Linear wave equations on Lorentzian manifolds

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Abstract

We summarize the analytic theory of linear wave equations on globally hyperbolic Lorentzian manifolds.

Contents

1 Introduction

In General Relativity spacetime is modelled by a Lorentzian manifold, see e. g. [8, 15]. Many physical phenomena, such as electro-magnetic radiation, are described by solutions to certain linear wave equations defined on this spacetime manifold. Thus a good understanding of the theory of wave equations is crucial. This includes initial value problems (the Cauchy problem), fundamental solutions, and inverse operators (Green's operators). The classical textbooks on partial differential equations contain the relevant results for small domains in Lorentzian manifolds or for very special manifolds such as Minkowski space.
In this text we summarize the global analytic results obtained in [4], see also Leray's unpublished lecture notes [13] and Choquet-Bruhat's exposition [7]. In order to obtain a good solution theory one has to impose certain geometric conditions on the underlying manifold. The situation is similar to the study of elliptic operators on Riemannian manifolds. In order to ensure that the Laplace-Beltrami operator on a Riemannian manifold M is essentially self-adjoint one may make the natural assumption that M be complete. Unfortunately, there is no good notion of completeness for Lorentzian manifolds. It will turn

out that the analysis of wave operators works out nicely if one assumes that the underlying Lorentzian manifold be globally hyperbolic. Completeness of Riemannian manifolds and global hyperbolicity of Lorentzian manifolds are indeed related. If (S,g_0) is a Riemannian manifold, then the Lorentzian cylinder $M = \mathbb{R} \times S$ with product metric $g = -dt^2 + g_0$ is globally hyperbolic if and only if (S,g_0) is complete.
We will start by collecting some material on distributional sections in vector bundles. Then we will summarize the theory of globally hyperbolic Lorentzian manifolds. Then we will define wave operators, also called normally hyperbolic operators, and give some examples. After that we consider the basic initial value problem, the Cauchy problem. It turns out that on a globally hyperbolic manifold solutions exist and are unique. They depend continuously on the initial data. The support of the solutions can be controlled which is physically nothing than the statement that a wave can never propagate faster than with the speed of light. In the subsequent section we use the results on the Cauchy problem to show existence and uniqueness of fundamental solutions. This is closely related to the existence and uniqueness of Green's operators.
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2 Distributional sections in vector bundles

Let us start by giving some definitions and by fixing the terminology for distributions on manifolds. We will confine ourselves to those facts that we will actually need later on. A systematic and much more complete introduction may be found e. g. in [9].

2.1 Preliminaries on distributional sections

Let M be a manifold equipped with a smooth volume density dV. Later on we will use the volume density induced by a Lorentzian metric but this is irrelevant for now. We consider a real or complex vector bundle $E \to M$. We will always write $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depending on whether E is real or complex. The space of compactly supported smooth sections in E will be denoted by $\mathscr{D}(M,E)$. We equip E and the cotangent bundle T^*M with connections, both denoted by ∇. They induce connections on the tensor bundles $T^*M \otimes \cdots \otimes T^*M \otimes E$, again denoted by ∇. For a continuously differentiable section $\varphi \in C^1(M,E)$ the covariant derivative is a continuous section in $T^*M \otimes E$, $\nabla\varphi \in C^0(M, T^*M \otimes E)$. More generally, for $\varphi \in C^k(M,E)$ we get $\nabla^k\varphi \in C^0(M, \underbrace{T^*M \otimes \cdots \otimes T^*M}_{k \text{ factors}} \otimes E)$.
We choose an auxiliary Riemannian metric on T^*M and an auxiliary Riemannian or Hermitian metric on E depending on whether E is real or complex. This induces metrics on all bundles $T^*M \otimes \cdots \otimes T^*M \otimes E$. Hence the norm of $\nabla^k\varphi$ is defined at all points of M.
For a subset $A \subset M$ and $\varphi \in C^k(M,E)$ we define the C^k-norm by

$$\|\varphi\|_{C^k(A)} := \max_{j=0,\ldots,k} \sup_{x\in A} |\nabla^j \varphi(x)|. \tag{1}$$

If A is compact, then different choices of the metrics and the connections yield equivalent norms $\|\cdot\|_{C^k(A)}$. For this reason there will be no need to explicitly specify the metrics and the connections.
The elements of $\mathscr{D}(M,E)$ are referred to as test sections in E. We define a notion of convergence of test sections.

Definition 2.1. Let $\varphi, \varphi_n \in \mathscr{D}(M,E)$. We say that the sequence $(\varphi_n)_n$ *converges to* φ *in* $\mathscr{D}(M,E)$ if the following two conditions hold:

1. There is a compact set $K \subset M$ such that the supports of φ and of all φ_n are contained in K, i. e. $\mathrm{supp}(\varphi), \mathrm{supp}(\varphi_n) \subset K$ for all n.

2. The sequence $(\varphi_n)_n$ converges to φ in all C^k-norms over K, i. e. for each $k \in \mathbb{N}$

$$\|\varphi - \varphi_n\|_{C^k(K)} \underset{n\to\infty}{\longrightarrow} 0.$$

We fix a finite-dimensional $\mathbb{K}$-vector space W. Recall that $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depending on whether E is real or complex. Denote by E^* the vector bundle over M dual to E.

Definition 2.2. A $\mathbb{K}$-linear map $F : \mathscr{D}(M,E^*) \to W$ is called a *distribution in E with values in W* or a *distributional section in E with values in W* if it is continuous in the sense that for all convergent sequences $\varphi_n \to \varphi$ in $\mathscr{D}(M,E^*)$ one has $F[\varphi_n] \to F[\varphi]$. We write $\mathscr{D}'(M,E,W)$ for the space of all W-valued distributions in E.

Note that since W is finite-dimensional all norms $|\cdot|$ on W yield the same topology on W. Hence there is no need to specify a norm on W for Definition 2.2 to make sense. Note moreover, that distributional sections in E act on test sections in E^*.

