Basic properties of vector bundles

The following text is based to a very large extent on a chapter of lecture notes on differential geometry [1] by Prof. Dr. Christian Bär. For an introduction to these topics see also the books by Conlon [3] or Lee [4].

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1 Vector bundles

Definition. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let E and M be differentiable manifolds. A smooth surjective map $\pi\colon E \to M$ is called a* real *or a* complex vector bundle *of rank N if*

(i) for all $p \in M$ the fiber $E_p := \pi^{-1}(p)$ has a structure of N-dimensional $\mathbb{K}$-vector space and

(ii) there exist an open covering $\mathcal{U}$ of M and diffeomorphisms

$$\Phi_\alpha : U_\alpha \times \mathbb{K}^N \to \pi^{-1}(U_\alpha), \quad U_\alpha \in \mathcal{U},$$

such that for all α we have $\pi \circ \Phi_\alpha = \mathrm{pr}_{U_\alpha}$ and for all $a, b \in \mathbb{K}$ and all $v, w \in \mathbb{K}^N$

$$\Phi_\alpha(p, av + bw) = a\Phi_\alpha(p, v) + b\Phi_\alpha(p, w).$$

Remark. *Since Φ_α is a diffeomorphism the restriction $\{p\} \times \mathbb{K}^N \stackrel{\Phi_\alpha}{\to} E_p$ is bijective and thus is an isomorphism of vector spaces.*

Definition. *E is called the* total space*, M is called the* base space *and π is called the* projection map*. The maps Φ_α are called* local trivializations.

Example. *(1) The trivial vector bundle. $E = M \times \mathbb{K}^N$ and $\pi = \mathrm{pr}_M$. We get a global trivialization by putting $U_\alpha = M$ and $\Phi_\alpha = \mathrm{id}$.*

(2) The tangent bundle. $E = TM := \cup_{p\in M} T_pM$. Let $x_\alpha\colon U_\alpha \to V_\alpha \subset \mathbb{R}^n$ be a chart of M and put

$$\begin{aligned} \Phi_\alpha : \quad & U_\alpha \times \mathbb{R}^n \to \pi^{-1}(U_\alpha) = \cup_{p\in U_\alpha} T_pM \\ & (p, v) \mapsto \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\Big|_p \end{aligned}$$

(3) The Möbius band. Define $F\colon [0,2\pi] \times (-1,1) \to \mathbb{R}^3$ *by*

$$F(u,v) := \begin{pmatrix} (1 + \frac{v}{2}\cos(\frac{u}{2}))\cos(u) \\ (1 + \frac{v}{2}\cos(\frac{u}{2}))\sin(u) \\ \frac{v}{2}\sin(\frac{u}{2}) \end{pmatrix}.$$

Define $E := \mathrm{Im}(F)$ *and* $M := \{F(u,0) \mid u \in [0,2\pi]\}$. *Then* M *is diffeomorphic to the unit circle* S^1 *and* $\pi\colon E \to M$, $\pi(F(u,v)) := F(u,0)$ *is a real vector bundle of rank* 1, *since the fiber* $(-1,1)$ *over every point of* S^1 *is diffeomorphic to* $\mathbb{R}$.

Definition. *A vector bundle of rank* 1 *is also called a* line bundle.

Definition. *A* vector subbundle *of a vector bundle* E *is a submanifold* $\tilde{E} \subset E$ *such that* $\pi|_{\tilde{E}}\colon \tilde{E} \to M$ *is a vector bundle. In particular for all* $p \in M$ *the fiber* $\tilde{E}_p \subset E_p$ *is a vector subspace.*

Example. *Let* M *be a differentiable manifold and let* $S \subset M$ *be a submanifold. Then* TS *is a vector subbundle of* TM.

Definition. *Let* $\pi\colon E \to M$ *and* $\tilde{\pi}\colon \tilde{E} \to \tilde{M}$ *be two* $\mathbb{K}$*-vector bundles. A* vector bundle homomorphism F *over* f *consists of two smooth maps* $F\colon E \to \tilde{E}$ *and* $f\colon M \to \tilde{M}$ *such that the diagram*

$$\begin{array}{ccc} E & \xrightarrow{F} & \tilde{E} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ \tilde{M} & \xrightarrow{f} & \tilde{M} \end{array}$$

commutes and for all $p \in M$ *the map* $F|_{E_p}\colon E_p \to \tilde{E}_{f(p)}$ *is a vector space homomorphism.*

Example. *(1)* $E = M \times \mathbb{K}^N$, $\tilde{E} = \tilde{M} \times \mathbb{K}^{\tilde{N}}$. *Let* $\varphi\colon M \to \mathrm{Mat}(N \times \tilde{N}, \mathbb{K})$ *be smooth and* $f\colon M \to \tilde{M}$ *be smooth. Then*

$$F: \quad E \to \tilde{E}, \quad F(p,v) := (f(p), \varphi(p) \cdot v)$$

is a vector bundle homomorphism over f.

(2) If $f\colon M \to \tilde{M}$ *is smooth, then* $df\colon TM \to T\tilde{M}$ *is a vector bundle homomorphism over* f.

Definition. *Let* $\pi\colon E \to M$ *be a vector bundle. A* section *of* E *is a map* $s\colon M \to E$ *such that* $\pi \circ s = \mathrm{id}_M$.

Example.
- *The sections of the tangent bundle of* M *are the vector fields on* M.
- *Sections of the trivial bundle* $M \times \mathbb{K}^N$ *have the form*

$$s(p) = (p, \varphi(p))$$

where $\varphi\colon M \to \mathbb{K}^N$ *is smooth.*

Definition. *A vector bundle homomorphism F over f is called a* vector bundle isomorphism *if F and f are diffeomorphisms.*
Two vector bundles $\pi\colon E \to M$ and $\tilde{\pi}\colon \tilde{E} \to \tilde{M}$ are called isomorphic *if there is a vector bundle isomorphism $E \to \tilde{E}$.*
We say that a vector bundle is trivial *if it is isomorphic to the trivial vector bundle. $M \times \mathbb{K}^N$.*

Lemma 1.1. *A vector bundle $\pi\colon E \to M$ of rank N is trivial if and only if there exist N smooth sections $s_1, ..., s_N$ of E such that for every $p \in M$ the vectors $s_1(p), ..., s_N(p)$ form a basis of E_p.*

Proof. „$\Rightarrow$" Let $\pi\colon E \to M$ be trivial and let $\Phi\colon E \to M \times \mathbb{K}^N$ be a vector bundle isomorphism. Let $e_1, ..., e_N$ be a basis of $\mathbb{K}^N$. Put $s_j(p) := \Phi^{-1}(p, e_j)$, $j = 1, ..., N$.
„$\Leftarrow$" Assume that $s_1, ..., s_N$ form a basis everywhere. Define $\Phi^{-1}\colon M \times \mathbb{K}^N \to E$ by

$$\Phi^{-1}(p, v) := \sum_{j=1}^{N} v^j \cdot s_j(p).$$

□

Example. *Is the vector bundle $TM \to M$ trivial? The answer depends on M.*

- *TS^1 is trivial since*

$$s(x, y) := (-y, x)^t, \quad (x, y) \in S^1 \subset \mathbb{R}^2$$

  gives a basis of every $T_{(x,y)}S^1$.

