Some basic notions from point-set topology

March 1, 2019

Topological spaces

Definition 0.1. Let X be a set.

1. A *topology* on M is a set $\mathcal{O}$ whose elements are subsets of X such that:
   a) $\emptyset \in \mathcal{O}$ and $X \in \mathcal{O}$,
   b) If $A_1, ..., A_n \in \mathcal{O}$ are finitely many elements of $\mathcal{O}$ then $A_1 \cap ... \cap A_n \in \mathcal{O}$,
   c) If $(A_i)_{i \in I}$ is an arbitrary family of elements of $\mathcal{O}$ then $\bigcup_{i \in I} A_i \in \mathcal{O}$.
2. A *topological space* is a pair $(X, \mathcal{O})$ where $\mathcal{O}$ is a topology on X.

Elements of $\mathcal{O}$ are called *open subsets* of X. If B is a subset of X such that $X \setminus B$ is open then B is called a *closed subset* of X.

Example 0.2. 1. If X is an arbitrary set, then $\mathcal{O} := \{\emptyset, X\}$ is a topology on X. It is called the *trivial topology* on X.

2. If X is an arbitrary set, then $\mathcal{O} := \mathcal{P}(X) := \{Y \mid Y \subset X\}$ the power set of X is a topology on X. It is called the *discrete topology* on X. All subsets of X are open with respect to the discrete topology.

3. Let $X = \mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$, $r > 0$, we define

$$B_r(x) := \{y \in \mathbb{R}^n \mid \|x - y\| < r\},$$

where $\|.\|$ denotes the Euclidean norm on $\mathbb{R}^n$. Now, for a subset $A \subset \mathbb{R}^n$ we define

$$A \in \mathcal{O} \Leftrightarrow \text{for every } x \in A \text{ there is } r > 0 \text{ such that } B_r(x) \subset A.$$

Then $\mathcal{O}$ defines a topology on $\mathbb{R}^n$. It is called the *standard topology on* $\mathbb{R}^n$.

Remark 0.3. 1. The subsets $\emptyset = X \setminus X$ and $X = X \setminus \emptyset$ are closed and open.

2. If $B_1, ..., B_n$ are finitely many closed subsets of X then $B_1 \cup ... \cup B_n$ is closed since with $A_i := X \setminus B_i$ we have $B_1 \cup ... \cup B_n = X \setminus (A_1 \cap ... \cap A_n)$ and $A_1 \cap ... \cap A_n$ is open.

3. If $(B_i)_{i \in I}$ is an arbitrary family of closed subsets of X then $\bigcap_{i \in I} B_i$ is closed since with $A_i := X \setminus B_i$ we have $\bigcap_{i \in I} B_i = X \setminus \bigcup_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$ is open.

Definition 0.4. Let X be a set. A topology $\mathcal{O}$ on X is called *Hausdorff* iff for any two points $x, y \in X$ with $x \neq y$ there exist open subsets $A, B \in \mathcal{O}$ of X such that $x \in A$, $y \in B$ and $A \cap B = \emptyset$.

Example 0.5.
1. If X has at least two distinct points then the trivial topology on X is not Hausdorff.

2. The discrete topology on X is Hausdorff.

3. The standard topology on $\mathbb{R}^n$ is Hausdorff.

Definition 0.6. Let X be a set with a topology $\mathcal{O}$. A *basis* of the topology $\mathcal{O}$ is a subset $\mathcal{B} \subset \mathcal{O}$ such that every non-empty open subset of X can be obtained as a union of sets in $\mathcal{B}$.

Example 0.7.
1. If $\mathcal{O}$ is the trivial topology on X, then $\mathcal{B} = \{X\}$ is a basis of $\mathcal{O}$.

2. If $\mathcal{O}$ is the discrete topology on X then $\mathcal{B} = \{\{x\} \mid x \in X\}$ is a basis of $\mathcal{O}$.

3. If $\mathcal{O}$ is the standard topology on $\mathbb{R}^n$ then $\mathcal{B} = \{B_r(x) \mid x \in \mathbb{R}^n, r > 0\}$ is a basis of $\mathcal{O}$.

Definition 0.8. Let $(X, \mathcal{O}_X)$ be a topological space and let $x \in X$. A subset $Y \subset X$ is called a *neighborhood* of x, if there is $A \in \mathcal{O}_X$ such that $x \in A$ and $A \subset Y$.

Interior and closure of a subset

If a subset $Y \subset X$ is not open then we want to consider those points in Y for which Y is a neighborhood. Similarly we want to consider those points $x \in X$ such that every neighborhood of x intersects Y. This leads to the following definitions.

Definition 0.9. Let $(X, \mathcal{O})$ be a topological space and let Y be a subset of X.