Example 2.3. Pick a bundle $E \to M$ and a point $x \in M$. The *delta-distribution* δ_x is a distributional section in E with values in E_x^*. For $\varphi \in \mathscr{D}(M,E^*)$ it is defined by

$$\delta_x[\varphi] = \varphi(x).$$

Example 2.4. Every locally integrable section $f \in L^1_{\mathrm{loc}}(M,E)$ can be regarded as a $\mathbb{K}$-valued distribution in E by setting for any $\varphi \in \mathscr{D}(M,E^*)$

$$f[\varphi] := \int_M \varphi(f)\ \mathrm{dV}.$$

Here $\varphi(f)$ denotes the $\mathbb{K}$-valued L^1-function with compact support on M obtained by pointwise application of $\varphi(x) \in E_x^*$ to $f(x) \in E_x$.

2.2 Differential operators acting on distributions

Let E and F be two $\mathbb{K}$-vector bundles over the manifold M, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Consider a linear differential operator $P : C^\infty(M,E) \to C^\infty(M,F)$. There is a unique linear differential operator $P^* : C^\infty(M,F^*) \to C^\infty(M,E^*)$ called the *formal adjoint of P* such that for any $\varphi \in \mathscr{D}(M,E)$ and $\psi \in \mathscr{D}(M,F^*)$

$$\int_M \psi(P\varphi)\ \mathrm{dV} = \int_M (P^*\psi)(\varphi)\ \mathrm{dV}. \tag{2}$$

If P is of order k, then so is P^* and (2) holds for all $\varphi \in C^k(M,E)$ and $\psi \in C^k(M,F^*)$ such that $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\psi)$ is compact. With respect to the canonical identification $E = (E^*)^*$ we have $(P^*)^* = P$.
Any linear differential operator $P : C^\infty(M,E) \to C^\infty(M,F)$ extends canonically to a linear operator $P : \mathscr{D}'(M,E,W) \to \mathscr{D}'(M,F,W)$ by

$$(PT)[\varphi] := T[P^*\varphi]$$

where $\varphi \in \mathscr{D}(M,F^*)$. If a sequence $(\varphi_n)_n$ converges in $\mathscr{D}(M,F^*)$ to 0, then the sequence $(P^*\varphi_n)_n$ converges to 0 as well because P^* is a differential operator. Hence $(PT)[\varphi_n] = T[P^*\varphi_n] \to 0$. Therefore PT is indeed again a distribution.
The map $P : \mathscr{D}'(M,E,W) \to \mathscr{D}'(M,F,W)$ is $\mathbb{K}$-linear. If P is of order k and φ is a C^k-section in E, seen as a $\mathbb{K}$-valued distribution in E, then the distribution $P\varphi$ coincides with the continuous section obtained by applying P to φ classically.
An important special case occurs when P is of order 0, i. e. $P \in C^\infty(M,\mathrm{Hom}(E,F))$. Then $P^* \in C^\infty(M,\mathrm{Hom}(F^*,E^*))$ is the pointwise adjoint. In particular, for a function $f \in C^\infty(M,\mathbb{K})$ we have

$$(fT)[\varphi] = T[f\varphi].$$

2.3 Supports

Definition 2.5. The *support* of a distribution $T \in \mathscr{D}'(M,E,W)$ is defined as the set

$$\begin{aligned}&\mathrm{supp}(T)\\ &\quad := \quad \{x \in M \mid \forall \text{ neighborhood } U \text{ of } x \; \exists \varphi \in \mathscr{D}(M,E) \text{ with } \mathrm{supp}(\varphi) \subset U \text{ and } T[\varphi] \neq 0\}.\end{aligned}$$

It follows from the definition that the support of T is a closed subset of M. In case T is a L^1_{loc}-section this notion of support coincides with the usual one for sections.
If for $\varphi \in \mathscr{D}(M,E^*)$ the supports of φ and T are disjoint, then $T[\varphi] = 0$. Namely, for each $x \in \mathrm{supp}(\varphi)$ there is a neighborhood U of x such that $T[\psi] = 0$ whenever $\mathrm{supp}(\psi) \subset U$. Cover the compact set $\mathrm{supp}(\varphi)$ by finitely many such open sets $U_1,\dots,U_k$. Using a partition of unity one can write $\varphi = \psi_1 + \cdots + \psi_k$ with $\psi_j \in \mathscr{D}(M,E^*)$ and $\mathrm{supp}(\psi_j) \subset U_j$. Hence

$$T[\varphi] = T[\psi_1 + \cdots + \psi_k] = T[\psi_1] + \cdots + T[\psi_k] = 0.$$

Be aware that it is not sufficient to assume that φ vanishes on $\mathrm{supp}(T)$ in order to ensure $T[\varphi] = 0$. For example, if $M = \mathbb{R}$ and E is the trivial $\mathbb{K}$-line bundle let $T \in \mathscr{D}'(\mathbb{R},\mathbb{K})$ be given by $T[\varphi] = \varphi'(0)$. Then $\mathrm{supp}(T) = \{0\}$ but $T[\varphi] = \varphi'(0)$ may well be nonzero while $\varphi(0) = 0$.
If $T \in \mathscr{D}'(M,E,W)$ and $\varphi \in C^\infty(M,E^*)$, then the evaluation $T[\varphi]$ can be defined if $\mathrm{supp}(T) \cap \mathrm{supp}(\varphi)$ is compact even if the support of φ itself is noncompact. To do this pick a function $\sigma \in \mathscr{D}(M,\mathbb{R})$ that is constant 1 on a neighborhood of $\mathrm{supp}(T) \cap \mathrm{supp}(\varphi)$ and put

$$T[\varphi] := T[\sigma\varphi].$$

This definition is independent of the choice of σ since for another choice σ' we have

$$T[\sigma\varphi] - T[\sigma'\varphi] = T[(\sigma - \sigma')\varphi] = 0$$

because $\mathrm{supp}((\sigma - \sigma')\varphi)$ and $\mathrm{supp}(T)$ are disjoint.
Let $T \in \mathscr{D}'(M,E,W)$ and let $\Omega \subset M$ be an open subset. Each test section $\varphi \in \mathscr{D}(\Omega,E^*)$ can be extended by 0 and yields a test section $\varphi \in \mathscr{D}(M,E^*)$. This defines an embedding $\mathscr{D}(\Omega,E^*) \subset \mathscr{D}(M,E^*)$. By the restriction of T to Ω we mean its restriction from $\mathscr{D}(M,E^*)$ to $\mathscr{D}(\Omega,E^*)$.