- *By the hairy ball theorem every smooth vector field on S^2 vanishes somewhere. Therefore TS^2 is not trivial.*

Algebraic constructions for vector bundles

Whitney sum of two vector bundles

Let $\pi_1\colon E_1 \to M$ and $\pi_2\colon E_2 \to M$ be two vector bundles. Put $E := \cup_{p\in M} E_{1,p} \oplus E_{2,p}$ and $\pi\colon E \to M$ such that

$$\pi(\underbrace{E_{1,p} \oplus E_{2,p}}_{=E_p}) = \{p\}.$$

It remains to define a topology and a differentiable structure on E such that π is smooth and such that there exist local trivializations with respect to the natural vector space structure on E_p.
To this end let $x\colon U \to V \subset \mathbb{R}^n$ be a chart of M. After possibly replacing U by an open subset of U there exist local trivializations

$$\begin{aligned} \Phi_1 &: U \times \mathbb{K}^{n_1} \to \pi_1^{-1}(U) \quad \text{and} \\ \Phi_2 &: U \times \mathbb{K}^{n_2} \to \pi_2^{-1}(U). \end{aligned}$$

Define π: $U \times (\mathbb{K}^{n-1} \oplus \mathbb{K}^{n_2}) \to \pi^{-1}(U)$ by

$$\Phi(p, v \oplus w) := \underbrace{\Phi_1(p,v)}_{\in E_{1,p}} \oplus \underbrace{\Phi_2(p,w)}_{\in E_{2,p}}.$$

Define φ: $\pi^{-1}(U) \to V \times (\mathbb{K}^{n_1} \oplus \mathbb{K}^{n_2}) \overset{\text{open}}{\subset} \begin{cases} \mathbb{R}^{n+n_1+n_2}, & \mathbb{K} = \mathbb{R} \\ \mathbb{R}^{n+2n_1+2n_2}, & \mathbb{K} = \mathbb{C} \end{cases}$ by

$$\varphi(q) := (x \times \mathrm{id})(\Phi^{-1}(q)).$$

The map φ is bijective. One checks that the set

$$A := \left\{ \varphi : \pi^{-1}(U) \to V \times (\mathbb{K}^{n_1} \oplus \mathbb{K}^{n_2}) \;\middle|\; \begin{array}{l} x : U \to V \text{ chart of } M, \\ \Phi_i : U \to \mathbb{K}^{n_i} \to \pi_i^{-1}(U) \text{ local triviali-} \\ \text{zations of } E_i,\ i = 1,2 \end{array} \right\}$$

satisfies the conditions of Theorem 1.1.10 in [2]. By this theorem and by the propositions following it there is a unique topology and a unique differentiable structure on E such that the subsets $\pi^{-1}(U) \subset E$ are open and the maps φ are charts of E. Then the maps Φ are diffeomorphisms and hence local trivializations of E. The vector bundle $E := E_1 \oplus E_2$ is called the *Whitney sum* of E_1 and E_2.
In an analogous way one constructs the following vector bundles over M:

(1) *Tensor bundle.* $E_1 \otimes E_2 := \cup_{p \in M} E_{1,p} \otimes E_{2,p}$

(2) *Dual bundle.* $E^* := \cup_{p \in M} E_p^*$

(3) *Exterior product bundle.* $\bigwedge^k E := \cup_{p \in M} \bigwedge^k E_p$

(4) *Homomorphism bundle.* $\mathrm{Hom}(E_1, E_2) := E_1^* \otimes E_2$

(5) *Quotient bundle.* Let $\tilde{E} \subset E$ be a vector subbundle. Define $E/\tilde{E} := \cup_{p \in M} E_p/\tilde{E}_p$.

Example.

- $T^*M := TM^*$ *is called the* cotangent bundle *of* M.
  If x*:* $U \to V$ *is a chart of* M*, then for* $p \in U$ *the linear forms* $dx^1|_p, ..., dx^n|_p$ *form a basis of* T_p^*M*. The map*

$$\begin{aligned} \Phi : \quad & U \times \mathbb{R}^n \to \pi^{-1}(U) = \cup_{p \in U} T_p^* M \\ & (p, \omega) \mapsto \sum_{i=1}^n \omega_i dx^i|_p \end{aligned}$$

  is a local trivialization of T^*M.

- $\bigwedge^k T^*M$, $k = 0, 1, .., n$. *If* x*:* $U \to V$ *is a chart of* M*, then for* $p \in U$ *the vectors*

$$dx^{i_1}|_p \wedge ... \wedge dx^{i_k}|_p \in \wedge^k T_p^* M, \quad 1 \le i_1 < ... < i_k \le n,$$

  form a basis of $\bigwedge^k T_p^*M$*. The sections of this bundle are called* differential k-forms *or* differential forms *of degree* k *on* M.

- $\underbrace{TM \otimes ... \otimes TM}_{r \text{ times}} \otimes \underbrace{T^*M \otimes ... \otimes T^*M}_{s \text{ times}}$.
  The sections of this bundle are called (r, s)-tensor fields on M.

geometric object	*is a section of*
vector field	TM
semi-Riemannian metric	$T^*M \otimes T^*M$
Riemann curvature tensor $\mathrm{R}(\cdot,\cdot)\cdot$ $g(\mathrm{R}(\cdot,\cdot)\cdot,\cdot)$	 $T^*M \otimes T^*M \otimes T^*M \otimes TM$ $T^*M \otimes T^*M \otimes T^*M \otimes T^*M$
Ricci curvature ric Ric	 $T^*M \otimes T^*M$ $T^*M \otimes TM$
scalar curvature	*trivial line bundle*

Restriction and pullback
Let $S \subset M$ be a submanifold, let $\pi\colon E \to M$ be a vector bundle. Define

$$E|_S := \bigcup_{p \in S} E_p = \pi^{-1}(S) \quad \text{and} \quad \pi_S := \pi|_{E|_S} : E|_S \to S.$$

Then $E|_S$ is a vector bundle over S and is called the *restriction* of E to S.

Example. *Let (M, g) be a semi-Riemannian manifold and let $S \subset M$ be a semi-Riemannian submanifold. For $p \in S$ define*

$$N_pS := \{y \in T_pM \mid g(y, z) = 0 \textit{ for all } z \in T_pS\}.$$

Then $NS := \cup_{p \in S} N_pS$ is a vector bundle over S and is called the normal bundle *of S in M. Obviously we have*

$$TM|_S = TS \oplus NS.$$

Remark. *The normal bundle NS may also be defined without using a semi-Riemannian metric. Namely put*

$$NS := (TM|_S)/TS.$$

But then NS is not a vector subbundle of $TM|_S$.

Let S, M be differentiable manifolds and let $f\colon S \to M$ be a smooth map. Let $\pi\colon E \to M$ be a vector bundle. Put

$$f^*E := \bigcup_{p \in S} \underbrace{(\{p\} \times E_{f(p)})}_{=(f^*E)_p}$$

and define $\tilde{\pi}\colon f^*E \to S$ by $\tilde{\pi}(p, v) := p$. Then $\tilde{\pi}\colon f^*E \to S$ is a vector bundle over S and is called the *pullback bundle* of $\pi\colon E \to M$.

