + r)$$

holds. □

In the following this property will be used often. For instance, to interchange the integral over the geometric support of a collared Voronoi proto d-cell with such a sum. Denote for any $\psi \in \mathcal{C}_c(\mathbb{R}^d)$ with corresponding $\mathcal{R} > 0$ the intersection $\mathcal{D}_t(p_j) \cap \overline{\mathcal{B}}_{\mathcal{R}}(0)$ by $\mathcal{D}_t^{\mathcal{R}}(p_j)$.

Using the results of chapter 3 and chapter 5 we define for $p_j \in \mathcal{P}^{(d)}$ a family $(\mathcal{E}_t(p_j))_{t \in \mathcal{T}}$ of closed subspaces of $L^2(\widehat{\mathbb{L}_{\mathcal{D}}})$ by

$$\mathcal{E}_t(p_j) := \overline{Lin\{f_\alpha : \widehat{\mathbb{L}_{\mathcal{D}}} \to \mathbb{C} \mid \alpha \in \mathcal{D}_t(p_j)\}}.$$

By Lemma 5.5 the term f_α is well-defined for $\alpha \in \mathcal{D}_t(p_j)$. Consider the Hilbert space

$$\mathcal{H}_t := \bigoplus_{j=1}^{N_d} L^2(\overline{\mathrm{V_j}}) \otimes \mathcal{E}_t(p_j)\,.$$

According to Statement A.13 we define for $F, G \in \mathcal{H}_t$ the scalar product on $\mathcal{H}_t$ by

$$\begin{aligned}
\langle F \mid G \rangle_{\mathcal{H}_t} &= \sum_{j=1}^{N_d} \langle F(p_j;\cdot,\cdot) \mid G(p_j;\cdot,\cdot)\rangle_{L^2(\overline{\mathrm{V_j}}) \otimes \mathcal{E}_t(p_j)} \\
&= \sum_{j=1}^{N_d} \int_{\overline{\mathrm{V_j}}} \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} \overline{F(p_j;r,k)} \cdot G(p_j;r,k)\, d\varrho(k)\, d\ell^d(r),
\end{aligned}$$

where ℓ^d denotes the Lebesgue measure on $\mathbb{R}^d$. Denote by $\mathcal{H}_t^{(p_j)}$ the tensor space $L^2(\overline{\mathrm{V_j}}) \otimes \mathcal{E}_t(p_j)$.

6.1 Definition and properties

Consider some function $\psi \in \mathcal{C}_c(\mathbb{R}^d)$. Then, for some $t \in \mathcal{T}$ and $p_j \in \mathcal{P}^{(d)}$ the *t-Wannier transform* $\mathcal{W}_t$ is defined by

$$(\mathcal{W}_t\,\psi)(p_j; s, k) := \sum_{\alpha \in \mathcal{D}_t(p_j)} \psi(\alpha + s) \cdot f_\alpha(k), \qquad s \in \overline{\mathrm{V_j}} \subset \mathbb{R}^d,\ k \in \widehat{\mathbb{L}_{\mathcal{D}}}$$

where V_j is the geometric support of the collared Voronoi proto d-cell p_j. Note that according to Lemma 6.2 this sum exists uniformly in $r \in \overline{V_j}$ and so the transformation is well-defined.

We define the *related shift map* $\tau : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ for $\psi \in L^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ by

$$\tau^{-x}\psi(y) = \psi(x + y), \qquad y \in \mathbb{R}^d .$$

Similar to the associated shift of a measure we get

$$\tau^{-x}\left(\tau^{-y}\psi\right) = \tau^{-(x+y)}\psi$$

and

$$\tau^{-0}\psi = \psi.$$

Lemma 6.3 ([BNM]). *Consider some fixed $t \in \mathcal{T}$. For any $\psi \in \mathcal{C}_c(\mathbb{R}^d)$ the t-Wannier transform $\mathcal{W}_t\,\psi$ is an element of the Hilbert space $\mathcal{H}_t$.*

Proof. By definition of the direct sum (see A.2 "Tensor product and direct sum") it suffices to show that

$$(\mathcal{W}_t\,\psi)(p_j; \cdot, \cdot) \in \mathcal{H}_t^{(p_j)}, \qquad p_j \in \mathcal{P}^{(d)}.$$

Consider some $p_j \in \mathcal{P}^{(d)}$ and $\psi \in \mathcal{C}_c(\mathbb{R}^d)$ with corresponding radius $\mathcal{R} > 0$ (Lemma 6.2). Then,

$$(\mathcal{W}_t\,\psi)(p_j; \cdot, \cdot) = \sum_{\alpha \in \mathcal{D}_t(p_j)} \tau^{-\alpha}\psi \otimes f_\alpha = \sum_{\alpha \in \mathcal{D}_t^{\mathcal{R}}(p_j)} \tau^{-\alpha}\psi \otimes f_\alpha.$$

For $\alpha \in \mathcal{D}_t^{\mathcal{R}}(p_j) \subseteq \mathcal{D}_t(p_j)$ the map f_α is an element of $\mathcal{E}_t(p_j)$. According to Lemma 6.2 this is a finite linear combination of elements of the set $\{f_\alpha \mid \alpha \in \mathcal{D}_t(p_j)\}$. Further, we have $\tau^{-\alpha}\psi \in L^2(\overline{\mathrm{V_j}})$ and so

$$(\mathcal{W}_t\,\psi)(p_j; \cdot, \cdot) \in L^2(\overline{\mathrm{V_j}}) \otimes \mathcal{E}_t(p_j) = \mathcal{H}_t^{(p_j)}$$

$\square$

Proposition 6.4 ([BNM]). *Consider some $p_j \in \mathcal{P}^{(d)}$ with geometric support V_j and some fixed $t \in \mathcal{T}$. Then, for $\psi \in \mathcal{C}_c(\mathbb{R}^d)$ and $\beta \in \mathcal{D}_t(p_j)$ the equation*

$$(\mathcal{W}_{\tau^\beta t}\,\psi)(p_j; s+\beta, k) = (\mathcal{W}_t\,\psi)(p_j; s, k) \cdot \overline{f_\beta(k)}, \qquad s \in \overline{\mathrm{V_j}}, k \in \widehat{\mathbb{L}_{\mathcal{D}}}$$

holds.

Proof. Let $\beta \in \mathcal{D}_t(p_j)$ be arbitrary and consider the corresponding radius $\mathcal{R} > 0$ of $\psi \in \mathcal{C}_c(\mathbb{R}^d)$ (Lemma 6.2). The statement follows for some $s \in \overline{\mathrm{V_j}}$ and $k \in \widehat{\mathbb{L}_{\mathcal{D}}}$ by

$$\begin{aligned}
(\mathcal{W}_{\tau^\beta t}\,\psi)(p_j; s+\beta, k) &= \sum_{\alpha \in \mathcal{D}^{\mathcal{R}}_{\tau^\beta t}(p_j)} \psi(s+\beta+\alpha) \cdot f_\alpha(k) \\
&= \sum_{\alpha \in \mathcal{D}^{\mathcal{R}}_{\tau^\beta t}(p_j)} \psi(s+(\alpha+\beta)) \cdot f_{\alpha+\beta-\beta}(k) \\
&\overset{\text{L. 3.16}}{=} \left(\sum_{\alpha \in \mathcal{D}^{\mathcal{R}}_{\tau^\beta t}(p_j)} \psi(s+(\alpha+\beta)) \cdot f_{\alpha+\beta}(k) \right) \cdot \overline{f_\beta(k)} \\
&\overset{\gamma=\alpha+\beta}{=} \left(\sum_{\gamma-\beta \in \mathcal{D}^{\mathcal{R}}_{\tau^\beta t}(p_j)} \psi(s+\gamma) \cdot f_\gamma(k) \right) \cdot \overline{f_\beta(k)} \\
&\overset{\text{L. 5.6}}{=} \left(\sum_{\gamma \in \mathcal{D}^{\mathcal{R}}_{t}(p_j)} \psi(s+\gamma) \cdot f_\gamma(k) \right) \cdot \overline{f_\beta(k)} \\
&= (\mathcal{W}_t\,\psi)(p_j; s, k) \cdot \overline{f_\beta(k)}.
\end{aligned}$$

□

This property of the Wannier transform is called *covariance condition*. This means that in a certain sense the transformation is translation invariant.

Proposition 6.5 ([BNM]). *Consider some $p_j \in \mathcal{P}^{(d)}$ with geometric support V_j and a fixed $t \in \mathcal{T}$. For $\beta \in \mathcal{D}_t(p_j)$ and $\psi \in \mathcal{C}_c(\mathbb{R}^d)$ the equality*

$$\psi(s+\beta) = \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} (\mathcal{W}_t\,\psi)(p_j; s, k) \cdot \overline{f_\beta(k)}\, d\varrho(k), \qquad s \in \overline{\mathrm{V_j}}$$

is true.

Proof. Let $\mathcal{R} > 0$ be the corresponding radius of $\psi \in \mathcal{C}_c(\mathbb{R}^d)$, see Lemma 6.2. Consider some $s \in \overline{\mathrm{V_j}}$ and $\beta \in \mathcal{D}_t(p_j)$. According to Lemma 5.6 the origin lies in $\mathcal{D}_{\tau^\beta t}(p_j)$ and because the Haar measure ϱ on $\widehat{\mathbb{L}_{\mathcal{D}}}$ is normed it follows

$$
\begin{aligned}
& \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} (\mathcal{W}_t\,\psi)(p_j; s, k) \cdot \overline{f_\beta(k)}\; d\varrho(k) \\
\overset{\text{P. 6.4}}{=} \quad & \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} (\mathcal{W}_{\tau^\beta t}\,\psi)(p_j; s+\beta, k)\; d\varrho(k) \\
= \quad & \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} \sum_{\alpha \in \mathcal{D}^{\mathcal{R}}_{\tau^\beta t}(p_j)} \psi(s+\beta+\alpha) \cdot f_\alpha(k)\; d\varrho(k) \\
\overset{\text{L. 6.2}}{=} \quad & \psi((s+\beta)) \cdot \underbrace{\int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} \underbrace{f_0(k)}_{=1}\; d\varrho(k)}_{=1} + \sum_{\alpha \in \mathcal{D}^{\mathcal{R}}_{\tau^\beta t}(p_j)\setminus\{0\}} \psi(s+\beta+\alpha) \cdot \underbrace{\int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} f_\alpha(k)\; d\varrho(k)}_{=0 \text{ because } \alpha\neq 0 \text{ (L. 3.13)}} \\
= \quad & \psi(s+\beta).
\end{aligned}
$$

□

We denote by ∇_r the gradient with respect to the variable of $\mathbb{R}^d$. For the sake of convenience we write $\nabla\psi$ for some $\psi \in \mathcal{C}^1_c(\mathbb{R}^d)$ instead of $\nabla_r\psi$.

Lemma 6.6 ([BNM])**.** *Consider some $\psi \in \mathcal{C}^1_c(\mathbb{R}^d)$ and some fixed $t \in \mathcal{T}$. Then, for $r \in \overline{\mathrm{V}_\mathrm{j}}$ the equation*

$$\nabla_r(\mathcal{W}_t\,\psi) = \mathcal{W}_t(\nabla\psi)$$

holds.

Proof. First note that since $\psi \in \mathcal{C}^1_c(\mathbb{R}^d)$ we know that $\nabla\psi \in \mathcal{C}_\mathrm{c}(\mathbb{R}^d)$ and so $\mathcal{W}_t(\nabla\psi)$ is well-defined. Let $\mathcal{R} > 0$ be the corresponding radius of ψ (Lemma 6.2). Then,

$$
\begin{aligned}
\nabla_r\left((\mathcal{W}_t\,\psi)(p_j; r, \cdot)\right) &= \nabla_r \sum_{\alpha \in \mathcal{D}^{\mathcal{R}}_t(p_j)} \psi(r+\alpha) \cdot f_\alpha \\
&\overset{\text{L. 6.2}}{=} \sum_{\alpha \in \mathcal{D}^{\mathcal{R}}_t(p_j)} \nabla_r\psi(r+\alpha) \cdot f_\alpha \\
&= \sum_{\alpha \in \mathcal{D}^{\mathcal{R}}_t(p_j)} (\nabla\psi)(r+\alpha) \cdot f_\alpha \\
&= \left(\mathcal{W}_t(\nabla\psi)\right)(p_j; r, \cdot).
\end{aligned}
$$

□

Theorem 6.7 ([BNM], Plancherel formula). *Consider some fixed $t \in \mathcal{T}$. Then, for $\psi \in \mathcal{C}_c(\mathbb{R}^d)$ the t-Wannier transform $\mathcal{W}_t : \mathcal{C}_c(\mathbb{R}^d) \to \mathcal{H}_t$ satisfies*

$$\int\limits_{\mathbb{R}^d} |\psi(r)|^2 \, d\ell^d(r) = \sum_{j=1}^{N_d} \int\limits_{\overline{\mathrm{V_j}}} \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} |(\mathcal{W}_t\,\psi)(p_j; r, k)|^2 \, d\varrho(k) \, d\ell^d(r).$$

Further, there exists a uniquely unitary extension $\widetilde{\mathcal{W}_t} : L^2(\mathbb{R}^d) \to \mathcal{H}_t$ of the t-Wannier transform $\mathcal{W}_t$.