Definition 2.6. The *singular support* $\mathrm{sing\,supp}(T)$ of a distribution $T \in \mathscr{D}'(M,E,W)$ is the set of points which do not have a neighborhood restricted to which T coincides with a smooth section.

The singular support is also closed and we always have $\mathrm{sing\,supp}(T) \subset \mathrm{supp}(T)$.

Example 2.7. For the delta-distribution δ_x we have $\mathrm{supp}(\delta_x) = \mathrm{sing\,supp}(\delta_x) = \{x\}$.

2.4 Convergence of distributions

The space $\mathscr{D}'(M,E)$ of distributions in E will always be given the *weak topology*. This means that $T_n \to T$ in $\mathscr{D}'(M,E,W)$ if and only if $T_n[\varphi] \to T[\varphi]$ for all $\varphi \in \mathscr{D}(M,E^*)$. Linear differential operators P are always continuous with respect to the weak topology. Namely, if $T_n \to T$, then we have for every $\varphi \in \mathscr{D}(M,E^*)$

$$PT_n[\varphi] = T_n[P^*\varphi] \to T[P^*\varphi] = PT[\varphi].$$

Hence

$$PT_n \to PT.$$

Remark 2.8. Let $T_n, T \in C^0(M,E)$ and suppose $\|T_n - T\|_{C^0(M)} \to 0$. Consider T_n and T as distributions. Then $T_n \to T$ in $\mathscr{D}'(M,E)$. In particular, for every linear differential operator P we have $PT_n \to PT$.

3 Globally hyperbolic Lorentzian manifolds

Next we summarize some notions and facts from Lorentzian geometry. More comprehensive introductions can be found in [2] and in [14].

By a *Lorentzian manifold* we mean a semi-Riemannian manifold whose metric has signature $(-,+,\cdots,+)$. We denote the Lorentzian metric by g or by $\langle\cdot,\cdot\rangle$. A tangent vector $X \in TM$ is called *timelike* if $\langle X,X\rangle < 0$, *lightlike* if $\langle X,X\rangle = 0$ and $X \neq 0$, *causal* if it is timelike or lightlike, and *spacelike* otherwise. At each point $p \in M$ the set of timelike vectors in T_pM decomposes into two connected components. A *timeorientation* on M is a choice of one of the two connected components of timelike vectors in T_pM which depends continuously on p. This means that we can find a continuous timelike vector field on M taking values in the chosen connected components. Tangent vectors in the chosen connected component are called *future directed*, those in the other component are called *past directed*.

Let M be a timeoriented Lorentzian manifold. A piecewise C^1-curve in M is called *timelike*, *lightlike*, *causal*, *spacelike*, *future directed*, or *past directed* if its tangent vectors are timelike, lightlike, causal, spacelike, future directed, or past directed respectively.

The *chronological future* $I_+^M(x)$ of a point $x \in M$ is the set of points that can be reached from x by future directed timelike curves. Similarly, the *causal future* $J_+^M(x)$ of a point $x \in M$ consists of those points that can be reached from x by causal curves and of x itself. The *chronological future* of a subset $A \subset M$ is defined to be $I_+^M(A) := \bigcup_{x\in A} I_+^M(x)$. Similarly, the *causal future* of A is $J_+^M(A) := \bigcup_{x\in A} J_+^M(x)$. The *chronological past* $I_-^M(A)$ and the *causal past* $J_-^M(A)$ are defined by replacing future directed curves by past directed curves. One has in general that $I_\pm^M(A)$ is the interior of $J_\pm^M(A)$ and that $J_\pm^M(A)$ is contained in the closure of $I_\pm^M(A)$. The chronological future and past are open subsets but the causal future and past are not always closed even if A is closed.

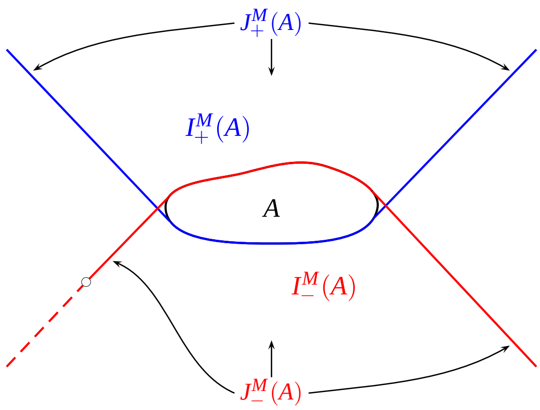

Fig. 1: Causal and chronological future resp. past of A

We will also use the notation $J^M(A) := J_-^M(A) \cup J_+^M(A)$. A subset $A \subset M$ is called *past compact* if $A \cap J_-^M(p)$ is compact for all $p \in M$. Similarly, one defines *future compact* subsets.

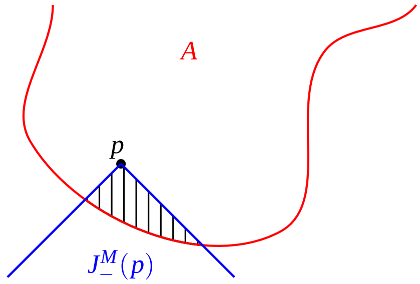

Fig. 2: Past compact subset

Definition 3.1. A subset S of a connected timeoriented Lorentzian manifold is called *achronal* if each timelike curve meets S in at most one point. A subset S of a connected timeoriented Lorentzian manifold is called *acausal* if each causal curve meets S in at most one point. A subset S of a connected timeoriented Lorentzian manifold is a *Cauchy hypersurface* if each inextendible timelike curve in M meets S at exactly one point.