Remark. *Let $S \subset M$ be a submanifold and let $\pi\colon E \to M$ be a vector bundle. The restriction $E|_S$ is isomorphic to the pullback bundle f^*E, where $f\colon S \to M$ is the inclusion map.*

Local trivializations of f^*E are obtained as follows:
Let $U \subset M$ be an open subset and let $\Phi\colon U \times \mathbb{K}^N \to \pi^{-1}(U)$ be a local trivialization of E. Let $\tilde{U} \subset S$ be an open subset with $\tilde{U} \subset f^{-1}(U)$. Put

$$\tilde{\Phi}: \quad \tilde{U} \times \mathbb{K}^N \to \tilde{\pi}^{-1}(U)$$
$$\tilde{\Phi}(p, v) := (p, \Phi(f(p), v)).$$

Using Theorem 1.1.10 in [2] one obtains a topology and a differentiable structure on f^*E such that the maps $\tilde{\Phi}$ are local trivializations of f^*E.

Example. *Let $f\colon S \to M$ be a smooth map. The sections of $f^*TM \to S$ are exactly the vector fields along f.*

2 Metrics and connections on vector bundles

Definition. *Let $E \to M$ be a $\mathbb{R}$-vector bundle. A* Riemannian metric *on E is a smooth section g of $E^* \otimes E^* \to M$, such that for all $p \in M$*

$$g(p) \in (E^* \otimes E^*)_p = E_p^* \otimes E_p^* \cong \{\textit{bilinear forms on } E_p\}$$

is symmetric and positive definite. A real vector bundle with a Riemannian metric g is called a Riemannian vector bundle*.*

Proposition 2.1. *On every real vector bundle there exists a Riemannian metric.*

Proof. (a) We first assume that the vector bundle $E \to M$ is trivial. Let $\Phi\colon M \times \mathbb{R}^N \to E$ be a global trivialization. For $p \in M$ and $v, w \in E_p$ write $\Phi^{-1}(v) = (p, x)$ and $\Phi^{-1}(w) = (p, y)$ with $x, y \in \mathbb{R}^N$. In order to define a Riemannian metric on E we use the standard Euclidean scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^N$ and define

$$g(p)(v, w) := \langle x, y \rangle.$$

(b) Let $E \to M$ be a vector bundle that is not necessarily trivial. There exists an open covering $\{U_\alpha\}$ of M and local trivializations $\Phi_\alpha\colon U_\alpha \times \mathbb{R}^N \to \pi^{-1}(U_\alpha)$ (In other words: the restrictions $E|_{U_\alpha}$ are trivial vector bundles. This is expressed by saying that every vector bundle is locally trivial).
Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to the open covering $\{U_\alpha\}$, i. e. $\varphi_\alpha\colon M \to \mathbb{R}$ is smooth, $0 \le \varphi_\alpha \le 1$, $\sum_\alpha \varphi_\alpha = 1$, for every $p \in M$ we have $\varphi_\alpha(p) \neq 0$ for only finitely many α and $\operatorname{supp}(\varphi_\alpha) \subset U_\alpha$.
By part (a) we know that there exist Riemannian metrics g_α on $E|_{U_\alpha}$. For $p \in M$ we put

$$g(p) := \sum_\alpha \varphi_\alpha(p) \cdot g_\alpha(p).$$

Note that $\varphi_\alpha \cdot g_\alpha$ is defined on all of M (identically 0 on $M \setminus U_\alpha$) and is smooth. Furthermore $g(p)$ is a symmetric bilinear form on E_p and moreover it is positive definite since for all $v \neq 0$ we have

$$g(p)(v, v) = \sum_\alpha \varphi_\alpha(p) g_\alpha(p)(v, v) > 0$$

since $\varphi_\alpha(p)g_\alpha(p)(v,v) \geq 0$ for all α and > 0 for some of the α.

$\square$

Remark. *Riemannian metrics on vector bundles $E, F \to M$ induce canonical Riemannian metrics on E^*, $\bigwedge^k E$, $E \oplus F$, $E \otimes F$ and E/F (in case $F \subset E$ is a vector subbundle).*

Let V, W be finite dimensional Euclidean vector spaces with orthonormal bases $v_1, ..., v_n$ and $w_1, ..., w_m$ respectively. Then there exist Euclidean scalar products

on	V^*	$\bigwedge^k V$	$V \oplus W$	$V \otimes W$
with orthonormal basis	$v_1^*, ..., v_n^*$ dual basis	$v_{i_1} \wedge ... \wedge v_{i_k}$, $1 \leq i_1 < ... < i_k \leq n$	$v_1, ..., v_n$, $w_1, ..., w_m$	$v_i \otimes w_j$, $1 \leq i \leq n$, $1 \leq j \leq m$
dimension	n	$\binom{n}{k}$	$n+m$	$n \cdot m$

In case $W \subset V$ and $v_j = w_j$ for $j = 1, ..., m$ there is a Euclidean scalar product on V/W such that $[v_{m+1}], ..., [v_n]$ form an orthonormal basis of V/W.
On the pullback bundle f^*E of a Riemannian vector bundle E we obtain the Riemannian metric

$$g^{f^*E}(p) := g^E(f(p)).$$

Definition. *Let $E \to M$ be a $\mathbb{K}$-vector bundle. A* connection *on E is a map*

$$\nabla : \quad C^\infty(M, TM) \times C^\infty(M, E) \to C^\infty(M, E), \quad (X, s) \mapsto \nabla_X s,$$

such that the following holds:

(i) For all $s \in C^\infty(M, E)$, $X_1, X_2 \in C^\infty(M, TM)$, $f_1, f_2 \in C^\infty(M)$:

$$\nabla_{f_1 X_1 + f_2 X_2} s = f_1 \nabla_{X_1} s + f_2 \nabla_{X_2} s.$$

(ii) For all $s_1, s_2 \in C^\infty(M, E)$ and $X \in C^\infty(M, TM)$:

$$\nabla_X (s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2.$$

(iii) For all $s \in C^\infty(M, E)$, $X \in C^\infty(M, TM)$ and $f \in C^\infty(M)$:

$$\nabla_X (f \cdot s) = \partial_X f \cdot s + f \cdot \nabla_X s.$$

Remark. *If ∇ is a connection on E, then the map $(X, s) \mapsto \nabla_X s$ is $C^\infty(M)$-linear in X and $\mathbb{R}$-linear in s. Thus ∇ can be considered as a map*

$$\nabla : \quad C^\infty(E) \to C^\infty(T^*M \otimes E).$$

Definition. *Let $E \to M$ be an $\mathbb{R}$-vector bundle with a Riemannian metric g. A connection ∇ on E is called a* metric connection *if we have*

(iv) For all $X \in C^\infty(M, TM)$ and $s_1, s_2 \in C^\infty(M, E)$:

$$\partial_X g(s_1, s_2) = g(\nabla_X s_1, s_2) + g(s_1, \nabla_X s_2).$$

Proposition 2.2. *Let $E \to M$ be a Riemannian vector bundle. Then there exists a metric connection on E.*