Proof. Consider a function $\psi \in \mathcal{C}_c(\mathbb{R}^d)$ with corresponding radius $\mathcal{R} > 0$ (Lemma 6.2). By a short computation we get for each $p_j \in \mathcal{P}^{(d)}$ and $r \in \overline{\mathrm{V_j}}$ that

$$\begin{aligned}
&\int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} |(\mathcal{W}_t\,\psi)(p_j; r, k)|^2 \, d\varrho(k) \\
&= \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} \overline{\left(\sum_{\alpha_1 \in \mathcal{D}_t^{\mathcal{R}}(p_j)} \psi(\alpha_1 + r) \cdot f_{\alpha_1}(k) \right)} \cdot \left(\sum_{\alpha_2 \in \mathcal{D}_t^{\mathcal{R}}(p_j)} \psi(\alpha_2 + r) \cdot f_{\alpha_2}(k) \right) d\varrho(k) \\
&\overset{\text{L. 6.2}}{=} \sum_{\alpha_1 \in \mathcal{D}_t^{\mathcal{R}}(p_j)} \overline{\psi(\alpha_1 + r)} \cdot \sum_{\alpha_2 \in \mathcal{D}_t^{\mathcal{R}}(p_j)} \psi(\alpha_2 + r) \cdot \underbrace{\int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} \overline{f_{\alpha_1}(k)} \cdot f_{\alpha_2}(k) \, d\varrho(k)}_{=\delta_{\alpha_1, \alpha_2}} \\
&= \sum_{\alpha_1 \in \mathcal{D}_t^{\mathcal{R}}(p_j)} |\psi(\alpha_1 + r)|^2
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{j=1}^{N_d} \int\limits_{\overline{\mathrm{V_j}}} \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} |(\mathcal{W}_t\,\psi)(p_j; r, k)|^2 \, d\varrho(k) \, d\ell^d(r) &\overset{\text{L. 6.2}}{=} \sum_{j=1}^{N_d} \sum_{\alpha_1 \in \mathcal{D}_t^{\mathcal{R}}(p_j)} \int\limits_{\overline{\mathrm{V_j}}} |\psi(\alpha_1 + r)|^2 \, d\ell^d(r) \\
&\overset{s = \alpha_1 + r}{=} \sum_{j=1}^{N_d} \sum_{\alpha_1 \in \mathcal{D}_t(p_j)} \int\limits_{\alpha_1 + \overline{\mathrm{V_j}}} |\psi(s)|^2 \, d\ell^d(s) \\
&= \int\limits_{\bigcup_{j=1}^{N_d} \left(\bigcup_{\alpha_1 \in \mathcal{D}_t(p_j)} (\alpha_1 + \overline{\mathrm{V_j}}) \right)} |\psi(s)|^2 \, d\ell^d(s) \\
&\overset{\text{L. 5.2}}{=} \int\limits_{\mathbb{R}^d} |\psi(s)|^2 \, d\ell^d(s).
\end{aligned}$$

According to Lemma 6.3 we know that $\mathcal{W}_t$ maps to $\mathcal{H}_t$. We would like to verify that the t-Wannier transform $\mathcal{W}_t$ has a dense image in $\mathcal{H}_t$.

Assume the contrary. Hence, there is an $F := \bigoplus_{\mathrm{j}=1}^{N_d} F(p_j; r, k) \neq 0$ of $\mathcal{H}_t$ such that for each $\psi \in \mathcal{C}_{\mathrm{c}}(\mathbb{R}^d)$ with radius $\mathcal{R} > 0$ the equation

$$\begin{aligned} 0 &\overset{\text{Ass.}}{=} \langle F \mid \mathcal{W}_t \, \psi \rangle_{\mathcal{H}_t} \\ &= \sum_{j=1}^{N_d} \int_{\overline{\mathrm{V}_\mathrm{j}}} \int_{\widehat{\mathbb{L}_\mathcal{D}}} \overline{F(p_j; r, k)} \cdot (\mathcal{W}_t \, \psi)(p_j; r, k) \; d\varrho(k) \; d\ell^d(r) \\ &\overset{\text{L. 6.2}}{=} \sum_{j=1}^{N_d} \sum_{\alpha \in \mathcal{D}_t^{\mathcal{R}}(p_j)} \int_{\overline{\mathrm{V}_\mathrm{j}}} \int_{\widehat{\mathbb{L}_\mathcal{D}}} \overline{F(p_j; r, k)} \cdot \psi(\alpha + r) \cdot f_\alpha(k) \; d\varrho(k) \; d\ell^d(r) \end{aligned}$$

holds. Suppose that $supp(\psi) \subseteq \alpha + \overline{\mathrm{V}_\mathrm{j}}$ for some $p_j \in \mathcal{P}^{(d)}$ and $\alpha \in \mathcal{D}_t(p_j)$. Thus,

$$\begin{aligned} 0 &= \int_{\overline{\mathrm{V}_\mathrm{j}}} \int_{\widehat{\mathbb{L}_\mathcal{D}}} \overline{F(p_j; r, k)} \cdot \psi(\alpha + r) \cdot f_\alpha(k) \; d\varrho(k) \; d\ell^d(r) \\ &= \int_{\overline{\mathrm{V}_\mathrm{j}}} \psi(\alpha + r) \cdot \int_{\widehat{\mathbb{L}_\mathcal{D}}} \overline{F(p_j; r, k)} \cdot f_\alpha(k) \; d\varrho(k) \; d\ell^d(r). \end{aligned}$$

Since this equality is true for each $\psi \in \mathcal{C}_{\mathrm{c}}(\mathbb{R}^d)$, where the support of ψ lies in $\alpha + \overline{\mathrm{V}_\mathrm{j}}$, we get for almost every $r \in \overline{\mathrm{V}_\mathrm{j}}$ that

$$0 = \int_{\widehat{\mathbb{L}_\mathcal{D}}} \overline{F(p_j; r, k)} \cdot f_\alpha(k) \; d\varrho(k).$$

Moreover, this works for each $\alpha \in \mathcal{D}_t(p_j)$. Because $\{f_\alpha \mid \alpha \in \mathcal{D}_t(p_j)\}$ is an orthonormal basis of $\mathcal{E}_t(p_j)$ it follows

$$F(p_j; \cdot, \cdot) \equiv 0,$$

for $p_j \in \mathcal{P}^{(d)}$ and almost every $r \in \overline{\mathrm{V}_\mathrm{j}}$. Since we get this conclusion for any $p_j \in \mathcal{P}^{(d)}$ the map F vanishes almost everywhere, which contradicts our assumption that $F \neq 0$.

Consequently, the range of the t-Wannier transform $\mathcal{W}_t$ has to be dense in $\mathcal{H}_t$. Altogether, $\mathcal{W}_t$ maps a dense subset $\mathcal{C}_{\mathrm{c}}(\mathbb{R}^d)$ of $L^2(\mathbb{R}^d)$ onto a dense subset of $\mathcal{H}_t$ and it is an isometry. Hence, there is a unitary extension $\widetilde{\mathcal{W}_t} : L^2(\mathbb{R}^d) \to \mathcal{H}_t$ of $\mathcal{W}_t$ such that

$$\widetilde{\mathcal{W}_t} \mid_{\mathcal{C}_{\mathrm{c}}(\mathbb{R}^d)} = \mathcal{W}_t \, .$$

$\square$

6.2 Properties of the Hilbert spaces $\mathcal{H}_t$

For the next considerations we have to recall the definitions and results of section 4.2.

Proposition 6.8 ([BNM]). *For each $p_j \in \mathcal{P}^{(d)}$ we can endow the family $(\mathcal{E}_t(p_j))_{t\in\mathcal{T}}$ with a structure of a continuous field of Hilbert spaces.*

Proof. Consider some $p_j \in \mathcal{P}^{(d)}$. We define

$$\Lambda^{\mathcal{E}_{(.)}(p_j)} := \{v \in \prod_{t\in\mathcal{T}} \mathcal{E}_t(p_j) \mid \forall t \in \mathcal{T}, \exists \left(v^{(\alpha)}\right)_{\alpha\in\mathcal{D}_t(p_j)} \subset \mathcal{C}(\mathcal{T}) \text{ finite family, such that } v_t = \sum_{\alpha\in\mathcal{D}_t(p_j)} v^{(\alpha)}(t)\cdot f_\alpha\}.$$

At first we have to prove that

$$\Lambda_t^{\mathcal{E}_{(.)}(p_j)} := \{v_t \mid v \in \Lambda^{\mathcal{E}_{(.)}(p_j)}\}$$

is a dense subset of $\mathcal{E}_t(p_j)$. Let t be some fixed element of $\mathcal{T}$. Because $(f_\alpha)_{\alpha\in\mathcal{D}_t(p_j)}$ is the basis of $\mathcal{E}_t(p_j)$ it follows that any v_t is an element of $\mathcal{E}_t(p_j)$. Since $\Lambda_t^{\mathcal{E}_{(.)}(p_j)}$ contains any finite linear combination of $(f_\alpha)_{\alpha\in\mathcal{D}_t(p_j)}$ it is a dense subset of $\mathcal{E}_t(p_j)$.

Secondly, we have to verify that for any $v \in \Lambda^{\mathcal{E}_{(.)}(p_j)}$ the map $t \in \mathcal{T} \mapsto \|v_t\|_{L^2(\widehat{\mathbb{L}_\mathcal{D}})}$ is continuous. Consider some $v \in \Lambda^{\mathcal{E}_{(.)}(p_j)}$ and so for each $t \in \mathcal{T}$ we have a corresponding finite family $\mathcal{F} := \left(v^{(\alpha)}\right)_{\alpha\in\mathcal{D}_t(p_j)}$ of continuous functions on $\mathcal{T}$ such that

$$v_t = \sum_{\alpha\in\mathcal{D}_t(p_j)} v^{(\alpha)}(t)\cdot f_\alpha.$$

Thus,

$$\begin{aligned}
\|v_t\|^2_{L^2(\widehat{\mathbb{L}_\mathcal{D}})} &= \int_{\widehat{\mathbb{L}_\mathcal{D}}} \overline{v_t(k)}\cdot v_t(k)\, d\varrho(k) \\
&\underset{\text{finite}}{\overset{\mathcal{F}}{=}} \sum_{\alpha_1\in\mathcal{D}_t(p_j)} \sum_{\alpha_2\in\mathcal{D}_t(p_j)} \overline{v^{(\alpha_1)}(t)}\cdot v^{(\alpha_2)}(t)\cdot \underbrace{\int_{\widehat{\mathbb{L}_\mathcal{D}}} \overline{f_{\alpha_1}(k)}\cdot f_{\alpha_2}(k)\, d\varrho(k)}_{=\delta_{\alpha_1,\alpha_2} \text{ by P. 3.15}} \\
&= \sum_{\alpha_1\in\mathcal{D}_t(p_j)} |v^{(\alpha_1)}(t)|^2
\end{aligned}$$

Since we have chosen a finite family $\left(v^{(\alpha)}\right)_{\alpha\in\mathcal{D}_t(p_j)} \subset \mathcal{C}(\mathcal{T})$ the sum is finite implying that $t \in \mathcal{T} \mapsto \|v_t\|_{L^2(\widehat{\mathbb{L}_\mathcal{D}})}$ is continuous. The statement follows by Proposition 4.3 $\square$

Consider now the topological groupoid $\Gamma_{\mathcal{T}} = (\Gamma^0_{\mathcal{T}}, \Gamma^1_{\mathcal{T}})$ described in Lemma 4.2. Let $\gamma = (t, \beta) \in \Gamma^1_{\mathcal{T}}$ be an arrow of the groupoid $\Gamma_{\mathcal{T}}$. We define for $p_j \in \mathcal{P}^{(d)}$ the linear operator

$$\mathfrak{O}_{p_j}(\gamma) : \mathcal{E}_{\tau^{-\beta}t}(p_j) \longrightarrow \mathcal{E}_t(p_j)$$

by

$$\mathfrak{O}_{p_j}(\gamma)\, f_\alpha = f_{\alpha-\beta}, \qquad f_\alpha \in \mathcal{E}_{\tau^{-\beta}t}(p_j).$$

In comparison to Lemma 5.6

$$\alpha \in \mathcal{D}_{\tau^{-\beta}t}(p_j) \Leftrightarrow \alpha - \beta \in \mathcal{D}_t(p_j).$$

and so

$$f_\alpha \in \mathcal{E}_{\tau^{-b}t}(p_j) \Leftrightarrow f_{\alpha-\beta} \in \mathcal{E}_t(p_j)$$

for $\beta \in \mathcal{D}_t(p_j)$. Thus, the operator $\mathfrak{O}_{p_j}(\gamma)$ is well-defined and by definition bijective.