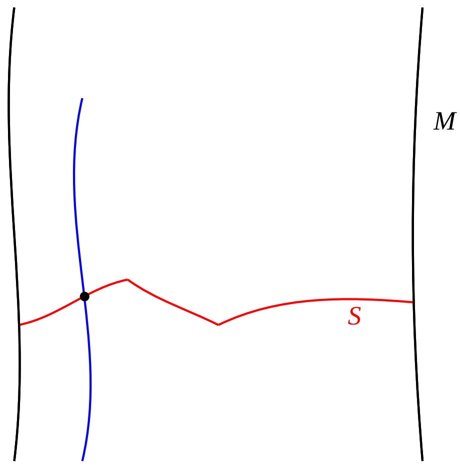

Fig. 3: Cauchy hypersurface

Obviously every acausal subset is achronal, but the reverse is wrong. Any Cauchy hypersurface is achronal. Moreover, it is a closed topological hypersurface and it is hit by each inextendible causal curve in at least one point. Any two Cauchy hypersurfaces in M are homeomorphic. Furthermore, the causal future and past of a Cauchy hypersurface is past and future compact respectively.

Definition 3.2. A Lorentzian manifold is said to satisfy the *causality condition* if it does not contain any closed causal curve.
A Lorentzian manifold is said to satisfy the *strong causality condition* if there are no almost closed causal curves. More precisely, for each point $p \in M$ and for each open neighborhood U of p there exists an open neighborhood $V \subset U$ of p such that each causal curve in M starting and ending in V is entirely contained in U.

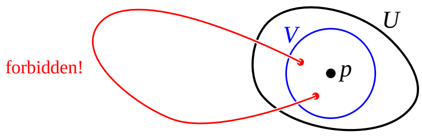

Fig. 4: Strong causality condition

Obviously, the strong causality condition implies the causality condition.
In order to get a good analytical theory for wave operators we must impose certain geometric conditions on the Lorentzian manifold. Here are several equivalent formulations.

Theorem 3.3. *Let M be a connected timeoriented Lorentzian manifold. Then the following are equivalent:*

(1) M satisfies the strong causality condition and for all $p,q \in M$ the intersection $J_+^M(p) \cap J_-^M(q)$ is compact.

(2) There exists a Cauchy hypersurface in M.

(3) There exists a smooth spacelike Cauchy hypersurface in M.

(4) M is foliated by smooth spacelike Cauchy hypersurfaces. More precisely, M is isometric to $\mathbb{R} \times S$ with metric $-\beta dt^2 + g_t$ where β is a smooth positive function, g_t is a Riemannian metric on S depending smoothly on $t \in \mathbb{R}$ and each $\{t\} \times S$ is a smooth spacelike Cauchy hypersurface in M.

That (1) implies (4) has been shown by Bernal and Sánchez in [5, Thm. 1.1] using work of Geroch [11, Thm. 11]. See also [8, Prop. 6.6.8] and [15, p. 209] for earlier mentionings of this fact. The implications $(4) \Rightarrow (3)$ and $(3) \Rightarrow (2)$ are trivial. That (2) implies (1) is well-known, see e. g. [14, Cor. 39, p. 422].

Definition 3.4. A connected timeoriented Lorentzian manifold satisfying one and hence all conditions in Theorem 3.3 is called *globally hyperbolic*.

Remark 3.5. If M is a globally hyperbolic Lorentzian manifold, then a nonempty open subset $\Omega \subset M$ is itself globally hyperbolic if and only if for any $p,q \in \Omega$ the intersection $J_+^\Omega(p) \cap J_-^\Omega(q) \subset \Omega$ is compact. Indeed non-existence of almost closed causal curves in M directly implies non-existence of such curves in Ω.

Remark 3.6. It should be noted that global hyperbolicity is a conformal notion. The definition of a Cauchy hypersurface requires only causal concepts. Hence if (M,g) is globally hyperbolic and we replace the metric g by a conformally related metric $\hat{g} = f \cdot g$, f a smooth positive function on M, then $(M,\hat{g})$ is again globally hyperbolic.

Examples 3.7. Minkowski space is globally hyperbolic. Every spacelike hyperplane is a Cauchy hypersurface. One can write Minkowski space as $\mathbb{R} \times \mathbb{R}^{n-1}$ with the metric $-\,dt^2 + g_t$ where g_t is the Euclidean metric on $\mathbb{R}^{n-1}$ and does not depend on t.
Let (S, g_0) be a connected Riemannian manifold and $I \subset \mathbb{R}$ an interval. The manifold $M = I \times S$ with the metric $g = -\,dt^2 + g_0$ is globally hyperbolic if and only if (S,g_0) is complete. This applies in particular if S is compact.
More generally, if $f : I \longrightarrow \mathbb{R}$ is a smooth positive function we may equip $M = I \times S$ with the metric $g = -\,dt^2 + f(t)^2 \cdot g_0$. Again, (M,g) is globally hyperbolic if and only if (S,g_0) is complete. *Robertson-Walker spacetimes* and, in particular, *Friedmann cosmological models*, are of this type. They are used to discuss big bang, expansion of the universe, and cosmological redshift, compare [14, Ch. 12]. Another example of this type is *deSitter*

spacetime, where $I = \mathbb{R}$, $S = S^{n-1}$, g_0 is the canonical metric of S^{n-1} of constant sectional curvature 1, and $f(t) = \cosh(t)$. But *Anti-deSitter spacetime* is not globally hyperbolic.
The interior and exterior *Schwarzschild spacetimes* are globally hyperbolic. They model the universe in the neighborhood of a massive static rotionally symmetric body such as a black hole. They are used to investigate perihelion advance of Mercury, the bending of light near the sun and other astronomical phenomena, see [14, Ch. 13].

Lemma 3.8. *Let S be a Cauchy hypersurface in a globally hyperbolic Lorentzian manifold M and let $K, K' \subset M$ be compact.*
Then $J_\pm^M(K) \cap S$, $J_\pm^M(K) \cap J_\mp^M(S)$, and $J_+^M(K) \cap J_-^M(K')$ are compact.