Proof. (a) Again, we first assume that E is trivial. By Lemma 1.1 there exist smooth sections $s_1, ..., s_N \in C^\infty(M, E)$ such that for all p the vectors $s_1(p), ..., s_N(p)$ form a basis of E_p. By the Gram-Schmidt process we obtain $e_1, ..., e_N \in C^\infty(M, E)$ that form an orthonormal basis at every point. We define ∇ by

$$\nabla_X\Big(\sum_{i=1}^N f_i e_i\Big) := \sum_{i=1}^N \partial_X f_i \cdot e_i.$$

One checks that ∇ satisfies $(i) - (iii)$ and thus is a connection on E. Moreover let $s_1 = \sum_{i=1}^N f_i e_i$, $s_2 = \sum_{j=1}^N h_j e_j$. Then we have

$$\begin{aligned}
\partial_X g(s_1, s_2) &= \partial_X\Big(\sum_{i,j=1}^N f_i h_j \overbrace{g(e_i, e_j)}^{=\delta_{ij}}\Big) \\
&= \partial_X\Big(\sum_{i=1}^N f_i h_i\Big) \\
&= \sum_{i=1}^N \partial_X f_i \cdot h_i + \sum_{i=1}^N f_i \cdot \partial_X h_i
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
g(\nabla_X s_1, s_2) &= g\Big(\sum_{i=1}^N \partial_X f_i \cdot e_i, \sum_{j=1}^N h_j e_j\Big) \\
&= \sum_{i,j=1}^N \partial_X f_i \cdot h_j \cdot g(e_i, e_j) \\
&= \sum_{i=1}^N \partial_X f_i \cdot h_i
\end{aligned}$$

and in the same way one obtains $g(s_1, \nabla_X s_2) = \sum_{i=1}^N f_i \cdot \partial_X h_i$. Therefore ∇ is a metric connection.

(b) Now let E be not necessarily trivial. Let $\{U_\alpha\}$ be an open covering of M such that for every α the restriction $E|_{U_\alpha}$ is trivial. Then for every α there exists a metric connection ${}^\alpha\nabla$ on $E|_{U_\alpha}$ by part (a). Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to the open covering $\{U_\alpha\}$. For $X \in T_pM$ and $s \in C_p^\infty(E)$ we put

$$\nabla_X s := \sum_\alpha \varphi_\alpha \cdot {}^\alpha\nabla_X s.$$

Then ∇ does the job.

□

Definition. *A Riemannian metric on TM is called torsion free if for every local coordinate system $x^1, ..., x^n$ of M we have*

$$\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i \quad \text{for all } i, j \in \{1, ..., n\},$$

where we have written $\partial_i := \frac{\partial}{\partial x^i}$ for all i.

Remark. *In case $E = TM$ the Levi-Civita connection is the unique connection that is metric and torsion-free. But the condition of being metric does not determine a connection uniquely. Moreover, on a general vector bundle the condition of being torsion-free does not make sense.*

Remark. *If $E, F \to M$ are vector bundles with connections ∇^E and ∇^F respectively then these connections induce connections on the vector bundles E^*, $\bigwedge^k E$, $E \oplus F$ and $E \otimes F$:*

(a) For $\omega \in C^\infty(M, E^*)$, $s \in C^\infty(M, E)$ and $X \in C^\infty(M, TM)$ we define

$$(\nabla_X^{E^*}\omega)(s) := \partial_X(\omega(s)) - \omega(\nabla_X^E s).$$

(Then the "product rule" $\partial_X(\omega(s)) = (\nabla_X^{E^*}\omega)(s) + \omega(\nabla_X^E s)$ holds.)

(b) For $s_{i_1}, ..., s_{i_k} \in C^\infty(M, E)$ and $X \in C^\infty(M, TM)$ we define

$$\nabla_X^{\bigwedge^k E}(s_{i_1} \wedge ... \wedge s_{i_k}) := (\nabla_X^E s_{i_1}) \wedge s_{i_2} \wedge ... \wedge s_{i_k} + s_{i_1} \wedge (\nabla_X^E s_{i_2}) \wedge ... \wedge s_{i_k} + ... + s_{i_1} \wedge s_{i_2} \wedge ... \wedge (\nabla_X^E s_{i_k}).$$

(c) For $s_1 \in C^\infty(M, E)$, $s_2 \in C^\infty(M, F)$ and $X \in C^\infty(M, TM)$ we define

$$\nabla_X^{E\oplus F}(s_1 \oplus s_2) := (\nabla_X^E s_1) \oplus (\nabla_X^F s_2).$$

(d) For $s_1 \in C^\infty(M, E)$, $s_2 \in C^\infty(M, F)$ and $X \in C^\infty(M, TM)$ we define

$$\nabla_X^{E\otimes F}(s_1 \otimes s_2) := (\nabla_X^E s_1) \otimes s_2 + s_1 \otimes (\nabla_X^F s_2).$$

Remark. *If ∇^E and ∇^F are metric connections then the induced connections are metric connections with respect to the induced Riemannian metrics.*

Definition. *Let $E \to M$ be a complex vector bundle.*

(1) We denote by $\overline{E}$ the complex conjugate vector bundle (i. e. the scalar multiplications $\bar{\cdot}$ on $\overline{E}$ and $\cdot$ on E are related by $\alpha \bar{\cdot} v = \overline{\alpha} \cdot v$ for $\alpha \in \mathbb{C}$, $v \in E_p$).

(2) A Hermitian metric on a complex vector bundle $E \to M$ is a smooth section h of $E^* \otimes \overline{E}^*$ such that for all $p \in M$ the sesquilinear form $h(p)$ satisfies $h(p)(w, v) = \overline{h(p)(v, w)}$ for all $v, w \in E_p$ and $h(p)(v, v) > 0$ for all $v \neq 0$.

(3) A complex vector bundle with a Hermitian metric is called a Hermitian vector bundle.

(4) A connection ∇ on a Hermitian vector bundle $E \to M$ is called a metric connection *if for all $s_1, s_2 \in C^\infty(M, E)$ and $X \in C^\infty(M, TM)$ we have*

$$\partial_X h(s_1, s_2) = h(\nabla_X s_1, s_2) + h(s_1, \nabla_X s_2).$$

Remark. *One can show that on every complex vector bundle there exist a Hermitian metric h and a connection that is metric with respect to h.*

Connections in local coordinates

Let $E \to M$ be a $\mathbb{K}$-vector bundle and let $x\colon U \to V \subset \mathbb{R}^n$ be a chart of M. Without loss of generality we may assume that $E|_U \to U$ is trivial. Let $s_1, ..., s_N$ be smooth sections of $E|_U$ which form a basis at every point. For $i = 1, ..., n$ and $\alpha = 1, ..., N$ we write

$$\nabla_{\frac{\partial}{\partial x^i}} s_\alpha =: \sum_{\beta=1}^{N} (\Gamma_{i\alpha}^{\beta} \circ x) \cdot s_\beta.$$