Proposition 6.9 ([BNM])**.** *For each $p_j \in \mathcal{P}^{(d)}$ the family $(\mathfrak{O}_{p_j}(\gamma))_{\gamma\in\Gamma^1_{\mathcal{T}}}$ is a strongly continuous, unitary, bounded representation of the topological groupoid $\Gamma_{\mathcal{T}}$.*

Proof. Let $p_j \in \mathcal{P}^{(d)}$ and $\gamma = (t, \beta) \in \Gamma^1_{\mathcal{T}}$ be fixed. Note that it is sufficient to prove that the operators match at the basis of the corresponding spaces in (R2) and (R3).

(R1) Recall that $e_t = (t, 0) \in \Gamma^1_{\mathcal{T}}$ is the unit of $t \in \mathcal{T}$. Consequently,

$$\mathfrak{O}_{p_j}(e_t) : \mathcal{E}_{\tau^{-0}t}(p_j) = \mathcal{E}_t(p_j) \mapsto \mathcal{E}_t(p_j)\,.$$

The operator $\mathfrak{O}_{p_j}(\gamma)$ satisfies

$$\mathfrak{O}_{p_j}(e_t) f_\alpha = f_{\alpha-0} = f_\alpha$$

for any $\alpha \in \mathcal{D}_t(p_j)$, which leads to the statement.

(R2) For any $(\gamma_1, \gamma_2) \in \Gamma^2_{\mathcal{T}}$ we know that $\gamma_1 = (t, \beta_1)$ and $\gamma_2 = (\tau^{-\beta_1}t, \beta_2)$ for $\beta_1, \beta_2 \in \mathbb{R}^d$. Further, the composition $\gamma_1 \circ \gamma_2$ is equal to $(t, \beta_1 + \beta_2)$.

First note that

$$\begin{aligned}\mathfrak{O}_{p_j}(\gamma_1 \circ \gamma_2) &: \mathcal{E}_{\tau^{-(\beta_1+\beta_2)}t}(p_j) \mapsto \mathcal{E}_t(p_j),\\ \mathfrak{O}_{p_j}(\gamma_1) &: \mathcal{E}_{\tau^{-\beta_1}t}(p_j) \mapsto \mathcal{E}_t(p_j),\\ \mathfrak{O}_{p_j}(\gamma_2) &: \mathcal{E}_{\tau^{-\beta_2}(\tau^{-\beta_1}t)}(p_j) = \mathcal{E}_{\tau^{-(\beta_1+\beta_2)}t}(p_j) \mapsto \mathcal{E}_{\tau^{-\beta_1}t}(p_j).\end{aligned}$$

Thus, the domain and the range of $\mathfrak{O}_{p_j}(\gamma_1 \circ \gamma_2)$ and $\mathfrak{O}_{p_j}(\gamma_1) \circ \mathfrak{O}_{p_j}(\gamma_2)$ coincide. Moreover, for any $\alpha \in \mathcal{D}_{\tau^{-(\beta_1+\beta_2)}t}(p_j)$ they satisfy

$$\mathfrak{O}_{p_j}(\gamma_1 \circ \gamma_2) f_\alpha = f_{\alpha-(\beta_1+\beta_2)} = \mathfrak{O}_{p_j}(\gamma_1) f_{\alpha-\beta_2} = \mathfrak{O}_{p_j}(\gamma_1)\left(\mathfrak{O}_{p_j}(\gamma_2) f_\alpha\right).$$

(R3) For any $\gamma := (t,\beta) \in \Gamma^1_{\mathcal{T}}$ the operator $\mathfrak{O}_{p_j}(\gamma)$ is bijective and linear and so it is invertible. Consider the operator

$$\mathcal{Q}_{p_j}(\gamma) : \mathcal{E}_t(p_j) \mapsto \mathcal{E}_{\tau^{-\beta}t}(p_j),$$

where

$$\mathcal{Q}_{p_j}(\gamma)\, f_\alpha := f_{\alpha+\beta}, \qquad f_\alpha \in \mathcal{E}_t(p_j)\,.$$

Similar to $\mathfrak{O}_{p_j}(\gamma)$ it follows that this linear operator is well-defined and bijective.

Obviously, the image of $\mathfrak{O}_{p_j}(\gamma)$ and the domain of $\mathcal{Q}_{p_j}(\gamma)$ are equal and vice versa. For $\alpha \in \mathcal{D}_{\tau^{-\beta}t}(p_j)$ and $\tilde{\alpha} \in \mathcal{D}_t(p_j)$ the equations

$$\mathcal{Q}_{p_j}(\gamma)(\mathfrak{O}_{p_j}(\gamma)\, f_\alpha) = \mathcal{Q}_{p_j}(\gamma)(f_{\alpha-\beta}) = f_{(\alpha-\beta)+\beta} = f_\alpha$$

and

$$\mathfrak{O}_{p_j}(\gamma)(\mathcal{Q}_{p_j}(\gamma)\, f_{\tilde{\alpha}}) = \mathfrak{O}_{p_j}(\gamma)(f_{\tilde{\alpha}+\beta}) = f_{(\tilde{\alpha}+\beta)-\beta} = f_{\tilde{\alpha}}$$

hold. Consequently, $\mathcal{Q}_{p_j}(\gamma)$ is the inverse operator of $\mathfrak{O}_{p_j}(\gamma)$. In our further considerations we write $\mathfrak{O}^{-1}_{p_j}(\gamma)$ instead of $\mathcal{Q}_{p_j}(\gamma)$.

By Lemma 4.2 the inverse element of $\gamma = (t,\beta)$ is given by $\gamma^{-1} = (\tau^{-\beta}t, -\beta)$. Hence,

$$\mathfrak{O}_{p_j}(\gamma^{-1}) : \mathcal{E}_{\tau^{-(-\beta)}(\tau^{-\beta}t)}(p_j) = \mathcal{E}_t(p_j) \mapsto \mathcal{E}_{\tau^{-\beta}t}(p_j).$$

Furthermore, for each $f_\alpha \in \mathcal{E}_t(p_j)$ they satisfy

$$\mathfrak{O}_{p_j}(\gamma^{-1}) f_\alpha = f_{\alpha-(-\beta)} = \mathfrak{O}^{-1}_{p_j}(\gamma)\, f_\alpha.$$

Next we would like to check if for each $\gamma \in \Gamma^1_{\mathcal{T}}$ the operators $\mathfrak{O}_{p_j}(\gamma)$ are unitary. Note that the norm of a unitary operators is always equal to one and so these operators are bounded.

Because $\mathfrak{O}_{p_j}(\gamma)$ is surjective it is sufficient to show

$$\langle \mathfrak{O}_{p_j}(\gamma)\, f_{\alpha_1} \mid \mathfrak{O}_{p_j}(\gamma)\, f_{\alpha_2} \rangle_{L^2(\widehat{\mathbb{L}_\mathcal{D}})} = \langle f_{\alpha_1} \mid f_{\alpha_2} \rangle_{L^2(\widehat{\mathbb{L}_\mathcal{D}})}$$

for each $f_{\alpha_1}, f_{\alpha_2} \in \mathcal{E}_{\tau^{-\beta}t}(p_j)$.

Consider some $f_{\alpha_1}, f_{\alpha_2} \in \mathcal{E}_{\tau^{-\beta}t}(p_j)$, then,

$$\begin{aligned}
\langle \mathfrak{O}_{p_j}(\gamma)\, f_{\alpha_1} \mid \mathfrak{O}_{p_j}(\gamma)\, f_{\alpha_2}\rangle_{L^2(\widehat{\mathbb{L}_\mathcal{D}})} &= \langle f_{\alpha_1-\beta} \mid f_{\alpha_2-\beta}\rangle_{L^2(\widehat{\mathbb{L}_\mathcal{D}})} \\
&= \int_{\widehat{\mathbb{L}_\mathcal{D}}} \overline{f_{\alpha_1-\beta}(k)} \cdot f_{\alpha_2-\beta}(k)\; d\varrho(k) \\
&\overset{\text{L. 3.16 (i)}}{=} \int_{\widehat{\mathbb{L}_\mathcal{D}}} \overline{f_{\alpha_1}(k)} \cdot \underbrace{f_\beta(k) \cdot \overline{f_\beta(k)}}_{=1 \text{ see L. 3.16 (ii)}} \cdot f_{\alpha_2}(k)\; d\varrho(k) \\
&= \int_{\widehat{\mathbb{L}_\mathcal{D}}} \overline{f_{\alpha_1}(k)} \cdot f_{\alpha_2}(k)\; d\varrho(k) \\
&= \langle f_{\alpha_1} \mid f_{\alpha_2}\rangle_{L^2(\widehat{\mathbb{L}_\mathcal{D}})}.
\end{aligned}$$

We still have to verify that this unitary representation is strongly continuous. According to Proposition 4.6 we have to prove that it is weakly continuous. By Proposition 4.4 it suffices to consider two vector fields $u, v \in \Lambda^{\mathcal{E}_{(.)}(p_j)}$. By definition for any $t \in \mathcal{T}$ there are two finite families

$$\mathcal{F}_u := \left(u^{(\alpha_1)}\right)_{\alpha_1 \in \mathcal{D}_t(p_j)}, \mathcal{F}_v := \left(v^{(\alpha_2)}\right)_{\alpha_2 \in \mathcal{D}_{\tau^{-\beta}t}(p_j)} \subset \mathcal{C}(\mathcal{T})$$

such that

$$\begin{aligned}
u_{r(\gamma)} \overset{\text{L. 4.2}}{=} u_t &= \sum_{\alpha_1 \in \mathcal{D}_t(p_j)} u^{(\alpha_1)}(t) \cdot f_{\alpha_1} \\
v_{s(\gamma)} \overset{\text{L. 4.2}}{=} v_{\tau^{-\beta}t} &= \sum_{\alpha_2 \in \mathcal{D}_{\tau^{-\beta}t}(p_j)} v^{(\alpha_1)}(\tau^{-\beta}t) \cdot f_{\alpha_2}
\end{aligned}$$

Hence,

$$\begin{aligned}
\langle u_{r(\gamma)} \mid \mathfrak{O}_{p_j}(\gamma)\, v_{s(\gamma)}\rangle_{r(\gamma)} &= \langle \sum_{\alpha_1 \in \mathcal{D}_t(p_j)} u^{(\alpha_1)}(t) \cdot f_{\alpha_1} \mid \sum_{\alpha_2 \in \mathcal{D}_{\tau^{-\beta}t}(p_j)} v^{(\alpha_2)}(\tau^{-\beta}t) \cdot \mathfrak{O}_{p_j}(\gamma)\, f_{\alpha_2}\rangle_{r(\gamma)} \\
&= \sum_{\alpha_1 \in \mathcal{D}_t(p_j)} \overline{u^{(\alpha_1)}(t)} \cdot \sum_{\alpha_2 \in \mathcal{D}_{\tau^{-\beta}t}(p_j)} v^{(\alpha_2)}(\tau^{-\beta}t) \cdot \underbrace{\langle f_{\alpha_1} \mid f_{\alpha_2-\beta}\rangle_{r(\gamma)}}_{=\delta_{\alpha_1,\alpha_2-\beta}} \\
&\overset{\tilde\alpha=\alpha_2-\beta}{=} \sum_{\alpha_1 \in \mathcal{D}_t(p_j)} \overline{u^{(\alpha_1)}(t)} \cdot \sum_{\tilde\alpha+\beta \in \mathcal{D}_{\tau^{-\beta}t}(p_j)} v^{(\tilde\alpha+\beta)}(\tau^{-\beta}t) \cdot \delta_{\alpha_1,\tilde\alpha} \\
&\overset{\text{L. 5.6}}{=} \sum_{\alpha_1 \in \mathcal{D}_t(p_j)} \overline{u^{(\alpha_1)}(t)} \cdot \sum_{\tilde\alpha \in \mathcal{D}_t(p_j)} v^{(\tilde\alpha+\beta)}(\tau^{-\beta}t) \cdot \delta_{\alpha_1,\tilde\alpha} \\
&= \sum_{\alpha_1 \in \mathcal{D}_t(p_j)} \overline{u^{(\alpha_1)}(t)} \cdot v^{(\alpha_1+\beta)}(\tau^{-\beta}t).
\end{aligned}$$

Thus, $\langle v^{(1)}_{r(\gamma)} \mid \mathfrak{O}_{p_j}(\gamma)\, v^{(2)}_{s(\gamma)}\rangle_{r(\gamma)}$ is a finite sum of continuous functions with respect to $\gamma \in \Gamma^1_\mathcal{T}$. Therefore, recall that the associated shift map τ of measures is continuous as well. This implies that the unitary representation is weakly continuous. □

Actually this unitary representation is continuous, see Proposition 4.7.

Theorem 6.10. *We can endow the family $(\mathcal{H}_t)_{t\in\mathcal{T}}$ with a structure of a continuous field of Hilbert spaces. Moreover, the family*

$$\left(U(\gamma) := \bigoplus_{j=1}^{N_d} \mathbb{1}_{L^2(\overline{\mathrm{V_j}})} \otimes \mathfrak{D}_{p_j}(\gamma)\right)_{\gamma\in\Gamma^1_{\mathcal{T}}}$$

defines a strongly continuous, unitary representation of the groupoid $\Gamma_{\mathcal{T}}$.