4 Wave operators

Let M be a Lorentzian manifold and let $E \to M$ be a real or complex vector bundle. A linear differential operator $P : C^\infty(M,E) \to C^\infty(M,E)$ of second order will be called a *wave operator* or a *normally hyperbolic operator* if its principal symbol is given by the metric,

$$\sigma_P(\xi) = -\langle \xi, \xi \rangle \cdot \mathrm{id}_{E_x}$$

for all $x \in M$ and all $\xi \in T_x^*M$. In other words, if we choose local coordinates $x^1, \dots, x^n$ on M and a local trivialization of E, then

$$P = -\sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^{n} A_j(x) \frac{\partial}{\partial x^j} + B(x)$$

where A_j and B are matrix-valued coefficients depending smoothly on x and $(g^{ij})_{ij}$ is the inverse matrix of $(g_{ij})_{ij}$ with $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$.

Example 4.1. Let E be the trivial line bundle so that sections in E are just functions. The d'Alembert operator $P = \Box = -\operatorname{div} \circ \operatorname{grad}$ is a wave operator.

Example 4.2. Let E be a vector bundle and let ∇ be a connection on E. This connection together with the Levi-Civita connection on T^*M induces a connection on $T^*M \otimes E$, again denoted ∇. We define the *connection-d'Alembert operator* $\Box^\nabla$ to be minus the composition of the following three maps

$$C^\infty(M,E) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes E) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes T^*M \otimes E) \xrightarrow{\mathrm{tr} \otimes \mathrm{id}_E} C^\infty(M,E)$$

where $\mathrm{tr} : T^*M \otimes T^*M \to \mathbb{R}$ denotes the metric trace, $\mathrm{tr}(\xi \otimes \eta) = \langle \xi, \eta \rangle$. We compute the principal symbol,

$$\sigma_{\Box^\nabla}(\xi)\varphi = -(\mathrm{tr} \otimes \mathrm{id}_E) \circ \sigma_\nabla(\xi) \circ \sigma_\nabla(\xi)(\varphi) = -(\mathrm{tr} \otimes \mathrm{id}_E)(\xi \otimes \xi \otimes \varphi) = -\langle \xi, \xi \rangle\, \varphi.$$

Hence $\Box^\nabla$ is a wave operator.

Example 4.3. Let $E = \Lambda^k T^*M$ be the bundle of k-forms. Exterior differentiation $d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$ increases the degree by one while the codifferential $\delta : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k-1} T^*M)$ decreases the degree by one. While d is independent of the metric, the codifferential δ does depend on the Lorentzian metric. The operator $P = d\delta + \delta d$ is a wave operator.

Example 4.4. If M carries a Lorentzian metric and a spin structure, then one can define the spinor bundle ΣM and the Dirac operator

$$D : C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M),$$

see [1] or [3] for the definitions. The principal symbol of D is given by Clifford multiplication,

$$\sigma_D(\xi)\psi = \xi^\sharp \cdot \psi.$$

Hence

$$\sigma_{D^2}(\xi)\psi = \sigma_D(\xi)\sigma_D(\xi)\psi = \xi^\sharp \cdot \xi^\sharp \cdot \psi = -\langle \xi, \xi \rangle \psi.$$

Thus $P = D^2$ is a wave operator.

5 The Cauchy problem

We now come to the basic initial value problem for wave operators, the *Cauchy problem*. The local theory of linear hyperbolic operators can be found in basically any textbook on partial differential equations. In [10] and [12] the local theory for wave operators on Lorentzian manifolds is developed. The results of this section are of global nature. They make statements about solutions to the Cauchy problem which are defined globally on a manifold. Proofs of the results of this section can be found in [4, Sec. 3.2].

Theorem 5.1 (Existence and uniqueness of solutions)**.** *Let M be a globally hyperbolic Lorentzian manifold and let $S \subset M$ be a smooth spacelike Cauchy hypersurface. Let ν be the future directed timelike unit normal field along S. Let E be a vector bundle over M and let P be a wave operator acting on sections in E.*
Then for each $u_0, u_1 \in \mathscr{D}(S,E)$ and for each $f \in \mathscr{D}(M,E)$ there exists a unique $u \in C^\infty(M,E)$ satisfying $Pu = f$, $u|_S = u_0$, and $\nabla_\nu u|_S = u_1$.

It is unclear how to even formulate the Cauchy problem on a Lorentzian manifold which is not globally hyperbolic. One would have to replace the concept of a Cauchy hypersurface by something different to impose the initial conditions upon. Here are two examples which illustrate what can typically go wrong.

Example 5.2. Let $M = S^1 \times \mathbb{R}^{n-1}$ with the metric $g = -d\theta^2 + g_0$ where $d\theta^2$ is the standard metric on S^1 of length 1 and g_0 is the Euclidean metric on $\mathbb{R}^{n-1}$. The universal covering of M is Minkowski space.
Let us try to impose a Cauchy problem on $\{\theta_0\} \times \mathbb{R}^{n-1}$ which is the image of a Cauchy hypersurface in Minkowski space. Such a solution would lift to Minkowski space where it indeed exists uniquely due to Theorem 5.1. But such a solution on Minkowski space is in general not time periodic, hence does not descend to a solution on M.
Therefore existence of solutions fails. The problem is here that M violates the causality condition, i. e. there are closed causal curves.

Remark 5.3. Compact Lorentzian manifolds always possess closed timelike curves and are therefore never well suited for the analysis of wave operators.

Example 5.4. Let M be a timelike strip in 2-dimensional Minkowski space, i. e. $M = \mathbb{R} \times (0,1)$ with metric $g = -dt^2 + dx^2$. Let $S := \{0\} \times (0,1)$. Given any $u_0, u_1 \in \mathscr{D}(S,E)$ and any $f \in \mathscr{D}(M,E)$, there exists a solution u to the Cauchy problem. One can simply take the solution in Minkowski space and restrict it to M. But this solution is not unique in M. Choose x in Minkowski space, $x \notin M$, such that $J_+^{\mathrm{Mink}}(x)$ intersects M in the future of S and of $\mathrm{supp}(f)$. The advanced fundamental solution $w = F_+(x)$ (see next section) has support contained in $J_+^{\mathrm{Mink}}(x)$ and satisfies $Pw = 0$ away from x. Hence $u + w$ restricted to M is again a solution to the Cauchy problem on M with the same initial data.