This defines smooth functions $\Gamma_{i\alpha}^{\beta}\colon V \to \mathbb{K}$. They are called the *Christoffel symbols* of ∇ with respect to x and $s = (s_1, ..., s_N)$. The Christoffel symbols determine ∇, since for every $s \in C^\infty(M, E)$ and $X \in C^\infty(M, TM)$ we may write

$$s = \sum_{\alpha=1}^{N} f^\alpha s_\alpha, \quad X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$$

and we compute

$$\begin{aligned}
\nabla_X s &= \nabla_{\sum_{i=1}^n X^i \frac{\partial}{\partial x^i}} \Big(\sum_{\alpha=1}^{N} f^\alpha s_\alpha \Big) \\
&= \sum_{i=1}^{n} \sum_{\alpha=1}^{N} X^i \nabla_{\frac{\partial}{\partial x^i}} (f^\alpha s_\alpha) \\
&= \sum_{i=1}^{n} \sum_{\alpha=1}^{N} X^i \Big(\frac{\partial f^\alpha}{\partial x^i} s_\alpha + f^\alpha \sum_{\beta=1}^{n} (\Gamma_{i\alpha}^{\beta} \circ x) \cdot s_\beta \Big) \\
&= \sum_{i=1}^{n} X^i \sum_{\beta=1}^{N} \Big(\frac{\partial f^\beta}{\partial x^i} + \sum_{\alpha=1}^{N} f^\alpha (\Gamma_{i\alpha}^{\beta} \circ x) \Big) s_\beta.
\end{aligned}$$

The pullback connection

Let M, S be differentiable manifolds, let $E \to M$ be a $\mathbb{K}$-vector bundle and let $f\colon S \to M$ be a smooth map. The map $F\colon f^*E \to E$, $F(p, v) := v$ is a vector bundle homomorphism over f and an isomorphism on every fiber. In particular the following diagram commutes:

$$\begin{array}{ccc}
f^*E & \xrightarrow{F} & E \\
{\scriptstyle \tilde{\pi}} \downarrow & & \downarrow {\scriptstyle \pi} \\
S & \xrightarrow{f} & M
\end{array}$$

Proposition 2.3. *Let ∇ be a connection on E. Then there exists a unique connection ∇^{f^*E} on f^*E such that for all $s \in C^\infty(M,E)$, $p \in S$ and $X \in T_pS$ we have:*

$$\nabla^{f^*E}_X(F^{-1} \circ s \circ f) = F^{-1}(\nabla^E_{df(X)} s).$$

This connection is called the pullback connection *on f^*E.*

Proof. Uniqueness. Let $y\colon U \to V$ be a chart of M, let $s_1, ..., s_N$ be smooth sections of $E|_U$ which form a basis at every point and let $\Gamma^\beta_{i\alpha}\colon V \to \mathbb{K}$ be the corresponding Christoffel symbols. Let $x\colon \tilde{U} \to W$ be a chart of S where we assume that $\tilde{U} \subset f^{-1}(U)$. The sections $\tilde{s}_\alpha := F^{-1} \circ s_\alpha \circ f|_{\tilde{U}}$ of $f^*E|_{\tilde{U}}$ form a basis at every point of $\tilde{U}$. Let $\tilde{\Gamma}^\beta_{j\alpha}\colon W \to \mathbb{K}$ be the corresponding Christoffel symbols. Then for all $p \in \tilde{U}$ we have

$$\begin{aligned}\sum_{\beta=1}^N \tilde{\Gamma}^\beta_{j\alpha}(x(p))\tilde{s}_\beta(p) &= \nabla^{f^*E}_{\frac{\partial}{\partial x^j}(p)}\tilde{s}_\alpha = \nabla^{f^*E}_{\frac{\partial}{\partial x^j}(p)}(F^{-1} \circ s_\alpha \circ f) = F^{-1}(\nabla^E_{df(\frac{\partial}{\partial x^j}(p))} s_\alpha) \\ &= F^{-1}(\nabla^E_{\sum_{i=1}^n \frac{\partial f^i}{\partial x^j}(p)\frac{\partial}{\partial y^j}(f(p))} s_\alpha) = \sum_{i=1}^n \frac{\partial f^i}{\partial x^j}(p) F^{-1}(\nabla^E_{\frac{\partial}{\partial y^j}(f(p))} s_\alpha) \\ &= \sum_{i=1}^n \frac{\partial f^i}{\partial x^j}(p) F^{-1}\Big(\sum_{\beta=1}^N \Gamma^\beta_{j\alpha}(y(f(p)))s_\beta(f(p))\Big) \\ &= \sum_{\beta=1}^N \sum_{i=1}^n \frac{\partial f^i}{\partial x^j}(p)\Gamma^\beta_{i\alpha}(y(f(p)))\tilde{s}_\beta(p)\end{aligned}$$

and therefore

$$\tilde{\Gamma}^\beta_{j\alpha}(x(p)) = \sum_{i=1}^n \frac{\partial f^i}{\partial x^j}(p)\Gamma^\beta_{i\alpha}(y(f(p))) \tag{1}$$

Thus the Christoffel symbols $\tilde{\Gamma}^\beta_{j\alpha}$ of ∇^{f^*E} are determined by those of ∇^E.
Existence. Define the Christoffel symbols by the formula (1). One checks that this defines a connection ∇^{f^*E} as in the assertion. □

Example.

- *If $E = TM \to M$ and ∇^E is the Levi-Civita connection then ∇^{f^*E} is the covariant derivative of vector fields along smooth maps (see Chapter 2.4 in [2]).*
- *If ∇^E is any connection on $E \to M$ and $f\colon S \to M$ is a constant map then for any basis $v_1, ..., v_N$ of $E_{f(x)}$ the sections*

$$s_\alpha(p) := (p, v_\alpha)$$

are smooth and form a global trivialization of f^*E, in particular f^*E is trivial. For ∇^{f^*E} we obtain $\tilde{\Gamma}^\beta_{j\alpha} \equiv 0$ and thus

$$\nabla^{f^*E}_X\Big(\sum_{\alpha=1}^N f^\alpha s_\alpha\Big) = \sum_{\alpha=1}^N \partial_X f^\alpha \cdot s_\alpha.$$

3 Curvature of a vector bundle

Definition. *Let $E \to M$ be a $\mathbb{K}$-vector bundle with a connection ∇. The* curvature tensor *R^∇ is for $X, Y \in C^\infty(M, TM)$ and $s \in C^\infty(M, E)$ given by*

$$R^\nabla(X,Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \in C^\infty(M, E).$$

Lemma 3.1. *For $p \in U$ the value $(R^\nabla(X,Y)s)(p)$ depends on X, Y and s only via the values $X(p)$, $Y(p)$ and $s(p)$.*