Proof. By definition of the direct sum it suffices to prove that $\left(\mathcal{H}_t^{(p_j)}\right)_{t\in\mathcal{T}}$ is endowed with a structure of a continuous field of Hilbert space for each $p_j \in \mathcal{P}^{(d)}$. Similarly, it is sufficient to verify that the family

$$\left(U^{(p_j)}(\gamma) := \mathbb{1}_{L^2(\overline{\mathrm{V_j}})} \otimes \mathfrak{D}_{p_j}(\gamma)\right)_{\gamma\in\Gamma^1_{\mathcal{T}}}$$

defines a strongly continuous, unitary representation of the groupoid $\Gamma_{\mathcal{T}}$.

First we will show that $\left(\mathcal{H}_t^{(p_j)}\right)_{t\in\mathcal{T}}$ can be endowed with a structure of a continuous field of Hilbert spaces. Define

$$\Lambda^{\mathcal{H}_{(.)}^{(p_j)}} := L^2(\overline{\mathrm{V_j}}) \otimes \Lambda^{\mathcal{E}_{(.)}(p_j)}$$

where $\Lambda^{\mathcal{E}_{(.)}(p_j)}$ is defined as in the proof of Proposition 6.8. Thus,

$$\Lambda_t^{\mathcal{H}_{(.)}^{(p_j)}} := \{v_t \mid v \in \Lambda^{\mathcal{H}_{(.)}^{(p_j)}}\}$$

is a dense subset of $\mathcal{H}_t^{(p_j)}$.

Further, for some $v \in \Lambda^{\mathcal{H}_{(.)}^{(p_j)}}$ there is a finite family $\left(v^{(\alpha)}\right)_{\alpha\in\mathcal{D}_t(p_j)}$ of continuous functions on $\mathcal{T}$ and a $\psi \in L^2(\overline{\mathrm{V_j}})$ such that

$$v_t = \psi \otimes \left(\sum_{\alpha\in\mathcal{D}_t(p_j)} v^{(\alpha)}(t)\cdot f_\alpha\right).$$

Hence,

$$\begin{aligned}\|v_t\|^2_{L^2(\overline{\mathrm{V_j}})\otimes\mathcal{E}_t(p_j)} &\overset{\text{St. A.13}}{=} \|\psi\|^2_{L^2(\overline{\mathrm{V_j}})}\cdot \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} \overline{\sum_{\alpha_1\in\mathcal{D}_t(p_j)} v^{(\alpha_1)}(t)\cdot f_{\alpha_1}(k)}\cdot \sum_{\alpha_2\in\mathcal{D}_t(p_j)} v^{(\alpha_2)}(t)\cdot f_{\alpha_2}(k)\, d\varrho(k)\\ &\underset{\text{P. 6.8}}{\overset{\text{see proof of}}{=}} \|\psi\|^2_{L^2(\overline{\mathrm{V_j}})}\cdot \sum_{\alpha_1\in\mathcal{D}_t(p_j)} |v^{(\alpha_1)}(t)|^2.\end{aligned}$$

Since we have chosen a finite family $\left(v^{(\alpha)}\right)_{\alpha\in\mathcal{D}_t(p_j)} \subset \mathcal{C}(\mathcal{T})$ and $\|\psi\|^2_{L^2(\overline{\mathrm{V}_\mathrm{j}})} < \infty$ the map

$$t \in \mathcal{T} \mapsto \|v_t\|_{L^2(\overline{\mathrm{V}_\mathrm{j}})\otimes\mathcal{E}_t(p_j)}$$

is continuous. According to Proposition 4.3 the family $\left(\mathcal{H}_t^{(p_j)}\right)_{t\in\mathcal{T}}$ can be endowed with a structure of a continuous field of Hilbert spaces.

Secondly, we will prove that $\left(U^{(p_j)}(\gamma)\right)_{\gamma\in\Gamma^1_{\mathcal{T}}}$ defines a strongly continuous unitary representation on $\Gamma_{\mathcal{T}}$. Let $\gamma = (t,\beta) \in \Gamma^1_{\mathcal{T}}$ be fixed but arbitrary. The proof works similarly to the proof of Proposition 6.9.

(R1) Consider the unit $e_t = (t,0) \in \Gamma^1_{\mathcal{T}}$ of $t \in \mathcal{T}$. Then,

$$\begin{aligned} U^{(p_j)}(e_t) &= \mathbb{1}_{L^2(\overline{\mathrm{V}_\mathrm{j}})} \otimes \mathfrak{O}_{p_j}(e_t) \\ &\overset{\text{P. 6.9}}{=} \mathbb{1}_{L^2(\overline{\mathrm{V}_\mathrm{j}})} \otimes id_{\mathcal{E}_t(p_j)} \\ &\overset{\text{L. A.12}}{=} id_{L^2(\overline{\mathrm{V}_\mathrm{j}})\otimes\mathcal{E}_t(p_j)}. \end{aligned}$$

(R2) For $(\gamma_1,\gamma_2) \in \Gamma^2_{\mathcal{T}}$ the operator $U^{(p_j)}$ satisfies

$$\begin{aligned} U^{(p_j)}(\gamma_1\circ\gamma_2) &= \mathbb{1}_{L^2(\overline{\mathrm{V}_\mathrm{j}})} \otimes \mathfrak{O}_{p_j}(\gamma_1\circ\gamma_2) \\ &\overset{\text{P. 6.9}}{=} \mathbb{1}_{L^2(\overline{\mathrm{V}_\mathrm{j}})} \otimes \left(\mathfrak{O}_{p_j}(\gamma_1)\circ\mathfrak{O}_{p_j}(\gamma_2)\right) \\ &\overset{\text{L. A.12}}{=} \left(\mathbb{1}_{L^2(\overline{\mathrm{V}_\mathrm{j}})} \otimes \mathfrak{O}_{p_j}(\gamma_1)\right) \circ \left(\mathbb{1}_{L^2(\overline{\mathrm{V}_\mathrm{j}})} \otimes \mathfrak{O}_{p_j}(\gamma_2)\right) \\ &= U^{(p_j)}(\gamma_1)\circ U^{(p_j)}(\gamma_2). \end{aligned}$$

(R3) First note that $\mathbb{1}^{-1}_{L^2(\overline{\mathrm{V}_\mathrm{j}})} = \mathbb{1}_{L^2(\overline{\mathrm{V}_\mathrm{j}})}$ is true. Hence, for $\gamma \in \Gamma^1_{\mathcal{T}}$

$$\begin{aligned} U^{(p_j)}(\gamma^{-1}) &= \mathbb{1}_{L^2(\overline{\mathrm{V}_\mathrm{j}})} \otimes \mathfrak{O}_{p_j}(\gamma^{-1}) \\ &\overset{\text{P. 6.9}}{=} \mathbb{1}^{-1}_{L^2(\overline{\mathrm{V}_\mathrm{j}})} \otimes \mathfrak{O}^{-1}_{p_j}(\gamma) \\ &\overset{\text{L. A.12}}{=} \left(\mathbb{1}_{L^2(\overline{\mathrm{V}_\mathrm{j}})} \otimes \mathfrak{O}_{p_j}(\gamma)\right)^{-1} \\ &= U^{(p_j)}(\gamma)^{-1}. \end{aligned}$$

Our next step is to verify that $U^{(p_j)}(\gamma)$ is unitary. Since $\mathbb{1}_{L^2(\overline{\mathrm{V}_\mathrm{j}})} \otimes \mathfrak{O}_{p_j}(\gamma)$ is surjective it suffices to show

$$\langle U^{(p_j)}(\gamma)F \mid U^{(p_j)}(\gamma)G\rangle_{L^2(\overline{\mathrm{V}_\mathrm{j}})\otimes\mathcal{E}_t(p_j)} = \langle F \mid G\rangle_{L^2(\overline{\mathrm{V}_\mathrm{j}})\otimes\mathcal{E}_{\tau^{-b_t}}(p_j)}$$

for any $F, G \in L^2(\overline{\mathrm{V}_\mathrm{j}}) \otimes \mathcal{E}_{\tau^{-\beta}t}(p_j)$.

Consider some fixed $F, G \in L^2(\overline{\mathrm{V_j}}) \otimes \mathcal{E}_{\tau^{-\beta}t}(p_j)$. By definition there are $\psi, \varphi \in L^2(\overline{\mathrm{V_j}})$ and $v, w \in \mathcal{E}_{\tau^{-\beta}t}(p_j)$ such that $F = \psi \otimes v$ and $G = \varphi \otimes w$. Thus,

$$\begin{aligned}
\left(U^{(p_j)}(\gamma)F\right)(r,k) &= \left(\mathbb{1}_{L^2(\overline{\mathrm{V_j}})} \otimes \mathfrak{O}_{p_j}(\gamma)\right)(\psi \otimes v) \\
&= \left(\mathbb{1}_{L^2(\overline{\mathrm{V_j}})}\, \psi \otimes \mathfrak{O}_{p_j}(\gamma)\, v\right) \\
&= \left(\psi \otimes \left(\mathfrak{O}_{p_j}(\gamma)\, v\right)\right)
\end{aligned}$$

and analogously

$$\left(U^{(p_j)}(\gamma)G\right)(r,k) = \left(\varphi \otimes \left(\mathfrak{O}_{p_j}(\gamma)\, w\right)\right)(r,k).$$

Hence,

$$\begin{aligned}
\langle U^{(p_j)}(\gamma)F \mid U^{(p_j)}(\gamma)G\rangle_{L^2(\overline{\mathrm{V_j}}) \otimes \mathcal{E}_t(p_j)} &= \langle \psi \mid \varphi\rangle_{L^2(\overline{\mathrm{V_j}})} \cdot \langle \mathfrak{O}_{p_j}(\gamma)\, v \mid \mathfrak{O}_{p_j}(\gamma)\, w\rangle_{\mathcal{E}_t(p_j)} \\
&\overset{\text{P. 6.9}}{=} \langle \psi \mid \varphi\rangle_{L^2(\overline{\mathrm{V_j}})} \cdot \langle v \mid w\rangle_{\mathcal{E}_{\tau^{-\beta}t}(p_j)} \\
&= \langle F \mid G\rangle_{L^2(\overline{\mathrm{V_j}}) \otimes \mathcal{E}_{\tau^{-\beta}t}(p_j)}.
\end{aligned}$$

Finally we will check if this unitary representation is strongly continuous. According to Proposition 4.6 we have to prove that it is weakly continuous. It is sufficient to consider $F, G \in \Lambda^{\mathcal{H}^{(p_j)}_{(.)}}$ only, see Proposition 4.4. By definition of $\Lambda^{\mathcal{H}^{(p_j)}_{(.)}}$ there are $(u^{(\alpha_1)})_{\alpha_1 \in \mathcal{D}_t(p_j)}, (v^{(\alpha_2)})_{\alpha_2 \in \mathcal{D}_t(p_j)} \subset \mathcal{C}(\mathcal{T})$ and $\psi, \varphi \in L^2(\overline{\mathrm{V_j}})$ such that

$$\begin{aligned}
F_{r(\gamma)} = F_t \quad &= \psi \otimes \sum_{\alpha_1 \in \mathcal{D}_t(p_j)} u^{(\alpha_1)}(t) \cdot f_{\alpha_1}, \\
G_{s(\gamma)} = G_{\tau^{-\beta}t} &= \varphi \otimes \sum_{\alpha_2 \in \mathcal{D}_{\tau^{-\beta}t}(p_j)} v^{(\alpha_2)}(\tau^{-\beta}t) \cdot f_{\alpha_2}.
\end{aligned}$$

Then,

$$\begin{aligned}
U^{(p_j)}(\gamma)\left(G_{s(\gamma)}\right) &= \left(\mathbb{1}_{L^2(\overline{\mathrm{V_j}})} \otimes \mathfrak{O}_{p_j}(\gamma)\right)\left(\varphi \otimes \sum_{\alpha_2 \in \mathcal{D}_{\tau^{-\beta}t}(p_j)} v^{(\alpha_2)}(\tau^{-\beta}t) \cdot f_{\alpha_2}\right) \\
&= \left(\mathbb{1}_{L^2(\overline{\mathrm{V_j}})}\, \varphi\right) \otimes \left(\sum_{\alpha_2 \in \mathcal{D}_{\tau^{-\beta}t}(p_j)} v^{(\alpha_2)}(\tau^{-\beta}t) \cdot \underbrace{\left(\mathfrak{O}_{p_j}(\gamma)\, f_{\alpha_2}\right)}_{=f_{\alpha_2-\beta}}\right) \\
&\overset{\tilde{\alpha}=\alpha_2-\beta}{=} \varphi \otimes \left(\sum_{\tilde{\alpha}+\beta \in \mathcal{D}_{\tau^{-\beta}t}(p_j)} v^{(\tilde{\alpha}+\beta)}(\tau^{-\beta}t) \cdot f_{\tilde{\alpha}}\right) \\
&\overset{\text{L. 5.6}}{=} \varphi \otimes \left(\sum_{\tilde{\alpha} \in \mathcal{D}_t(p_j)} v^{(\tilde{\alpha}+\beta)}(\tau^{-\beta}t) \cdot f_{\tilde{\alpha}}\right).
\end{aligned}$$