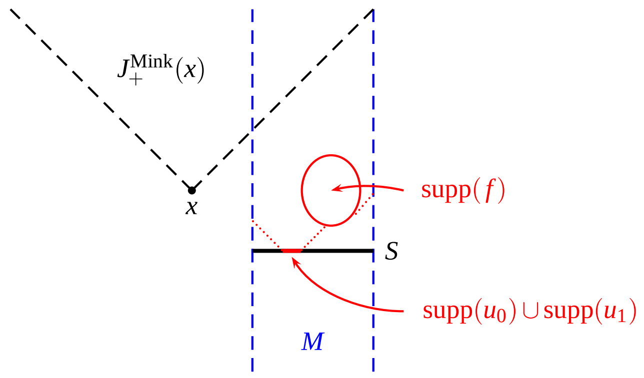

Fig. 5: Nonunique solution to Cauchy problem

The problem is here that S is acausal but not a Cauchy hypersurface. Physically, a wave "from outside the manifold" enters into M.

The physical statement that a wave can never propagate faster than with the speed of light is contained in the following.

Theorem 5.5 (Finite propagation speed)**.** *The solution u from Theorem 5.1 satisfies* $\mathrm{supp}(u) \subset J^M(K)$ *where* $K = \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)$.

The solution to the Cauchy problem depends continuously on the data.

Theorem 5.6 (Stability)**.** *Let M be a globally hyperbolic Lorentzian manifold and let $S \subset M$ be a smooth spacelike Cauchy hypersurface. Let ν be the future directed timelike unit normal field along S. Let E be a vector bundle over M and let P be a wave operator acting on sections in E.*
Then the map $\mathscr{D}(M,E) \oplus \mathscr{D}(S,E) \oplus \mathscr{D}(S,E) \to C^\infty(M,E)$ sending (f, u_0, u_1) to the unique solution u of the Cauchy problem $Pu = f$, $u|_S = u|_0$, $\nabla_\nu u = u_1$ is linear continuous.

This is essentially an application of the open mapping theorem for Fréchet spaces.

6 Fundamental solutions

Definition 6.1. Let M be a timeoriented Lorentzian manifold, let $E \to M$ be a vector bundle and let $P : C^\infty(M,E) \to C^\infty(M,E)$ be a wave operator. Let $x \in M$. A *fundamental solution* of P at x is a distribution $F \in \mathscr{D}'(M,E,E_x^*)$ such that

$$PF = \delta_x.$$

In other words, for all $\varphi \in \mathscr{D}(M,E^*)$ we have

$$F[P^*\varphi] = \varphi(x).$$

If $\mathrm{supp}(F(x)) \subset J_+^M(x)$, then we call F an *advanced fundamental solution*, if $\mathrm{supp}(F(x)) \subset J_-^M(x)$, then we call F a *retarded fundamental solution*.

Using the knowlegde about the Cauchy problem from the previous section it is now not hard to find global fundamental solutions on a globally hyperbolic manifold.

Theorem 6.2. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a wave operator acting on sections in a vector bundle E over M.*
Then for every $x \in M$ there is exactly one fundamental solution $F_+(x)$ for P at x with past compact support and exactly one fundamental solution $F_-(x)$ for P at x with future compact support. They satisfy

1. $\operatorname{supp}(F_\pm(x)) \subset J^M_\pm(x)$,
2. *for each $\varphi \in \mathscr{D}(M,E^*)$ the maps $x \mapsto F_\pm(x)[\varphi]$ are smooth sections in E^* satisfying the differential equation $P^*(F_\pm(\cdot)[\varphi]) = \varphi$.*

Sketch of proof. We do not do the uniqueness part. To show existence fix a foliation of M by spacelike Cauchy hypersurfaces S_t, $t \in \mathbb{R}$ as in Theorem 3.3. Let ν be the future directed unit normal field along the leaves S_t. Let $\varphi \in \mathscr{D}(M,E^*)$. Choose t so large that $\operatorname{supp}(\varphi) \subset I^M_-(S_t)$. By Theorem 5.1 there exists a unique $\chi_\varphi \in C^\infty(M,E^*)$ such that $P^*\chi_\varphi = \varphi$ and $\chi_\varphi|_{S_t} = (\nabla_\nu \chi_\varphi)|_{S_t} = 0$. One can check that χ_φ does not depend on the choice of t.
Fix $x \in M$. By Theorem 5.6 χ_φ depends continuously on φ. Since the evaluation map $C^\infty(M,E) \to E_x$ is continuous, the map $\mathscr{D}(M,E^*) \to E^*_x$, $\varphi \mapsto \chi_\varphi(x)$, is also continuous. Thus $F_+(x)[\varphi] := \chi_\varphi(x)$ defines a distribution. By definition $P^*(F_+(\cdot)[\varphi]) = P^*\chi_\varphi = \varphi$.
Now $P^*\chi_{P^*\varphi} = P^*\varphi$, hence $P^*(\chi_{P^*\varphi} - \varphi) = 0$. Since both $\chi_{P^*\varphi}$ and φ vanish along S_t the uniqueness part which we have omitted shows $\chi_{P^*\varphi} = \varphi$. Thus

$$(PF_+(x))[\varphi] = F_+(x)[P^*\varphi] = \chi_{P^*\varphi}(x) = \varphi(x) = \delta_x[\varphi].$$

Hence $F_+(x)$ is a fundamental solution of P at x.
It remains to show $\operatorname{supp}(F_+(x)) \subset J^M_+(x)$. Let $y \in M \setminus J^M_+(x)$. We have to construct a neighborhood of y such that for each test section $\varphi \in \mathscr{D}(M,E^*)$ whose support is contained in this neighborhood we have $F_+(x)[\varphi] = \chi_\varphi(x) = 0$. Since M is globally hyperbolic $J^M_+(x)$ is closed and therefore $J^M_+(x) \cap J^M_-(y') = \emptyset$ for all y' sufficiently close to y. We choose $y' \in I^M_+(y)$ and $y'' \in I^M_-(y)$ so close that $J^M_+(x) \cap J^M_-(y') = \emptyset$ and $\left(J^M_+(y'') \cap \bigcup_{t \le t'} S_t\right) \cap J^M_+(x) = \emptyset$ where $t' \in \mathbb{R}$ is such that $y' \in S_{t'}$.