Proof. We write $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$, $Y = \sum_{j=1}^n Y^j \frac{\partial}{\partial x^j}$ and $s = \sum_{\alpha=1}^N f^\alpha s_\alpha$. We compute (and use the Einstein summation convention)

$$\begin{aligned}
\nabla_X \nabla_Y s &= \nabla_X (Y^i \nabla_{\frac{\partial}{\partial x^i}} (f^\alpha s_\alpha)) \\
&= \nabla_X \Big(Y^i \Big(\frac{\partial f^\alpha}{\partial x^i} s_\alpha + f^\alpha \Gamma^\beta_{i\alpha} s_\beta \Big)\Big) \\
&= \nabla_X \Big(Y^i \Big(\frac{\partial f^\beta}{\partial x^i} + f^\alpha \Gamma^\beta_{i\alpha} \Big) s_\beta \Big) \\
&= X^j \Big[\frac{\partial Y^i}{\partial x^j} \Big(\frac{\partial f^\gamma}{\partial x^i} + f^\alpha \Gamma^\gamma_{i\alpha} \Big) + Y^i \Big(\frac{\partial^2 f^\gamma}{\partial x^j \partial x^i} + \frac{\partial f^\beta}{\partial x^j} \Gamma^\gamma_{i\beta} + f^\alpha \frac{\partial \Gamma^\gamma_{i\alpha}}{\partial x^j} \Big) \\
&\qquad + Y^i \Big(\frac{\partial f^\beta}{\partial x^i} + f^\alpha \Gamma^\beta_{i\alpha} \Big) \Gamma^\gamma_{j\beta} \Big] s_\gamma
\end{aligned}$$

When computing $\nabla_X \nabla_Y s - \nabla_Y \nabla_X s$ the terms symmetric in i, j cancel and we get

$$\begin{aligned}
\nabla_X \nabla_Y s - \nabla_Y \nabla_X s = \Big[\Big(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \Big) \Big(\frac{\partial f^\gamma}{\partial x^i} + f^\alpha \Gamma^\gamma_{i\alpha} \Big) \\
+ X^j Y^i f^\alpha \Big(\frac{\partial \Gamma^\gamma_{i\alpha}}{\partial x^j} - \frac{\partial \Gamma^\gamma_{j\alpha}}{\partial x^i} + \Gamma^\beta_{i\alpha} \Gamma^\gamma_{j\beta} - \Gamma^\beta_{j\alpha} \Gamma^\gamma_{i\beta} \Big) \Big] s_\gamma
\end{aligned}$$

and thus $\nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s = X^j Y^i f^\alpha R^\gamma_{ji\alpha} \cdot s_\gamma$, where

$$R^\gamma_{ji\alpha} = \frac{\partial \Gamma^\gamma_{i\alpha}}{\partial x^j} - \frac{\partial \Gamma^\gamma_{j\alpha}}{\partial x^i} + \Gamma^\beta_{i\alpha} \Gamma^\gamma_{j\beta} - \Gamma^\beta_{j\alpha} \Gamma^\gamma_{i\beta}.$$

□

Corollary 3.2. *The curvature tensor R^∇ is a smooth section of $T^*M \otimes T^*M \otimes E^* \otimes E$.*

Proposition 3.3. *The curvature tensor R^∇ has the following symmetries:*

(i) For all $X, Y \in C^\infty(M, TM)$: $R^\nabla(X,Y) = -R^\nabla(Y,X)$

(ii) If ∇ is a metric connection with respect to a Riemannian or Hermitian metric g, then for all $X, Y \in C^\infty(M, TM)$ and $s_1, s_2 \in C^\infty(M, E)$:

$$g(R^\nabla(X,Y)s_1, s_2) = -g(s_1, R^\nabla(X,Y)s_2).$$

Proof. (i) is clear by definition.

(ii) We have

$$
\begin{aligned}
0 &= (\partial_X \partial_Y - \partial_Y \partial_X - \partial_{[X,Y]}) g(s_1, s_2) \\
&= g(R^\nabla(X,Y)s_1, s_2) + g(s_1, R^\nabla(X,Y)s_2)
\end{aligned}
$$

where in the last step we have used that ∇ is a metric connection.

□

Corollary 3.4. *R^∇ is a smooth section of $\bigwedge^2 T^*M \otimes E^* \otimes E$.*

Proposition 3.5 (Bianchi identity)**.** *Let $E \to M$ be a $\mathbb{K}$-vector bundle with a connection ∇. Assume that M is equipped with a semi-Riemannian metric and TM with the Levi-Civita connection. Then for all $X, Y, Z \in C^\infty(M, TM)$ and for $R := R^\nabla$ we have*

$$
(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0.
$$

Proof. By definition for all $s \in C^\infty(M, E)$ we have

$$
(\nabla_X R)(Y,Z)s := \nabla_X(R(Y,Z)s) - R(\nabla_X Y, Z)s - R(Y, \nabla_X Z)s - R(Y,Z)\nabla_X s
$$

where we denote both the connection on E and the Levi-Civita connection by ∇.
Let $p \in M$, let $X, Y, Z \in T_pM$ and $e \in E_p$. We extend e to a smooth section s of E in an open neighborhood of p (this is possible since E is locally trivial). Furthermore we extend X, Y, Z to smooth vector fields X, Y, Z on an open neighborhood of p by parallel translation of X, Y, Z along the radial geodesics emanating from p. Then we have

$$
\nabla_. X|_p = \nabla_. Y|_p = \nabla_. Z|_p = 0
$$

and in particular

$$
[X,Y]|_p = \nabla_X Y|_p - \nabla_Y X|_p = 0, \quad [Y,Z]|_p = [X,Z]|_p = 0.
$$

We calculate

$$
\begin{aligned}
(\nabla_X R)(Y,Z)e &= \nabla_X(R(Y,Z)s)|_p - R(\nabla_X Y, Z)s|_p - R(Y, \nabla_X Z)s|_p - R(Y,Z)\nabla_X s|_p \\
&= (\nabla_X(\nabla_Y \nabla_Z s - \nabla_Z \nabla_Y s - \nabla_{[Y,Z]} s))|_p - R(Y,Z)\nabla_X s|_p.
\end{aligned}
$$

We conclude

$$
\begin{aligned}
&(\nabla_X R)(Y,Z)e + (\nabla_Y R)(Z,X)e + (\nabla_Z R)(X,Y)e \\
&\quad = (\nabla_X \nabla_Y \nabla_Z s - \nabla_X \nabla_Z \nabla_Y s - \nabla_X \nabla_{[Y,Z]} s - R(Y,Z)\nabla_X s \\
&\qquad + \nabla_Y \nabla_Z \nabla_X s - \nabla_Y \nabla_X \nabla_Z s - \nabla_Y \nabla_{[Z,X]} s - R(Z,X)\nabla_Y s \\
&\qquad + \nabla_Z \nabla_X \nabla_Y s - \nabla_Z \nabla_Y \nabla_X s - \nabla_Z \nabla_{[X,Y]} s - R(X,Y)\nabla_Z s)|_p \\
&\quad = \big(R(\underbrace{[Y,Z]}_{=0 \text{ at } p}, X)s + \nabla_{[[Y,Z],X]} s + R(\underbrace{[Z,X]}_{=0 \text{ at } p}, Y)s + \nabla_{[[Z,X],Y]} s \\
&\qquad + R(\underbrace{[X,Y]}_{=0 \text{ at } p}, Z)s + \nabla_{[[X,Y],Z]} s\big)|_p
\end{aligned}
$$

Since R depends only on the values of its arguments at the point p we get

$$\begin{aligned}&(\nabla_X R)(Y,Z)e + (\nabla_Y R)(Z,X)e + (\nabla_Z R)(X,Y)e \\ &\quad = \nabla_{[[Y,Z],X]+[[Z,X],Y]+[[X,Y],Z]} s|_p = 0\end{aligned}$$

by the Jacobi identity. □

Remark. *This Bianchi identity has nothing to do with the "first Bianchi identity" for the Levi-Civita connection* $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ *(see Proposition 3.1.7 in [2]). In the case of the Levi-Civita connection the above Bianchi identity is also called the "second Bianchi identity".*

Definition. *A connection* ∇ *is called* flat *if its curvature tensor satisfies* $R^\nabla = 0$.