Consequently,

$$
\begin{aligned}
&\langle F_t \mid U^{p_j}(\gamma)(G_t)\rangle_{L^2(\overline{\mathrm{V_j}})\otimes\mathcal{E}_t(p)} \\
= \;& \underbrace{\langle \psi \mid \varphi\rangle_{L^2(\overline{\mathrm{V_j}})}}_{=:K} \cdot \langle \sum_{\alpha_1\in\mathcal{D}_t(p_j)} u^{(\alpha_1)}(t)\cdot f_{\alpha_1} \mid \sum_{\tilde\alpha\in\mathcal{D}_t(p_j)} v^{(\tilde\alpha+\beta)}(\tau^{-\beta}t)\cdot f_{\tilde\alpha}\rangle_{\mathcal{E}_t(p_j)} \\
\overset{\text{P. 3.15}}{=} & K\cdot\left(\sum_{\alpha_1\in\mathcal{D}_t(p_j)} \overline{u^{(\alpha_1)}(t)}\cdot v^{(\alpha_1+\beta)}(\tau^{-\beta}t)\right),
\end{aligned}
$$

where K is some constant and independent of $\gamma\in\Gamma^1_{\mathcal{T}}$ (Cauchy–Schwarz inequality). Likewise, in Proposition 6.9 the unitary representation is weakly continuous. □

Proposition 6.11. *For each $\gamma=(t,\beta)\in\Gamma^1_{\mathcal{T}}$ we have*

$$U(\gamma)\circ\widetilde{\mathcal{W}}_{\tau^{-\beta}t}=\widetilde{\mathcal{W}}_t\circ U_{reg}(\beta)$$

where $U_{reg}: L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ is the regular representation of $\mathbb{R}^d$. More precisely, for $\psi\in L^2(\mathbb{R}^d)$ the regular representation is defined by

$$U_{reg}(\beta)\psi=\tau^{-\beta}\psi.$$

Proof. Similar to the proof of Theorem 6.7 it is sufficient to show the equality component-by-component with respect to $\mathcal{P}^{(d)}$. Consider some function $\psi\in\mathcal{C}_{\mathrm{c}}(\mathbb{R}^d)$ and for each $p_j\in\mathcal{P}^{(d)}$ we get

$$
\begin{aligned}
U^{(p_j)}(\gamma)\left((\mathcal{W}_{\tau^{-\beta}t}\,\psi)(p_j;\cdot,\cdot)\right) &= \left(\mathbb{1}_{L^2(\overline{\mathrm{V_j}})}\otimes\mathfrak{D}_{p_j}\gamma\right)\left(\sum_{\alpha\in\mathcal{D}_{\tau^{-\beta}t}(p_j)}(\tau^{-\alpha}\psi)\otimes f_\alpha\right)\\
&= \sum_{\alpha\in\mathcal{D}_{\tau^{-\beta}t}(p_j)} \mathbb{1}_{L^2(\overline{\mathrm{V_j}})}(\tau^{-\alpha}\psi)\otimes\mathfrak{D}_{p_j}(\gamma)(f_\alpha)\\
&= \sum_{\alpha\in\mathcal{D}_{\tau^{-\beta}t}(p_j)} (\tau^{-\alpha}\psi)\otimes f_{\alpha-\beta}\\
&\overset{\tilde\alpha=\alpha-\beta}{=} \sum_{\tilde\alpha+\beta\in\mathcal{D}_{\tau^{-\beta}t}(p_j)} (\tau^{-\tilde\alpha-\beta}\psi)\otimes f_{\tilde\alpha}\\
&\overset{\text{L. 5.6}}{=} \sum_{\tilde\alpha\in\mathcal{D}_t(p_j)} (\tau^{-\tilde\alpha}(\tau^{-\beta}\psi))\otimes f_{\tilde\alpha}\\
&= \sum_{\tilde\alpha\in\mathcal{D}_t(p_j)} \tau^{-\tilde\alpha}(U_{reg}(\beta)\psi)\otimes f_{\tilde\alpha}\\
&= (\mathcal{W}_t(U_{reg}(\beta)\psi))(p_j;\cdot,\cdot).
\end{aligned}
$$

Since U and U_{reg} are continuous the statement follows for the extension $\widetilde{\mathcal{W}}_t$ of $\mathcal{W}_t$. □

Chapter 7

The Schrödinger operator

7.1 Projection on $L^2(\widehat{\mathbb{L}_{\mathcal{D}}})$

The main ideas of this chapter are based on [BNM]. We will carry out the corresponding arguments in a detailed and rigorous way.

Recall for $p_j \in \mathcal{P}^{(d)}$ and some fixed $t \in \mathcal{T}$ the definition of the closed subspaces

$$\mathcal{E}_t(p_j) := \overline{Lin\{f_\alpha \in L^2(\widehat{\mathbb{L}_{\mathcal{D}}}) \mid \alpha \in \mathcal{D}_t(p_j)\}}$$

of $L^2(\widehat{\mathbb{L}_{\mathcal{D}}})$. In this chapter we follow the considerations of [BNM]. We define for each closed subspace the *projection* $\Pi_{t,p_j} \colon L^2(\widehat{\mathbb{L}_{\mathcal{D}}}) \to \mathcal{E}_t(p_j)$ for $h \in L^2(\mathbb{R}^d)$ by

$$\Pi_{t,p_j}(h) := \sum_{\alpha \in \mathcal{D}_t(p_j)} \langle f_\alpha \mid h \rangle_{L^2(\widehat{\mathbb{L}_{\mathcal{D}}})} \cdot f_\alpha.$$

Since the family $(f_\alpha))_{\alpha \in \mathcal{D}_t(p_j)}$ forms an orthonormal basis of $\mathcal{E}_t(p_j)$ the map Π_{t,p_j} is well-defined and an orthonormal projection by definition. Consequently, it is self-adjoint and for a fixed $t \in \mathcal{T}$ and two distinct elements p_{j_1}, p_{j_2} of $\mathcal{P}^{(d)}$ the maps $\Pi_{t,p_{j_1}}$ and $\Pi_{t,p_{j_2}}$ are orthogonal, see [WER].

Lemma 7.1. *Consider some fixed $t \in \mathcal{T}$ and $p_j \in \mathcal{P}^{(d)}$. For each $\psi \in \mathcal{C}_c^1(\mathbb{R}^d)$ and $r \in \overline{\mathrm{V_j}}$ the equality*

$$\Pi_{t,p_j}\left(\nabla_r(\mathcal{W}_t\,\psi)(p_j;\cdot,\cdot)\,|_r\right) = \nabla_r\left(\mathcal{W}_t\,\psi\right)(p_j;\cdot,\cdot)\,|_r$$

is true.

Proof. By Lemma 6.6 the equation

$$\nabla_r(\mathcal{W}_t\,\psi) = \mathcal{W}_t(\nabla\psi)$$

holds. According to Lemma 6.3 the Wannier transform $(\mathcal{W}_t\,\psi)(p_j;\cdot,\cdot)$ is an element of $L^2(\overline{\mathrm{V_j}})\otimes\mathcal{E}_t(p_j)$. Hence,

$$\begin{aligned}\Pi_{t,p_j}\left((\nabla_r(\mathcal{W}_t\,\psi))(p_j;\cdot,\cdot)\,|_r\right) &= \Pi_{t,p_j}\left((\mathcal{W}_t\,\nabla\psi)(p_j;\cdot,\cdot)\,|_r\right)\\ &= (\mathcal{W}_t\,\nabla\psi)(p_j;\cdot,\cdot)\,|_r\\ &= \left(\nabla_r\,(\mathcal{W}_t\,\psi)\right)(p_j;\cdot,\cdot)\,|_r\,.\end{aligned}$$

□

7.2 Representation of the Schrödinger operator

As mentioned above the main aim in this work is to decompose the Schrödinger operator in a useful way by using the Wannier transform. Consider the Schrödinger operator

$$\mathcal{S}_t := -\frac{\hbar}{2m}\triangle + V_t$$

where

$$V_t(x) := \sum_{j=1}^{N_d}\sum_{\alpha\in\mathcal{D}_t(p_j)} v_{at}^{(p_j)}(x-\alpha)$$

and $v_{at}^{(p_j)}\in\mathcal{C}_c(\mathbb{R}^d)$.

We interpret the elements of $\mathcal{D}$ as the positions of the atoms of the material. The closed Voronoi cell describe the area of influence $v_{at}^{(p_j)}$ of an atom. In our setting we have seen that there are a finite number of different Voronoi cells. Thus, the material contains a finite number of various species of atoms only. Then, the potential V_t is a convenient description of the potential of the material.

Theorem 7.2. *Let $\mathcal{S}$ be a Schrödinger operator which is given by $(\mathcal{S}_t)_{t\in\mathcal{T}}$. Further, let $supp(v_{at}^{(p_j)})\subseteq\overline{\mathrm{V_j}}$ where V_j is the geometric support of the collared Voronoi proto d-cell p_j. Then, for each $\psi\in\mathcal{C}_c^2(\mathbb{R}^d)$ and some fixed $t\in\mathcal{T}$*

$$\langle\psi\mid\mathcal{S}_t\,\psi\rangle = \sum_{j=1}^{N_d}\int_{\overline{\mathrm{V_j}}}\int_{\widehat{\mathbb{L}_\mathcal{D}}}\frac{\hbar}{2m}\cdot|\nabla_r(\mathcal{W}_t\,\psi)(p_j;r,k)|^2 + v_{at}^{(p_j)}(r)\cdot|(\mathcal{W}_t\,\psi)(p_j;r,k)|^2\,d\varrho(k)\,d\ell^d(r).$$

Proof. Consider some $t\in\mathcal{T}$ and $\psi\in\mathcal{C}_c^2(\mathbb{R}^d)$. Let $\mathcal{R}>0$ be the corresponding radius of $\psi\in\mathcal{C}_c^2(\mathbb{R}^d)$, see Lemma 6.2. Because of the considerations in A.3 "The quadratic form of the Laplacian" we have

$$\langle\psi\mid(-\triangle)\psi\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\overline{(\nabla\psi)(x)}\cdot(\nabla\psi)(x)\,d\ell^d(x)$$

Thus,

$$\begin{aligned}
\langle \psi \mid (-\triangle)\psi \rangle_{L^2(\mathbb{R}^d)} &= \int\limits_{\mathbb{R}^d} \overline{\nabla \psi(x)} \cdot \nabla \psi(x) \, d\ell^d(x) \\
&\overset{\text{L. 5.2}}{=} \int\limits_{\bigcup_{j=1}^{N_d} \left(\bigcup_{\alpha \in \mathcal{D}_t(p_j)} (\alpha + \overline{\mathrm{V}_\mathrm{j}}) \right)} |\nabla \psi(x)|^2 \, d\ell^d(x) \\
&= \sum_{j=1}^{N_d} \sum_{\alpha \in \mathcal{D}_t(p_j)} \int\limits_{\alpha + \overline{\mathrm{V}_\mathrm{j}}} |\nabla \psi(x)|^2 \, d\ell^d(x) \\
&\overset{x = \alpha + r}{=} \sum_{j=1}^{N_d} \sum_{\alpha \in \mathcal{D}_t(p_j)} \int\limits_{\overline{\mathrm{V}_\mathrm{j}}} |\nabla \psi(\alpha + r)|^2 \, d\ell^d(r)
\end{aligned}$$

According to Proposition 6.5 for $\psi \in \mathcal{C}_c^2(\mathbb{R}^d)$, $\alpha \in \mathcal{D}_t(p_j)$ and $r \in \overline{\mathrm{V}_\mathrm{j}}$ the equation

$$\psi(\alpha + r) = \int\limits_{\widehat{\mathbb{L}_\mathcal{D}}} (\mathcal{W}_t \, \psi)(p_j; r, k) \cdot \overline{f_\alpha(k)} \, d\varrho(k)$$

holds. Hence,

$$\begin{aligned}
\nabla \psi(\alpha + r) &= \nabla \left(\int\limits_{\widehat{\mathbb{L}_\mathcal{D}}} (\mathcal{W}_t \, \psi)(p_j; r, k) \cdot \overline{f_\alpha(k)} \, d\varrho(k) \right) \\
&\overset{\text{L. 6.2}}{=} \nabla \left(\sum_{\beta \in \mathcal{D}_t^{\mathcal{R}}(p_j)} \psi(\beta + r) \cdot \int\limits_{\widehat{\mathbb{L}_\mathcal{D}}} f_\beta(k) \cdot \overline{f_\alpha(k)} \, d\varrho(k) \right) \\
&= \sum_{\beta \in \mathcal{D}_t^{\mathcal{R}}(p_j)} \nabla \psi(\beta + r) \cdot \int\limits_{\widehat{\mathbb{L}_\mathcal{D}}} f_\beta(k) \cdot \overline{f_\alpha(k)} \, d\varrho(k) \\
&\overset{\text{L. 6.2}}{=} \int\limits_{\widehat{\mathbb{L}_\mathcal{D}}} \left(\sum_{\beta \in \mathcal{D}_t^{\mathcal{R}}(p_j)} \nabla \psi(\beta + r) \cdot f_\beta(k) \right) \cdot \overline{f_\alpha(k)} \, d\varrho(k) \\
&\overset{\text{L. 6.6}}{=} \int\limits_{\widehat{\mathbb{L}_\mathcal{D}}} \nabla_r \, (\mathcal{W}_t \, \psi)(p_j; r, k) \cdot \overline{f_\alpha(k)} \, d\varrho(k).
\end{aligned}$$