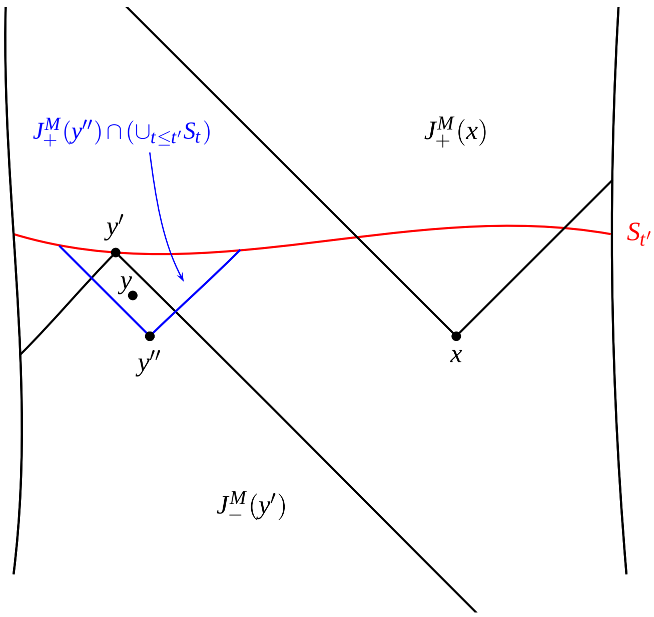

Fig. 6: Construction of y, y' and y''

Now $K := J^M_-(y') \cap J^M_+(y'')$ is a compact neighborhood of y. Let $\varphi \in \mathscr{D}(M,E^*)$ be such that $\operatorname{supp}(\varphi) \subset K$. By Theorem 5.1 $\operatorname{supp}(\chi_\varphi) \subset J^M_+(K) \cup J^M_-(K) \subset J^M_+(y'') \cup J^M_-(y')$. By the independence of χ_φ of the choice of $t > t'$ we have that χ_φ vanishes on $\bigcup_{t>t'} S_t$. Hence $\operatorname{supp}(\chi_\varphi) \subset \left(J^M_+(y'') \cap \bigcup_{t\le t'} S_t\right) \cup J^M_-(y')$ and is therefore disjoint from $J^M_+(x)$. Thus $F_+(x)[\varphi] = \chi_\varphi(x) = 0$ as required. $\square$

For a complete proof see [4, Sec. 3.3].

7 Green's operators

Now we want to find "solution operators" for a given wave operator P. More precisely, we want to find operators which are inverses of P when restricted to suitable spaces of sections. We will see that existence of such operators is basically equivalent to the existence of fundamental solutions.

Definition 7.1. Let M be a timeoriented connected Lorentzian manifold. Let P be a wave operator acting on sections in a vector bundle E over M. A linear map $G_+ : \mathscr{D}(M,E) \to C^\infty(M,E)$ satisfying

(i) $P \circ G_+ = \mathrm{id}_{\mathscr{D}(M,E)}$,

(ii) $G_+ \circ P|_{\mathscr{D}(M,E)} = \mathrm{id}_{\mathscr{D}(M,E)}$,

(iii) $\mathrm{supp}(G_+\varphi) \subset J_+^M(\mathrm{supp}(\varphi))$ for all $\varphi \in \mathscr{D}(M,E)$,

is called an *advanced Green's operator for P*. Similarly, a linear map $G_- : \mathscr{D}(M,E) \to C^\infty(M,E)$ satisfying (i), (ii), and

(iii') $\mathrm{supp}(G_-\varphi) \subset J_-^M(\mathrm{supp}(\varphi))$ for all $\varphi \in \mathscr{D}(M,E)$

instead of (iii) is called a *retarded Green's operator for P*.

Fundamental solutions and Green's operators are closely related.

Theorem 7.2. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a wave operator acting on sections in a vector bundle E over M.*
Then there exist unique advanced and retarded Green's operators $G_\pm : \mathscr{D}(M,E) \to C^\infty(M,E)$ *for P.*

Proof. By Theorem 6.2 there exist families $F_\pm(x)$ of advanced and retarded fundamental solutions for the adjoint operator P^* respectively. We know that $F_\pm(x)$ depend smoothly on x and the differential equation $P(F_\pm(\cdot)[\varphi]) = \varphi$ holds. By definition we have

$$P(G_\pm\varphi) = P(F_\mp(\cdot)[\varphi]) = \varphi$$

thus showing (i). Assertion (ii) follows from the fact that the $F_\pm(x)$ are fundamental solutions,

$$G_\pm(P\varphi)(x) = F_\mp(x)[P\varphi] = P^*F_\mp(x)[\varphi] = \delta_x[\varphi] = \varphi(x).$$

To show (iii) let $x \in M$ such that $(G_+\varphi)(x) \neq 0$. Since $\mathrm{supp}(F_-(x)) \subset J_-^M(x)$ the support of φ must hit $J_-^M(x)$. Hence $x \in J_+^M(\mathrm{supp}(\varphi))$ and therefore $\mathrm{supp}(G_+\varphi) \subset J_+^M(\mathrm{supp}(\varphi))$. The argument for G_- is analogous. □

We have seen that existence of fundamental solutions for P^* depending nicely on x implies existence of Green's operators for P. This construction can be reversed. Then uniqueness of fundamental solutions in Theorem 6.2 implies uniqueness of Green's operators.