Example.
- *Every metric connection on a Riemannian line bundle E is flat. Namely $R^\nabla(X,Y)$ is a skew-symmetric endomorphism of the* 1*-dimensional vector space E_p and thus vanishes.*
- *Every trivial vector bundle has a flat connection. Namely let $s_1, ..., s_N$ be a global trivialization and define*

$$\nabla_X\Big(\sum_{\alpha=1}^N f^\alpha s_\alpha\Big) := \sum_{\alpha=1}^N \partial_X f^\alpha \cdot s_\alpha.$$

By definition of the Lie bracket we get $R^\nabla = 0$.

Orientation and the Hodge star operator

Let V be an n-dimensional real vector space with a non-degenerate symmetric bilinear form g: $V \times V \to \mathbb{R}$ of *index* s (i. e. s is the maximal dimension of a negative definite subspace). Then $\bigwedge^p V$ has a non-degenerate symmetric bilinear form $g^{\bigwedge^p V}$: $\bigwedge^p V \times \bigwedge^p V \to \mathbb{R}$ characterized by the following:

Let $e_1, ..., e_n$ be a generalized orthonormal basis of V such that $g(e_i, e_j) = \varepsilon_i \delta_{ij}$ with $\varepsilon_i = \pm 1$. Then the vectors $e_{i_1} \wedge ... \wedge e_{i_p}$, $1 \le i_1 < ... < i_p \le n$, form a generalized orthonormal basis of $\bigwedge^p V$, where

$$g^{\bigwedge^p V}(e_{i_1} \wedge ... \wedge e_{i_p}, e_{j_1} \wedge ... \wedge e_{j_p}) = \varepsilon_{i_1} \cdots \varepsilon_{i_p} \delta_{i_1 j_1} \cdots \delta_{i_p j_p}.$$

In the same way, V^* has a non-degenerate symmetric bilinear form g^{V^*}: $V^* \times V^* \to \mathbb{R}$ characterized by

$$g^{V^*}(e_i^*, e_j^*) = \varepsilon_i \delta_{ij}.$$

Definition. *If V is oriented, we define the* volume form $\omega := e_1 \wedge ... \wedge e_n \in \bigwedge^n V$, *where $e_1, ..., e_n$ is a positively oriented generalized orthonormal basis of V.*

Remark. *The volume form is independent of the choice of the positively oriented generalized orthonormal basis. If one reverses the orientation of V, then ω gets replaced by $-\omega$. Moreover we have*

$$g(\omega, \omega) = (-1)^s,$$

where s is the index of g.

Definition. *Let M be a differentiable manifold. Let $x\colon U \to V$ and $y\colon \tilde{U} \to \tilde{V}$ be two charts of M. Then x and y are called* orientation compatible *if on $x(U \cap \tilde{U})$ we have*

$$\det D(y \circ x^{-1}) > 0.$$

A C^∞-atlas $\mathcal{A}$ of M is called oriented *if any two charts contained in $\mathcal{A}$ are orientation compatible. An* orientation *of M is a maximal oriented C^∞-atlas of M. A differentiable manifold equipped with an orientation is called an* oriented manifold.

Remark. *If M is an oriented manifold, then every tangent space T_pM (and thus every cotangent space T_p^*M) is equipped with an orientation in the sense of linear algebra.*

Namely, if $x\colon U \to V$ is a chart contained in the oriented atlas with $p \in U$, then $\frac{\partial}{\partial x^1}|_p, ..., \frac{\partial}{\partial x^n}|_p$ is a positively oriented basis of T_pM (and $dx^1|_p, ..., dx^n|_p$ is a positively oriented basis of T_p^*M).

Proposition 3.6. *Let M be an n-dimensional differentiable manifold. Then the following are equivalent:*

(i) M is orientable

(ii) The real line bundle $\bigwedge^n T_p^*M$ is trivial.

Proof. *(ii)* $\Rightarrow$ *(i)*: Assume that $\bigwedge^n T_p^*M$ is trivial. Then there exists a smooth section $\omega \in C^\infty(M, T^*M)$ such that $\omega(p) \neq 0$ for all $p \in M$. Let $\mathcal{A}$ be the differentiable structure of M. We put

$$\mathcal{A}_\omega := \left\{ (x : U \to V) \in \mathcal{A} \,\middle|\, \omega\Big(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\Big) > 0 \text{ on } U \right\}.$$

We show that $\mathcal{A}_\omega$ is an oriented atlas of M.

(a) We show that the charts contained in $\mathcal{A}_\omega$ cover all of M. Namely, let $p \in M$, let $(x : U \to V) \in \mathcal{A}$ with $p \in U$. W. l. o. g. we may assume that U (and therefore V) is connected.
If $\omega(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}) > 0$, then $(x : U \to V) \in \mathcal{A}_\omega$. Otherwise we have $\omega(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}) < 0$ on all of U (since $\omega \neq 0$ and U is connected). We put

$$y^1 := -x^1, \quad y^2 := x^2, ..., y^n := x^n, \quad \tilde{V} := \{(x^1, ..., x^n) \in \mathbb{R}^n \mid (-x^1, x^2, ..., x^n) \in V\}.$$

The chart $y\colon U \to \tilde{V}$ contains p and satisfies

$$\omega\Big(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\Big) > 0 \text{ on } U,$$

thus $(y : U \to \tilde{V}) \in \mathcal{A}_\omega$.

(b) It remains to show that any two charts $x\colon U \to V$ and $y\colon U \to \tilde{V}$ contained in $\mathcal{A}_\omega$ are orientation compatible. On $x(U \cap \tilde{U})$ we have

$$\begin{aligned} 0 &< \omega\Big(\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}\Big) \\ &= \omega\Big(\sum_{i_1=1}^{n} \frac{\partial(x^{i_1} \circ y^{-1})}{\partial y^1} \frac{\partial}{\partial x^{i_1}}, ..., \sum_{i_n=1}^{n} \frac{\partial(x^{i_n} \circ y^{-1})}{\partial y^n} \frac{\partial}{\partial x^{i_n}}\Big) \\ &= \det D(x \circ y^{-1}) \cdot \underbrace{\omega\Big(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\Big)}_{>0} \end{aligned}$$

and thus $\det D(x \circ y^{-1}) > 0$.