Consequently, for each $p_j \in \mathcal{P}^{(d)}$ the equality

$$
\begin{aligned}
& \sum_{\alpha \in \mathcal{D}_t^{\mathcal{R}}(p_j)} \int\limits_{\mathrm{V_j}} \overline{\nabla\psi(\alpha + r)} \cdot \nabla\psi(\alpha + r)\ d\ell^d(r) \\
= & \sum_{\alpha \in \mathcal{D}_t^{\mathcal{R}}(p_j)} \int\limits_{\mathrm{V_j}} \left(\int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} \overline{\nabla_r \left(\mathcal{W}_t\, \psi\right)(p_j; r, k)} \cdot f_\alpha(k)\ d\varrho(k) \right) \\
& \qquad \cdot \left(\int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} \nabla_r \left(\mathcal{W}_t\, \psi\right)(p_j; r, \tilde{k}) \cdot \overline{f_\alpha(\tilde{k})}\ d\varrho(\tilde{k}) \right)\ d\ell^d(r) \\
= & \sum_{\alpha \in \mathcal{D}_t^{\mathcal{R}}(p_j)} \int\limits_{\mathrm{V_j}} \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} \overline{\nabla_r \left(\mathcal{W}_t\, \psi\right)(p_j; r, k)} \\
& \qquad \cdot \left(\int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} \overline{f_\alpha(\tilde{k})} \cdot \nabla_r \left(\mathcal{W}_t\, \psi\right)(p_j; r, \tilde{k})\ d\varrho(\tilde{k}) \right) \cdot f_\alpha(k)\ d\varrho(k)\ d\ell^d(r) \\
\overset{\text{L. 6.2}}{=} & \int\limits_{\mathrm{V_j}} \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} \overline{\nabla_r \left(\mathcal{W}_t\, \psi\right)(p_j; r, k)} \\
& \qquad \cdot \left(\sum_{\alpha \in \mathcal{D}_t(p_j)} \left\langle f_\alpha \mid \nabla_r \left(\mathcal{W}_t\, \psi\right)(p_j; r, \cdot) \right\rangle_{L^2(\widehat{\mathbb{L}_{\mathcal{D}}})} \cdot f_\alpha(k) \right)\ d\varrho(k)\ d\ell^d(r) \\
= & \int\limits_{\mathrm{V_j}} \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} \overline{\nabla_r \left(\mathcal{W}_t\, \psi\right)(p_j; r, k)} \cdot \left(\Pi_{t,p_j} \left(\nabla_r \left(\mathcal{W}_t\, \psi\right)(p_j; r, \cdot)\right)\right)(k)\ d\varrho(k)\ d\ell^d(r) \\
\overset{\text{L. 7.1}}{=} & \int\limits_{\mathrm{V_j}} \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} \overline{\nabla_r \left(\mathcal{W}_t\, \psi\right)(p_j; r, k)} \cdot \nabla_r \left(\mathcal{W}_t\, \psi\right)(p_j; r, k)\ d\varrho(k)\ d\ell^d(r) \\
= & \int\limits_{\mathrm{V_j}} \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} \left|\nabla_r \left(\mathcal{W}_t\, \psi\right)(p_j; r, k)\right|^2\ d\varrho(k)\ d\ell^d(r)
\end{aligned}
$$

is true. Denote this calculation by $(\star)$. Summing up, we get

$$
\langle \psi \mid (-\triangle)\psi \rangle = \sum_{j=1}^{N_d} \int\limits_{\mathrm{V_j}} \int\limits_{\widehat{\mathbb{L}_{\mathcal{D}}}} \left|\nabla_r \left(\mathcal{W}_t\, \psi\right)(p_j; r, k)\right|^2\ d\varrho(k)\ d\ell^d(r).
$$

Further, for the potential $V_t(x)$ we have

$$
\begin{aligned}
\langle \psi \mid V_t\, \psi \rangle &= \int_{\mathbb{R}^d} \overline{\psi(x)} \cdot V_t(x) \cdot \psi(x)\, d\ell^d(x) \\
&\overset{\text{L. 5.2}}{=} \sum_{j=1}^{N_d} \sum_{\alpha \in \mathcal{D}_t(p_j)} \int_{\alpha + \overline{\mathrm{V_j}}} V_t(x) \cdot |\psi(x)|^2\, d\ell^d(x) \\
&\overset{x=\alpha+r}{=} \sum_{j=1}^{N_d} \sum_{\alpha \in \mathcal{D}_t(p_j)} \int_{\overline{\mathrm{V_j}}} \underbrace{V_t(\alpha + r)}_{\substack{= v_{at}^{(p_j)}(r) \\ \text{since } supp(v_{at}^{(p_j)}) \subseteq \overline{\mathrm{V_j}}}} \cdot |\psi(\alpha + r)|^2)\, d\ell^d(x) \\
&= \sum_{j=1}^{N_d} \int_{\overline{\mathrm{V_j}}} v_{at}^{(p_j)}(r) \cdot \left(\sum_{\alpha \in \mathcal{D}_t(p_j)} |\psi(\alpha + r)|^2 \right) d\ell^d(x) \\
&\overset{\substack{\text{as in} \\ (\star)}}{=} \sum_{j=1}^{N_d} \int_{\overline{\mathrm{V_j}}} v_{at}^{(p_j)}(r) \cdot \int_{\widehat{\mathbb{L}_{\mathcal{D}}}} |\left(\mathcal{W}_t\, \psi\right)(p_j; r, k)|^2\, d\varrho(k)\, d\ell^d(x).
\end{aligned}
$$

Altogether, this leads to the assertion. $\square$

Appendix A

Backgrounds

A.1 Continuous functions with compact support

A topological space X is called *normal* if the following two assertions hold.

(N1) Let x and y be two different elements of X. Then, there exists an open neighborhood V_x of x and V_y of y such that

$$x \notin V_y \text{ and } y \notin V_x.$$

(N2) Let A and B be some closed disjoint subsets of X. Then, we can find two open neighborhoods V_A of A and V_B of B such that

$$V_A \cap V_B = \emptyset.$$

Consider a locally compact space G. We define the *one-point compactification* or *Alexandroff-compactification* by the group $\tilde{G} := G \cup \{\infty\}$ with the topology characterized by

$$\mathcal{O}(\tilde{G}) := \{U \subseteq \tilde{G} \mid U \subseteq G \text{ open or } U = \tilde{G} \setminus K \text{ for some compact } K \subseteq G\}.$$

That $\tilde{G}$ is a compact space with respect to this topology is been shown in [QUE].

Lemma A.1. *Consider some locally compact space X and its one-point compactification $\tilde{X}$. If X is Hausdorff the group $\tilde{X}$ is also Hausdorff.*

Proof. Obviously we only have to verify that for any $x \in X$ there are two disjoint, open neighborhoods U_x of x and V_∞ of ∞. Let $x \in X$ be arbitrary. Because X is locally compact there is a compact set $K \subseteq X$ and an open set $U_x \subseteq X$ such that

$$x \in U_x \subseteq K.$$

Further, by definition $V_\infty := \tilde{X} \setminus K$ is open with respect to the topology on $\tilde{X}$. Moreover,

$$V_\infty \cap U_x \subseteq \tilde{X} \setminus K \cap K = \emptyset$$

which leads to the statement. $\square$

Lemma A.2 ([QUE]). *Let X be a Hausdorff space and $K \subseteq X$ some compact set. Then, for each $x \in X \setminus K$ we can find an open neighborhood U of K and an open neighborhood V of x such that they are disjoint.*

Proof. Fix some $x \in X \setminus K$. Since X is Hausdorff there are for any $y \in K$ two open, disjoint neighborhoods U_y of y and $V_x^{(y)}$ of x. Because $(U_y)_{y \in K}$ is an open covering of the compact set K there are $y_1, \ldots, y_n \in K$ such that

$$K \subseteq \bigcup_{j=1}^{n} U_{y_j} =: U.$$

Set $V := \bigcap_{j=1}^{n} V_x^{(y_j)}$ which is an open neighborhood of x. Furthermore,

$$V \cap U = \emptyset.$$

$\square$

Lemma A.3 ([QUE]). *A compact, Hausdorff space X is normal.*

Proof. Consider two closed, disjoint sets $A, B \subseteq G$. Because X is compact the sets A and B are compact as well. According to Lemma A.2 there are for any $x \in A$ two open, disjoint neighborhoods V_x of x and U_x of B. Since $(V_x)_{x \in A}$ is an open covering of the compact set A there are $x_1, \ldots, x_n \in A$ with

$$A \subseteq \bigcup_{j=1}^{n} V_{x_j} =: V.$$

The set $U := \bigcap_{j=1}^{n} U_{x_j}$ is an open neighborhood of B and

$$U \cap V = \emptyset.$$

Hence, the space X satisfies condition (N2). Condition (N1) follows immediately by the fact that X is a Hausdorff space. $\square$

A family $(U_i)_{i \in I}$ of open subsets of X is called *point-finite* if for each $x \in X$ it is true that

$$\sharp\{U_i \mid x \in U_i, i \in I\} < \infty.$$

Lemma A.4 ([QUE]). *Consider a normal topological space X, a closed subset $F \subseteq X$ and a point-finite cover $(U_i)_{i\in I}$ of F. Then, there exists an open cover $(U'_i)_{i\in I}$ of F such that for each $i \in I$ we have $\overline{U'_i} \subseteq U_i$.*

An open covering $(U_i)_{i\in I}$ of a topological space X is called *locally finite* if for each $x \in X$ there is an open neighborhood $V \subseteq X$ of x such that

$$\sharp\{U_i \mid i \in I \text{ where } U_i \cap V \neq \emptyset\}$$

is finite.

Proposition A.5 ([QUE]). *Let X be a normal, topological space and consider some locally finite cover $(V_i)_{i\in I}$ of X. Then, there is a family $(\phi_i)_{i\in I} \subseteq \mathrm{C}_c(\mathrm{G})$ with*

$$supp(\phi_i) \subseteq V_i, \qquad i \in I$$

and

$$\sum_{i\in I} \phi_i(x) = 1.$$

Proposition A.6. *Let G be a locally compact, Hausdorff space. Consider some function $f \in \mathrm{C}_c(\mathrm{G})$ and an open set $U \subseteq G$ which contains the neutral element 0 of G. For some $n \in \mathbb{N}$ there are $x_1, \ldots, x_n$ and $\psi_1, \ldots, \psi_n \in \mathrm{C}_c(\mathrm{G})$ such that for each $j \in \{1, \ldots, n\}$ it is true that*

$$supp(\psi_j) \subseteq x_j + U$$

and

$$f = \sum_{j=1}^{n} \psi_j.$$

Proof. Denote by K the support of $f \in \mathcal{C}_c(G)$. Because of the compactness of K there are $x_1, \ldots, x_n \in K$ such that

$$K \subseteq \bigcup_{j=1}^{n} (x_j + U).$$

Now consider the one-point compactification $\tilde{G} := G \cup \{\infty\}$ of G. We set $V_j := x_j + U$ and $V_{n+1} := K^C \cup \{\infty\}$. By definition these sets are open with respect to the topology on $\tilde{G}$ and

$$G = \bigcup_{j=1}^{n+1} V_j.$$

According to Lemma A.1 and Lemma A.3 the space $\tilde{G}$ is normal. By Lemma A.4 there is an open cover $V_1', \ldots, V_{n+1}'$ of $\tilde{G}$ such that for $j \in \{1, \ldots, n+1\}$

$$\overline{V_j'} \subseteq V_j.$$

Therefore, note that $(V_j)_{j=1}^{n+1}$ is obviously a point-finite cover of $\tilde{G}$. The set $\overline{V_j'}$ denotes the closure of V_j' with respect to the topology on $\tilde{G}$. Hence, $\overline{V_j'}$ is a compact set.

According to Proposition A.5 we can find continuous functions ϕ_j' with compact support in $\tilde{V}_j$ such that

$$\sum_{j=1}^{n+1} \phi_j(x) = 1, \qquad x \in \tilde{G}.$$

By definition the restriction ϕ_j of ϕ_j' on K is still continuous. Further, for $j \in \{1, \ldots, n\}$

$$supp(\phi_j) = supp(\phi_j') \cap G \subseteq V_j' \cap G = V_j' \subseteq \overline{V_j'}.$$

Since each $\overline{V_j'}$ is compact any function ϕ_j has compact support.