1. The *interior of* Y with respect to $(X, \mathcal{O})$ is defined as

$$\mathring{Y} := \{y \in Y \mid \text{there is } A \in \mathcal{O} \text{ such that } y \in A \text{ and } A \subset Y\}.$$

2. The *closure of* Y with respect to $(X, \mathcal{O})$ is defined as

$$\overline{Y} := \{x \in X \mid \text{for every } A \in \mathcal{O} \text{ such that } x \in A \text{ we have } A \cap Y \neq \emptyset\}.$$

Example 0.10. Let $X := \{1, 2\}$ and let $Y := \{1\} \subset X$.

1. If $\mathcal{O}$ is the trivial topology on X, then $\mathring{Y} = \emptyset$ and $\overline{Y} = \{1, 2\}$.
2. If $\mathcal{O}$ is the discrete topology on X, then $\mathring{Y} = \{1\}$ and $\overline{Y} = \{1\}$.

Lemma 0.11. Let $(X, \mathcal{O})$ be a topological space and let Y be a subset of X.

1. The interior of Y is the union of all open subsets of Y

$$\mathring{Y} = \bigcup_{A \text{ open}, A \subset Y} A.$$

2. The closure of Y is the intersection of all closed supersets of Y

$$\overline{Y} = \bigcap_{B \text{ closed}, Y \subset B} B.$$

Proof. 1. For all $y \in X$ we have

$$\begin{aligned} y \in \mathring{Y} &\Leftrightarrow \text{there is } A \in \mathcal{O} \text{ such that } y \in A \text{ and } A \subset Y \\ &\Leftrightarrow y \in \bigcup_{A \text{ open}, A \subset Y} A. \end{aligned}$$

2. "$\subset$" Let $x \in \overline{Y}$ and let B be a closed subset of X such that $Y \subset B$. If we had $x \notin B$, then we would have $x \in X \setminus B =: A$. Now, A is open and since $Y \subset B$ we have $A \cap Y = \emptyset$, which is a contradiction to $x \in \overline{Y}$.
"$\supset$" Let $x \in \bigcap_{B \text{ closed}, Y \subset B} B$ and let $A \in \mathcal{O}$ such that $x \in A$. If we had $A \cap Y = \emptyset$ then we would have $Y \subset X \setminus A =: B$. Now, B is closed and since $x \in A$ we have $x \notin B$, which is a contradiction to $x \in \bigcap_{B \text{ closed}, Y \subset B} B$. □

Remark 0.12.

1. From $\mathring{Y} = \bigcup_{A \text{ open}, A \subset Y} A$ we see that $\mathring{Y}$ is a union of open sets and thus $\mathring{Y}$ is an open set. Moreover we see that $\mathring{Y}$ is the largest subset of Y that is open.
2. Similarly, from $\overline{Y} = \bigcap_{B \text{ closed}, Y \subset B} B$ we see that $\overline{Y}$ is a closed set and that it is the smallest superset of Y that is closed.
3. Let $Y \subset X$ be any subset. From Lemma 0.11 we see that Y is open if and only if $\mathring{Y} = Y$. Similarly, we see that Y is closed if and only if $\overline{Y} = Y$.

Subspace topology

If X is a set with a topology and $Y \subset X$ is a subset, then the topology of X induces a topology on Y as follows.

Lemma 0.13. Let X be a set with a topology $\mathcal{O}_X$ and let $Y \subset X$ be a subset of X. Then

$$\mathcal{O}_Y := \{A \cap Y \mid A \in \mathcal{O}_X\}$$

is a topology on Y.

Proof. We verify that properties a), b), c) from the definition of a topology are satisfied.
a) Obviously, we have $\emptyset = \emptyset \cap Y \in \mathcal{O}_Y$ and $Y = X \cap Y \in \mathcal{O}_Y$.
b) If $A_1 \cap Y, ..., A_n \cap Y \in \mathcal{O}$ are finitely many elements of $\mathcal{O}_Y$ then

$$\bigcap_{i=1}^{n}(A_i \cap Y) = \Big(\bigcap_{i=1}^{n} A_i\Big) \cap Y \in \mathcal{O}_Y.$$

c) If $(A_i \cap Y)_{i \in I}$ is an arbitrary family of elements of $\mathcal{O}_Y$ then

$$\bigcup_{i \in I}(A_i \cap Y) = \Big(\bigcup_{i \in I} A_i\Big) \cap Y \in \mathcal{O}_Y.$$

□

Definition 0.14. Let X be a set with a topology $\mathcal{O}_X$ and let $Y \subset X$ be a subset of X. The topology

$$\mathcal{O}_Y := \{A \cap Y \mid A \in \mathcal{O}_X\}$$

on Y is called the *subspace topology of Y induced by* $(X, \mathcal{O}_X)$.