(i) $\Rightarrow$ *(ii)*: Choose a Riemannian metric g on M. For $p \in M$ let $\omega(p) \in \bigwedge^n T_p^*M$ be the volume form of T_p^*M, i. e. we have

$$g^{\bigwedge^n}(\omega, \omega) := g^{\bigwedge^n T_p^*M}(\omega, \omega) = 1.$$

Thus ω is a nowhere vanishing section of $\bigwedge^n T^*M$.
Now let $p \in M$. Choose a positively oriented chart $x\colon U \to V$ around p. Then $dx^1, ..., dx^n$ are smooth sections of $T^*M|_U$ that form a positively oriented basis at every point of U. Using the Gram-Schmidt process we obtain smooth sections $e_1, ..., e_n$ of $T^*M|_U$ that form a positively oriented orthonormal basis at every point of U. We have $\omega = e_1 \wedge ... \wedge e_n$ on U and thus ω is smooth on U.
We conclude that ω is a nowhere vanishing smooth section of $\bigwedge^n T^*M$. By Lemma 1.1 the real line bundle $\bigwedge^n T^*M$ is trivial. □

Lemma 3.7. *Let V be an oriented n-dimensional real vector space with a non-degenerate symmetric bilinear form g. Let $p \in \{0, ..., n\}$. Then there is a unique isomorphism of vector spaces $*\colon \bigwedge^p V \to \bigwedge^{n-p} V$, such that for all $\alpha, \beta \in \bigwedge^p V$ we have*

$$\alpha \wedge (*\beta) = g^{\bigwedge^p V}(\alpha, \beta) \cdot \omega.$$

Proof. Uniqueness. Let $*_1$ and $*_2$ be two such isomorphisms. Then for all $\alpha, \beta \in \bigwedge^p V$ we have

$$\alpha \wedge ((*_1 - *_2)\beta) = g^{\bigwedge^p V}(\alpha, \beta) \cdot \omega - g^{\bigwedge^p V}(\alpha, \beta) \cdot \omega = 0.$$

If $\gamma \in \bigwedge^{n-p} V$ and for all $\alpha \in \bigwedge^p V$ we have $\alpha \wedge \gamma = 0$, then we have $\gamma = 0$. Namely we write

$$\gamma = \sum_I \gamma^I e_I$$

where $e_1, ..., e_n$ is a generalized orthonormal basis, $e_I := e_{i_1} \wedge ... \wedge e_{i_{n-p}}$ for any multi-index $I = (1 \le i_1 < ... < i_{n-p} \le n)$ and we sum over all multi-indices I of length $n - p$ and have $\gamma^I \in \mathbb{R}$.
We fix a multi-index I_0 and consider its complementary multi-index I_0^C. For $\alpha = e_{I_0^C} \in \bigwedge^p V$ we have

$$0 = \alpha \wedge \gamma = e_{I_0^C} \wedge \sum_I \gamma^I e_I = \gamma^{I_0} e_{I_0^C} \wedge e_{I_0} = \pm\gamma^{I_0}\omega.$$

We conclude that $\gamma^{I_0} = 0$ for all multi-indices I_0. Thus we have $\gamma = 0$, i. e. $*_1 = *_2$.
Existence. Let $e_1, ..., e_n$ be a positively oriented generalized orthonormal basis of V. We define $*\colon \bigwedge^p V \to \bigwedge^{n-p} V$ by

$$*e_I := g^{\bigwedge^p V}(e_I, e_I) \cdot \mathrm{sign}(I, I^C) \cdot e_{I^C}.$$

Here (I, I^C) is considered as a permutation of $\{1, ..., n\}$ and $\mathrm{sign}(I, I^C)$ is the sign of this permutation. □

Remark. *If we reverse the orientation of* V *then* $*$ *gets replaced by* $-*$.

Definition. *The operator* $*$ *is called the* Hodge star operator.

Proposition 3.8. *The Hodge star operator has the following properties:*

(i) $*1 = \omega$ *and* $*\omega = (-1)^s$.

(ii) For $\alpha \in \bigwedge^p V$ *and* $\beta \in \bigwedge^{n-p} V$ *we have* $g^{\bigwedge^p V}(\alpha, *\beta) = (-1)^{p(n-p)} g^{\bigwedge^{n-p}}(*\alpha, \beta)$.

(iii) On $\bigwedge^p V$ *we have* $*^2 = (-1)^{p(n-p)+s}\mathrm{id}_{\bigwedge^p V}$.

Here s *is the index of the symmetric bilinear form* g *on* V.

Proof. (i) We have

$$\begin{aligned} *1 = *e_\emptyset &= \underbrace{g^{\bigwedge^0 V}(1,1)}_{=1} \underbrace{\mathrm{sign}(1 \cdots n)}_{=1} e_{1 \cdots n} = \omega \\ *\omega = *e_{1\cdots n} &= \underbrace{g^{\bigwedge^n V}(\omega, \omega)}_{=(-1)^s} \underbrace{\mathrm{sign}(1 \cdots n)}_{=1} e_\emptyset = (-1)^s. \end{aligned}$$

(iii) We compute

$$\begin{aligned} *^2 e_I &= *(g^{\bigwedge^p V}(e_I, e_I)\mathrm{sign}(I, I^C) e_{I^C}) \\ &= g^{\bigwedge^p V}(e_I, e_I) \cdot \mathrm{sign}(I, I^C) \cdot g^{\bigwedge^{n-p} V}(e_{I^C}, e_{I^C}) \cdot \mathrm{sign}(I^C, I) \cdot e_I \\ &= \underbrace{g^{\bigwedge^p V}(e_I, e_I) \cdot g^{\bigwedge^{n-p} V}(e_{I^C}, e_{I^C})}_{=(-1)^s} \cdot \underbrace{\mathrm{sign}(I, I^C) \cdot \mathrm{sign}(I^C, I)}_{=(-1)^{p(n-p)}} \cdot e_I \\ &= (-1)^{p(n-p)+s} e_I. \end{aligned}$$

(ii) On the one hand we have

$$\alpha \wedge \beta \overset{(iii)}{=} (-1)^{(n-p)p+s} \alpha \wedge *^2\beta = (-1)^{(n-p)p+s} g^{\bigwedge^p V}(\alpha, *\beta) \cdot \omega$$

on the other hand

$$\begin{aligned} \alpha \wedge \beta &= (-1)^{p(n-p)} \beta \wedge \alpha \overset{(iii)}{=} (-1)^{p(n-p)}(-1)^{p(n-p)+s} \beta \wedge *^2\alpha \\ &= (-1)^{2p(n-p)+s} g^{\bigwedge^{n-p} V}(\beta, *\alpha) \cdot \omega. \end{aligned}$$

□

Definition. *Let M be an oriented semi-Riemannian manifold. Let*

$$d: \quad C^\infty(M, \wedge^p T^*M) \to C^\infty(M, \wedge^{p+1} T^*M)$$

be the exterior derivative of differential p-forms on M. Then

$$\delta := (-1)^{np+1+s} * d* : \quad C^\infty(M, \wedge^{p+1} T^*M) \to C^\infty(M, \wedge^p T^*M)$$

is called the codifferential.

Remark. *For the definition of d neither the semi-Riemannian metric nor the orientation are used. If one reverses the orientation, then $*$ gets replaced by $-*$, thus δ remains unchanged. Therefore δ depends on the semi-Riemannian metric but not on the orientation. Hence, δ can also be defined on non-orientable manifolds.*

Remark. *We have $\delta^2 = \pm * d * * d* = \pm * d^2 * = 0$ since $*^2 = \pm 1$ and $d^2 = 0$.*

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