Moreover, because $supp(\phi_{n+1}) \subseteq V_{n+1} = K^C \cup \{\infty\}$ the equation

$$\sum_{j=1}^{n} \phi_j(x) = 1, \qquad x \in K$$

holds. Thus,

$$f(x) = \left(\sum_{j=1}^{n} \phi_j(x) \right) \cdot f(x) = \sum_{j=1}^{n} \underbrace{\phi_j(x) \cdot f(x)}_{=:\psi_j(x)},$$

because f vanishes on K^C. Obviously, the function ψ_j is continuous and has compact support in $(x_j + U)$ which leads to the statement. $\square$

A.2 Tensor products and direct sums

The following section is inspired by [WEI] and [KR]. Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be some Hilbert spaces and

$$\mathcal{H} := \{(x_1, \ldots, x_n) \mid x_i \in \mathcal{H}_i\}$$

the set of all n-tuples. This set gets a Hilbert space with the following operation

$$(x_1, \ldots, x_n) + \lambda \cdot (y_1, \ldots, y_n) := (x_1 + \lambda \cdot y_1, \ldots, x_n + \lambda \cdot y_n)$$

and scalar product

$$\langle (x_1,\ldots,x_n) \mid (y_1,\ldots,y_n)\rangle_{\mathcal{H}} := \sum_{j=1}^{n} \langle x_j \mid y_j\rangle_{\mathcal{H}_j}$$

for some $\lambda \in \mathbb{C}$ and $(x_1,\ldots,x_n),(y_1,\ldots,y_n) \in \mathcal{H}$, see [KR]. The norm on $\mathcal{H}$ is induced by the scalar product, which means for $(x_1,\ldots,x_n) \in \mathcal{H}$ that

$$\|(x_1,\ldots,x_n)\|_{\mathcal{H}} = \left(\sum_{j=1}^{n} \|x_j\|^2_{\mathcal{H}_j}\right)^{\frac{1}{2}}.$$

This Hilbert space is called direct sum of the Hilbert spaces $\mathcal{H}_1,\ldots,\mathcal{H}_n$ and is denoted by $\bigoplus_{j=1}^{n}\mathcal{H}_j$.

Lemma A.7 ([KR])**.** *The set $\mathcal{H}'_k := \{(x_1,\ldots,x_n) \in \mathcal{H} \mid x_i = 0 \text{ for } i \neq k\}$ defines a closed subspace of $\bigoplus_{j=1}^{n}\mathcal{H}_j$ and for $i \neq k$ the subspaces $\mathcal{H}'_i$ and $\mathcal{H}'_k$ are orthogonal.*

Consider now the Hilbert spaces $\mathcal{H}_1,\ldots,\mathcal{H}_n$ and $\tilde{\mathcal{H}}_1,\ldots,\tilde{\mathcal{H}}_n$. For $j \in \{1,\ldots,n\}$ let S_j be a bounded, linear operator from $\mathcal{H}_j$ to $\tilde{\mathcal{H}}_j$. We define an operator $S : \bigoplus_{j=1}^{n}\mathcal{H}_j \to \bigoplus_{j=1}^{n}\tilde{\mathcal{H}}_j$ by

$$S(x_1,\ldots,x_n) := (Sx_1,\ldots,Sx_n), \qquad (x_1,\ldots,x_n) \in \bigoplus_{j=1}^{n}\mathcal{H}_j.$$

Lemma A.8 ([KR])**.** *The operator S is a linear, bounded operator and*

$$\|S\| = \sup\{\|S_j\| \mid j \in \{1,\ldots,n\}\}.$$

Proof. The linearity of S follows immediately by a short computation

$$\begin{aligned} S((x_1,\ldots,x_n) + \lambda\cdot(y_1,\ldots,y_n)) =& (S(x_1+\lambda\cdot y_1),\ldots,S(x_n+\lambda\cdot y_n)) \\ =& (Sx_1+\lambda\cdot Sy_1,\ldots,Sx_n+\lambda\cdot Sy_n) \\ =& (Sx_1,\ldots,Sx_n)+\lambda\cdot(Sy_1,\ldots,Sy_n) \\ =& S(x_1,\ldots,x_n)+\lambda\cdot S(y_1,\ldots,y_n). \end{aligned}$$

Set $K := \sup\{\|S_j\| \mid j \in \{1,\ldots,n\}\}$, then,

$$\begin{aligned} \|S(x_1,\ldots,x_n)\| \le& \left(\sum_{j=1}^{n} \|S_j\|^2 \cdot \|x_j\|^2_{\mathcal{H}_j}\right)^{\frac{1}{2}} \\ \le& K\cdot\|(x_1,\ldots,x_n)\|_{\mathcal{H}} \end{aligned}$$

and

$$\begin{aligned}\|S_j x_j\|_{\mathcal{H}_j} =& \|(0,\ldots 0,, S_j x_j, 0,\ldots,0\|_{\mathcal{H}} \\ =& \|S(0,\ldots,0,x_j,0,\ldots,0)\|_{\mathcal{H}} \\ \leq& \|S\|\cdot\|x_j\|_{\mathcal{H}_j}.\end{aligned}$$

□

Note that it is also possible to introduce the direct sum for some infinite family $(\mathcal{H}_j)_{j\in I}$ of Hilbert spaces.

Now we introduce the concept of a tensor product. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces over $\mathbb{C}$. Consider the set

$$L(\mathcal{H}_1\times\mathcal{H}_2) := \{\sum_{i=1}^{n}\alpha_i\cdot(f_i,g_i) \mid \alpha_i\in\mathbb{C},\ f_i\in\mathcal{H}_1, g_i\in\mathcal{H}_2, n\in\mathbb{N}\}$$

and

$$N := Lin\{v\in L(\mathcal{H}_1\times\mathcal{H}_2) \mid v = \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\cdot\beta_j\cdot(f_i,g_j)\ -1\times\left(\sum_{i=1}^{n}\alpha_i\cdot f_i,\sum_{j=1}^{m}\beta_j\cdot g_j\right), \\ \alpha_i,\beta_i\in\mathbb{C}, f_i\in\mathcal{H}_1, g_i\in\mathcal{H}_2\}.$$

Then, the quotion space $\mathcal{H}_1\otimes\mathcal{H}_2 := \overline{L(\mathcal{H}_1\times\mathcal{H}_2)/N}$ with the operation

$$\left(\sum_{i=1}^{n}\alpha_i\cdot f_i\right)\otimes\left(\sum_{j=1}^{m}\beta_j\cdot g_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\cdot\beta_j\cdot(f_i\otimes g_j)$$

and scalar product

$$\langle f_1\otimes g_1 \mid f_2\otimes g_2\rangle_{\mathcal{H}_1\otimes\mathcal{H}_2} := \langle f_1\mid f_2\rangle_{\mathcal{H}_1}\cdot\langle g_1\mid g_2\rangle_{\mathcal{H}_2}, \qquad f_1\otimes g_1, f_2\otimes g_2\in(\mathcal{H}_1\otimes\mathcal{H}_2)$$

gets a Hilbert space, see [WEI]. This Hilbert space $\mathcal{H}_1\otimes\mathcal{H}_2$ is called tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$.

Lemma A.9 ([WEI])**.** *Let $\{e_i \mid i\in I\}$ and $\{\tilde{e}_j \mid j\in J\}$ be an orthonormal basis of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then,*

$$\{e_i\otimes\tilde{e}_j \mid i\in I \text{ and } j\in J\}$$

is an orthonormal basis of $\mathcal{H}_1\otimes\mathcal{H}_2$.

In the following lemma is given an example of a tensor space.

Lemma A.10 ([KR]). *Consider two σ-finite measure spaces $(X,\mathcal{B},\mu)$ and $(X',\mathcal{B}',\mu')$ and its L^2-spaces. Then, the tensor product of these L^2-spaces can be identified with the space $\tilde{\mathcal{H}} := L^2(X\times X',\mathcal{B}\times\mathcal{B}',\mu\times\mu')$.*

More precisely, any $f\otimes g\in\mathcal{H} := L^2(X,\mathcal{B},\mu)\otimes L^2(X',\mathcal{B}',\mu')$ is characterized by $\theta_{f,g}\in L^2(X\times X',\mathcal{B}\times\mathcal{B}',\mu\times\mu')$ where

$$\theta_{f,g}(x,y) := f(x)\cdot g(y), \qquad (x,y)\in X\times X'.$$

Proof. First note that the function $\theta_{f,g}$ is complex-valued and measurable. Since

$$\begin{aligned}\int_{X\times X'} |\theta_{f,g}(x,y)|^2\, d\mu(x)\, d\mu'(y) &= \int_{X\times X'} |f(x)\cdot g(y)|^2\, d\mu(x)\, d\mu'(y)\\ &= \int_{X\times X'} \overline{f(x)\cdot g(y)}\cdot (f(x)\cdot g(y))\, d\mu(x)\, d\mu'(y)\\ &= \left(\int_X \overline{f(x)}\cdot f(x)\, d\mu(x)\right)\cdot\left(\int_{X'} \overline{g(y)}\cdot g(y)\, d\mu'(y)\right)\\ &=\|f\|_{L^2(X,\mathcal{B},\mu)}\cdot\|g\|_{L^2(X',\mathcal{B}',\mu')} < \infty.\end{aligned}$$

Thus, the function $\theta_{f,g}$ is contained in $\tilde{\mathcal{H}}$. Note that by Lemma A.9 the set

$$\{\theta_{f,g} \mid f\otimes g\in\mathcal{H}\}$$

is a dense set of $\tilde{\mathcal{H}}$.

Moreover, for $f_1\otimes g_1, f_2\otimes g_2\in\mathcal{H}$ we get

$$\begin{aligned}\langle\theta_{f_1,g_1}\mid\theta_{f_2,g_2}\rangle_{\tilde{\mathcal{H}}} &= \int_{X\times X'} \overline{\theta_{f_1,g_1}(x,y)}\cdot\theta_{f_2,g_2}(x,y)\, d\mu(x)\, d\mu'(y)\\ &= \int_{X\times X'} \overline{f_1(x)\cdot g_1(y)}\cdot f_2(x)\cdot g_2(y)\, d\mu(x)\, d\mu'(y)\\ &= \left(\int_X \overline{f_1(x}\cdot f_2(x)\, d\mu(x)\right)\cdot\left(\int_{X'} \overline{g_1(y}\cdot g_2(y)\, d\mu'(y)\right)\\ &=\langle f_1\mid f_2\rangle_{L^2(X,\mathcal{B},\mu)}\cdot\langle g_1\mid g_2\rangle_{L^2(X',\mathcal{B}',\mu')}\\ &=\langle f_1\otimes g_1\mid f_2\otimes g_2\rangle_{\mathcal{H}}.\end{aligned}$$

Thus, for $\lambda_j\in\mathbb{C}$ and $f\otimes g\in\mathcal{H}$ the equation

$$\|\sum_{j=1}^n \lambda_j\cdot\theta_{f,g}\|_{\tilde{\mathcal{H}}} = \|\sum_{j=1}^n \lambda_j\cdot(f\otimes g)\|_{\mathcal{H}}$$

holds. Hence, there is a linear map $\tilde{U}$ form $\{\theta_{f,g} \mid f \otimes g \in \mathcal{H}\}$ onto the linear span of all simple tensors which preserves the norm and

$$\tilde{U}\theta_{f,g} = f \otimes g.$$

Then, we can extend this map $\tilde{U}$ by continuity to an isomorphismus U from $\tilde{H}$ onto $\mathcal{H}$. □

Proposition A.11 ([KR]). *Consider the Hilbert spaces $\mathcal{H}_1, \tilde{\mathcal{H}}_1, \mathcal{H}_2$ and $\tilde{\mathcal{H}}_2$. For some $S \in B(\mathcal{H}_1, \tilde{\mathcal{H}}_1)$ and $T \in B(\mathcal{H}_2, \tilde{\mathcal{H}}_2)$ there is a unique operator $O \in B(\mathcal{H}_1 \otimes \mathcal{H}_2, \tilde{\mathcal{H}}_1 \otimes \tilde{\mathcal{H}}_2)$ such that for each $f \otimes g \in \mathcal{H}_1 \otimes \mathcal{H}_2$ the equation*

$$O(f \otimes g) = (Sf) \otimes (Tg)$$

is true. Furthermore, we have $\|O\| = \|S\| \cdot \|T\|$.

Because of the uniqueness we denote the operator O by $S \otimes T$. Using this proposition we get the following lemma.

Lemma A.12. *Consider the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. For some $S \in B(\mathcal{H}_1)$ and $T, R \in B(\mathcal{H}_2)$ we get the following assertions.*

(i) The equality $id_{\mathcal{H}_1 \otimes \mathcal{H}_2} = id_{\mathcal{H}_1} \otimes id_{\mathcal{H}_2}$ is valid.

(ii) If S and T are invertible it follows that $(S \otimes T)^{-1} = S^{-1} \otimes T^{-1}$.