By definition of the subspace topology, for any subset $B \subset Y$ we have

$$B \in \mathcal{O}_Y \Leftrightarrow \text{there is } A \in \mathcal{O}_X \text{ such that } B = A \cap Y.$$

Example 0.15. Let $X = \mathbb{R}$ equipped with the standard topology $\mathcal{O}_X$ and let $Y = [0, 2]$. Then for the subset $B = [0, 1) \subset Y$ we have $B \in \mathcal{O}_Y$ since $B = (-1, 1) \cap Y$ and $(-1, 1) \in \mathcal{O}_X$.
Note that $B \notin \mathcal{O}_X$. Therefore the statement "B is open" can be confusing and one should rather say "B is open in Y with respect to the subspace topology".

Product topology

If X and Y are sets with topologies, then we can construct a topology on the Cartesian product $X \times Y$ as follows.

Lemma 0.16. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then

$$\mathcal{O}_{X \times Y} := \{A \subset X \times Y \mid A = \bigcup_{i \in I} U_i \times V_i \text{ where } U_i \in \mathcal{O}_X \text{ and } V_i \in \mathcal{O}_Y \text{ for all } i\}$$

is a topology on $X \times Y$. It is called the *product topology* on $X \times Y$.

Proof. Again, we verify that properties a), b), c) from the definition of a topology are satisfied.
a) We have $\emptyset \in \mathcal{O}_X$, $X \in \mathcal{O}_X$ and $Y \in \mathcal{O}_Y$ and thus $\emptyset = \emptyset \times Y \in \mathcal{O}_{X \times Y}$ and $X \times Y \in \mathcal{O}_{X \times Y}$.
b) For all $U_i, U_j \subset X$ and $V_i, V_j \subset Y$ we have

$$(U_i \times V_i) \cap (U_j \times V_j) = (U_i \cap U_j) \times (V_i \cap V_j).$$

By proceeding inductively, we get that the intersection of finitely many sets in $\mathcal{O}_{X \times Y}$ is again in $\mathcal{O}_{X \times Y}$.
c) It follows immediately from the definition of $\mathcal{O}_{X \times Y}$ that the union of arbitrarily many sets in $\mathcal{O}_{X \times Y}$ is again in $\mathcal{O}_{X \times Y}$. □

Note that the product topology on $X \times Y$ has a lot more open sets than just the products $U \times V$ with $U \in \mathcal{O}_X$, $V \in \mathcal{O}_Y$. In fact, the set

$$\mathcal{B} := \{A \subset X \times Y \mid A = U \times Y \text{ where } U \in \mathcal{O}_X \text{ and } V \in \mathcal{O}_Y\}$$

is not a topology on $X \times Y$. But $\mathcal{B}$ is a basis of the product topology on $X \times Y$.

Example 0.17. Let $X = Y = \mathbb{R}$ and let $\mathcal{O}_X = \mathcal{O}_Y$ be the standard topology. The product topology is then a topology on $X \times Y = \mathbb{R}^2$. By proceeding inductively we get the product topology on $\mathbb{R}^n$ for every n. It is not difficult to show that the product topology on $\mathbb{R}^n$ coincides with the standard topology on $\mathbb{R}^n$.

Continuous maps

Definition 0.18. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be topological spaces and let $f\colon X \to Y$ be a map.

1. Let $x \in X$. f is called *continuous at x* iff for every neighborhood V of $f(x)$ in Y the preimage $f^{-1}(V)$ is a neighborhood of x in X.
2. f is called *continuous* iff f is continuous at every $x \in X$.
3. f is called a *homeomorphism* iff f is bijective and both f and f^{-1} are continuous.

It is easy to see that f is continuous if and only if for every open set V in Y the preimage $f^{-1}(V)$ is open in X.

Example 0.19. $f\colon [0, 2\pi) \to S^1 := \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$, $f(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$ is continuous and bijective but not a homeomorphism since f^{-1} is not continuous at $(1, 0)^t \in S^1$.

Definition 0.20. Let $(X, \mathcal{O})$ be a topological space.

1. A subset $\mathcal{U}$ of the power set of X is called a *cover* of X iff
$$X = \bigcup_{U \in \mathcal{U}} U.$$
2. Let $\mathcal{U}$ and $\mathcal{V}$ be covers of X. If $\mathcal{V} \subset \mathcal{U}$ then $\mathcal{V}$ is called a *subcover* of $\mathcal{U}$.
3. The topological space $(X, \mathcal{O})$ is called *compact* iff every cover of X has a finite subcover.

Remark 0.21. 1. Let $X = \mathbb{R}^n$ be equipped with the standard topology. The Theorem of Heine-Borel says: A subset $Y \subset X$ with the subspace topology is compact if and only if Y is bounded and closed in X.
2. It is not difficult to show: If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are topological spaces, $(X, \mathcal{O})$ is compact and $f\colon X \to Y$ is continuous then $f(X) \subset Y$ equipped with the subspace topology is compact.