(iii) It is true that $S \otimes (T \circ R) = (S \otimes T) \circ (id_{\mathcal{H}_1} \otimes R) = (id_{\mathcal{H}_1} \otimes T) \circ (S \otimes R)$.

Proof. It is given the proof of (i). The other proofs works similarly. Consider some $f \otimes g \in (\mathcal{H}_1 \otimes \mathcal{H}_2)$. Then,

$$(id_{\mathcal{H}_1} \otimes id_{\mathcal{H}_2})(f \otimes g) = (id_{\mathcal{H}_1} f) \otimes (id_{\mathcal{H}_2} g) = f \otimes g = id_{\mathcal{H}_1 \otimes \mathcal{H}_2}(f \otimes g).$$

□

The following statement follows by our previous considerations.

Statement A.13. *Consider some Hilbert spaces $\mathcal{H}_1 \ldots, \mathcal{H}_n$ and $\tilde{\mathcal{H}}_1, \ldots, \tilde{\mathcal{H}}_n$. We define the Hilbert space $\mathcal{H}$ by*

$$\mathcal{H} := \bigoplus_{j=1}^{n} (\mathcal{H}_j \otimes \tilde{\mathcal{H}}_j).$$

Some $F \in \mathcal{H}$ has the form

$$F = (F^{(1)} \otimes \tilde{F}^{(1)}, \ldots, F^{(n)} \otimes \tilde{F}^{(n)}).$$

Then, the scalar product on $\mathcal{H}$ is given for $F, G \in \mathcal{H}$ by

$$\langle F \mid G\rangle_{\mathcal{H}} = \sum_{j=1}^{n} \langle F^{(j)} \mid G^{(j)}\rangle_{\mathcal{H}_j} \cdot \langle \tilde{F}^{(j)} \mid \tilde{G}^{(j)}\rangle_{\tilde{\mathcal{H}}_j}.$$

A.3 The quadratic form of the Laplacian

The following considerations can be found in [DAV]. Consider the operator H_0 defined as

$$H_0\psi(x) := (-\triangle)\psi(x) := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}\psi(x)$$

with domain $D(H_0) = \mathcal{C}_c^2(\mathbb{R}^d)$. Then, integration by parts gives

$$\langle \varphi \mid (-\triangle)\psi(x)\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \overline{\nabla\varphi(x)} \cdot \nabla\psi(x)\, d\ell^d(x) =: Q_0(\varphi, \psi)$$

for all $\varphi, \psi \in \mathcal{C}_c^2(\mathbb{R}^d)$ whence $-\triangle$ is a symmetric and positive operator on $L^2(\mathbb{R}^d)$. Moreover, let $(\varphi_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_c^2(\mathbb{R}^d)$ such that $Q_0(\varphi_n - \varphi_m) \to 0$ and $\|\varphi_n\|_{L^2(\mathbb{R}^d)} \to 0$, then, clearly $Q_0(\varphi_n) \to 0$. Thus, the form Q_0 is closable, i.e. there exists a closed extension Q of Q_0. It can be shown that $D(Q)$ is equal to $W^{1,2}(\mathbb{R}^d)$ the Sobolev space of order 1 over $L^2(\mathbb{R}^d)$.

A famous result of Friedrichs extension is then: There exists a unique self-adjoint operator H in $L^2(\mathbb{R}^d)$ such that

$$\begin{aligned} H\psi(x) &= (-\triangle)\psi(x), \\ D(H) &\subseteq D(Q) \end{aligned}$$

and for each $\psi \in D(H)$ and $v \in D(Q)$ the equality

$$Q(v, \psi) = \langle v \mid (-\triangle)\psi\rangle_{L^2(\mathbb{R}^d)}$$

holds. The domain is given by

$$D(H) = \{\psi \in D(Q) \mid \exists\varphi \in L^2(\mathbb{R}^d),\ \forall v \in D(Q) \text{ is true } Q(v, f) = \langle v \mid g\rangle_{L^2(\mathbb{R}^d)}\}.$$

In the setting described above the domain $D(H)$ equals $W^{2,2}(\mathbb{R}^d)$, the Sobolev space of order 2 over $L^2(\mathbb{R}^d)$.

Altogether, we can use the characterization

$$\langle \psi \mid (-\triangle)\psi\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \overline{(\nabla\psi)(x)} \cdot (\nabla\psi)(x)\, d\ell^d(x)$$

for $\psi \in \mathcal{C}_c^2(\mathbb{R}^d)$.

A.4 The direct integral

Vector fields

The definitions and statements in this section are motivated by [DIXC] and [DIXN].

Consider some measure space $(\Sigma, \mathfrak{B}, \mu)$. A field of Hilbert spaces $(\mathcal{H}(\sigma))_{\sigma\in\Sigma}$ over Σ is a map

$$\sigma \mapsto \mathcal{H}(\sigma), \qquad \sigma \in \Sigma$$

such that $\mathcal{H}(\sigma)$ is a Hilbert space. We would like to construct a Hilbert space which we call direct integral $\int_\Sigma^\oplus \mathcal{H}(\sigma)\, d\mu$ of the family $(\mathcal{H}(\sigma))_{\sigma\in\Sigma}$ over Σ. The definition of this space works similar to the construction of $L^2(\Sigma, \mathfrak{B}, \mu)$. The motivation behind this construction is to consider difficult operators, as the multiplication operator, in some L^2-spaces.

For $\sigma \in \Sigma$ the scalar product in $\mathcal{H}(\sigma)$ is denoted by $\langle \cdot \mid \cdot \rangle_\sigma$. We write $\|\cdot\|_\sigma$ for the induced norm of the scalar product on $\mathcal{H}(\sigma)$. Recall the definitions and results of section 4.2.
Let $(\Sigma, \mathfrak{B}, \mu)$ and $(\Sigma', \mathfrak{B}', \mu')$ be two measure spaces. Consider two families $(\mathcal{H}(\sigma))_{\sigma\in\Sigma}$ and $(\mathcal{H}'(\sigma'))_{\sigma'\in\Sigma'}$ of Hilbert spaces. Suppose that they can be endowed with a structure of a continuous field of Hilbert spaces with generator S and S', respectively. Let $\Phi : (\Sigma, \mathfrak{B}) \to (\Sigma', \mathfrak{B}')$ be a Borel isomorphism which transform μ to μ'. A Φ-isomorphism between these structures of a continuous field of Hilbert spaces is a family $(\Upsilon(\sigma))_{\sigma\in\Sigma}$ such that the following assertions are true.

(i) For each $\sigma \in \Sigma$ the map $\Upsilon(\sigma)$ is an isomoprhism from $\mathcal{H}(\sigma)$ onto $\mathcal{H}'(\Phi(\sigma))$.

(ii) For a vector field $v \in S$ the map $\Upsilon(\sigma) \mapsto \Upsilon(\sigma)v_\sigma \in \mathcal{H}'(\Phi(\sigma))$ belongs to S'.

Consider some sequence $\left(v^{(n)}\right)_{n\in\mathbb{N}}$ in $\mathcal{V}(\Sigma, \mathcal{H}(.))$ such that it satisfies the following two conditions.

(F1) For each $\sigma \in \Sigma$ the set $\{v_\sigma^{(n)} \mid 1 \leq n \leq dim(\mathcal{H}(\sigma))\}$ forms an orthonormal basis in $\mathcal{H}(\sigma)$.

(F2) For any $\sigma \in \Sigma$ and $n > dim(\mathcal{H}(\sigma))$ the component $v_\sigma^{(n)}$ is equal to zero.

A sequence of vector fields which satisfies (F1) and (F2) is called *a sequence of vector fields of orthonormal basis*. The existence of such sequences is been shown in [DIXN].

Consider such a sequence of vector fields of orthonormal basis $\left(v^{(n)}\right)_{n\in\mathbb{N}}$. An element $v \in \mathcal{V}(\Sigma, \mathcal{H}(.))$ is continuous with respect to $\left(v^{(n)}\right)_{n\in\mathbb{N}}$ if

$$\sigma \mapsto \langle v_\sigma \mid v_\sigma^{(n)} \rangle_\sigma$$

is continuous for any $n \in \mathbb{N}$.

Analogously an operator field $A \in \mathfrak{O}(\Sigma, B(\mathcal{H}(.)))$ is called continuous with respect to $\left(v^{(n)}\right)_{n\in\mathbb{N}}$ if

$$\sigma \mapsto \langle A_\sigma v_\sigma^{(n)} \mid v_\sigma^{(m)} \rangle_\sigma$$

is continuous for $n, m \in \mathbb{N}$. That these conditions are sufficient is proven in [DIXN].

Statement A.14 ([DIXN])**.** *Consider a family of Hilbert spaces $(\mathcal{H}(\sigma))_{\sigma\in\Sigma}$ over Σ. Let $\left(v^{(n)}\right)_{n\in\mathbb{N}}$ be a sequence of vector fields of orthonormal basis such that*

$$\sigma \rightarrow \langle v_\sigma^{(n)} \mid v_\sigma^{(m)} \rangle_\sigma$$

is continuous. Then, there exists exactly one structure of a continuous field of Hilbert spaces such that the vector fields v_n are continuous.

Definition and properties

Now consider the vector space

$$\mathcal{L}^2(\Sigma, \mathfrak{B}, \mathcal{H}(.)) := \{v \in \mathcal{V}(\Sigma, \mathcal{H}(.)) \mid v \text{ measurable}, \int_\Sigma \|v_\sigma\|_\sigma \, d\mu < \infty\}.$$

This space is endowed with a semiscalarproduct defined by

$$\langle v \mid w \rangle := \int_\Sigma \langle v_\sigma \mid w_\sigma \rangle_\sigma \, d\mu, \qquad v, w \in \mathcal{L}^2(\Sigma, \mathfrak{B}, \mathcal{H}(.)).$$

Similar to the construction of the L^2-space we get for $N := \{v \in \mathcal{L}^2(\Sigma, \mathfrak{B}, \mathcal{H}(.)) \mid \|v\| = 0\}$ that

$$\int_\Sigma^\oplus \mathcal{H}(\sigma) \, d\mu := L^2(\Sigma, \mathfrak{B}, \mathcal{H}(.)) := \mathcal{L}^2(\Sigma, \mathfrak{B}, \mathcal{H}(.))\big/N$$

is a Hilbert space with scalar product $\langle \cdot \mid \cdot \rangle$ and induced norm $\| \cdot \| := \langle \cdot \mid \cdot \rangle^{\frac{1}{2}}$. It turns out that $\int_\Sigma^\oplus \mathcal{H}(\sigma) \, d\mu$ is a complete normed vector space. This Hilbert space is called *direct integral of the family* $(\mathcal{H}(\sigma))_{\sigma\in\Sigma}$ *over* Σ. We call the Hilbert space $\mathcal{H}(\sigma)$ the *fibers* over Σ. This definition is motivated by the imagination of attaching at any $\sigma \in \Sigma$ the Hilbert space $\mathcal{H}(\sigma)$, see Figure 10 at page 60. It turns out that the direct integral does not depends on the structure of the measure space $(\Sigma, \mathfrak{B}, \mu)$, besides, a unitary isomorphism.

Statement A.15 ([DIXN]). *Consider a sequence of vector fields of orthonormal basis* $\left(v^{(n)}\right)_{n\in\mathbb{N}}$.

(i) A vector field w *is an element of* $\int_\Sigma^{\oplus} \mathcal{H}(\sigma)\, d\mu$ *iff the functions*

$$\sigma \mapsto \langle w_\sigma \mid v_\sigma^{(n)} \rangle_\sigma$$

are square integrable and

$$\sum_{n=1}^{\infty} \int_\Sigma |\langle w_\sigma \mid v_\sigma^{(n)} \rangle_\sigma|^2 \, d\mu < \infty.$$

(ii) The scalar product of $w, u \in \int_\Sigma^{\oplus} \mathcal{H}(\sigma)\, d\mu$ *is equal to*

$$\sum_{n=1}^{\infty} \int_\Sigma \overline{\langle w_\sigma \mid v_\sigma^{(n)} \rangle_\sigma} \cdot \langle u_\sigma \mid v_\sigma^{(n)} \rangle_\sigma \, d\mu.$$

(iii) For any $w \in \int_\Sigma^{\oplus} \mathcal{H}(\sigma)\, d\mu$ *and for each* $\sigma \in \Sigma$ *we define*

$$w_\sigma^{(n)} := \sum_{j=1}^{n} \langle w_\sigma \mid v_\sigma^{(j)} \rangle_\sigma \cdot v_\sigma^{(j)}.$$

Then, $\|w - w^{(n)}\|$ *tends to zero iff* n *goes to infinity.*

List of symbols

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Selbstständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Diplomarbeit mit dem Thema

Generalized Bloch Theory for Quasicrystals

selbständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet wurden.
Es bestehen meinerseits keine Einwände die vorliegende Arbeit für die öffentliche Benutzung im Universitätsarchiv zur Verfügung zu stellen.

Ort, Datum	Siegfried Beckus