Lemma 7.3. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a wave operator acting on sections in a vector bundle E over M. Let* $G_\pm$ *be the Green's operators for P and* $G_\pm^*$ *the Green's operators for the adjoint operator* P^**. Then*

$$\int_M (G_\pm^*\varphi) \cdot \psi \,\mathrm{dV} = \int_M \varphi \cdot (G_\mp\psi) \,\mathrm{dV} \tag{3}$$

holds for all $\varphi \in \mathscr{D}(M,E^*)$ *and* $\psi \in \mathscr{D}(M,E)$.

Proof. For the Green's operators we have $PG_\pm = \mathrm{id}_{\mathscr{D}(M,E)}$ and $P^*G^*_\pm = \mathrm{id}_{\mathscr{D}(M,E^*)}$ and hence

$$\begin{aligned}
\int_M (G^*_\pm \varphi)\cdot\psi\, \mathrm{dV} &= \int_M (G^*_\pm\varphi)\cdot(PG_\mp\psi)\,\mathrm{dV}\\
&= \int_M (P^*G^*_\pm\varphi)\cdot(G_\mp\psi)\,\mathrm{dV}\\
&= \int_M \varphi\cdot(G_\mp\psi)\,\mathrm{dV}.
\end{aligned}$$

Notice that $\mathrm{supp}(G_\pm\varphi)\cap\mathrm{supp}(G_\mp\psi)\subset J^M_\pm(\mathrm{supp}(\varphi))\cap J^M_\mp(\mathrm{supp}(\psi))$ is compact in a globally hyperbolic manifold so that the partial integration in the second equation is justified. □

Notation 7.4. We write $C^\infty_{\mathrm{sc}}(M,E)$ for the set of all $\varphi\in C^\infty(M,E)$ for which there exists a compact subset $K\subset M$ such that $\mathrm{supp}(\varphi)\subset J^M(K)$. Obviously, $C^\infty_{\mathrm{sc}}(M,E)$ is a vector subspace of $C^\infty(M,E)$.
The subscript "sc" should remind the reader of "space-like compact". Namely, if M is globally hyperbolic and $\varphi\in C^\infty_{\mathrm{sc}}(M,E)$, then for every Cauchy hypersurface $S\subset M$ the support of $\varphi|_S$ is contained in $S\cap J^M(K)$ hence compact by Lemma 3.8. In this sense sections in $C^\infty_{\mathrm{sc}}(M,E)$ have space-like compact support.

Definition 7.5. We say that a sequence of elements $\varphi_j\in C^\infty_{\mathrm{sc}}(M,E)$ *converges in $C^\infty_{\mathrm{sc}}(M,E)$ to $\varphi\in C^\infty_{\mathrm{sc}}(M,E)$* if there exists a compact subset $K\subset M$ such that

$$\mathrm{supp}(\varphi)\subset J^M(K) \text{ and } \mathrm{supp}(\varphi_j)\subset J^M(K)$$

for all j and

$$\|\varphi_j-\varphi\|_{C^k(K',E)}\to 0$$

for all $k\in\mathbb{N}$ and all compact subsets $K'\subset M$.

If G_+ and G_- are advanced and retarded Green's operators for P respectively, then we get a linear map

$$G := G_+ - G_- : \mathscr{D}(M,E)\to C^\infty_{\mathrm{sc}}(M,E).$$

Much of the solution theory of wave operators on globally hyperbolic Lorentzian manifolds is collected in the following theorem.

Theorem 7.6. *Let M be a globally hyperbolic Lorentzian manifold. Let P be a wave operator acting on sections in a vector bundle E over M. Let G_+ and G_- be advanced and retarded Green's operators for P respectively.*
Then

$$0\to\mathscr{D}(M,E)\xrightarrow{P}\mathscr{D}(M,E)\xrightarrow{G}C^\infty_{\mathrm{sc}}(M,E)\xrightarrow{P}C^\infty_{\mathrm{sc}}(M,E) \tag{4}$$

is an exact sequence of linear maps.

Proof. Properties (i) and (ii) in Definition 7.1 of Green's operators directly yield $G\circ P=0$ and $P\circ G=0$, both on $\mathscr{D}(M,E)$. Properties (iii) and (iii') ensure that G maps $\mathscr{D}(M,E)$ to $C^\infty_{\mathrm{sc}}(M,E)$. Hence the sequence of linear maps forms a complex.
Exactness at the first $\mathscr{D}(M,E)$ means that

$$P:\mathscr{D}(M,E)\to\mathscr{D}(M,E)$$

is injective. To see injectivity let $\varphi\in\mathscr{D}(M,E)$ with $P\varphi=0$. Then $\varphi=G_+P\varphi=G_+0=0$.
Next let $\varphi\in\mathscr{D}(M,E)$ with $G\varphi=0$, i. e. $G_+\varphi=G_-\varphi$. We put $\psi:=G_+\varphi=G_-\varphi\in C^\infty(M,E)$ and we see $\mathrm{supp}(\psi)=\mathrm{supp}(G_+\varphi)\cap\mathrm{supp}(G_-\varphi)\subset J^M_+(\mathrm{supp}(\varphi))\cap$

$J^M_-(\mathrm{supp}(\varphi))$. Since (M,g) is globally hyperbolic $J^M_+(\mathrm{supp}(\varphi)) \cap J^M_-(\mathrm{supp}(\varphi))$ is compact, hence $\psi \in \mathscr{D}(M,E)$. From $P(\psi) = P(G_+(\varphi)) = \varphi$ we see that $\varphi \in P(\mathscr{D}(M,E))$. This shows exactness at the second $\mathscr{D}(M,E)$.
Finally, let $\varphi \in C^\infty_{\mathrm{sc}}(M,E)$ such that $P\varphi = 0$. Without loss of generality we may assume that $\mathrm{supp}(\varphi) \subset I^M_+(K) \cup I^M_-(K)$ for a compact subset K of M. Using a partition of unity subordinated to the open covering $\{I^M_+(K), I^M_-(K)\}$ write φ as $\varphi = \varphi_1 + \varphi_2$ where $\mathrm{supp}(\varphi_1) \subset I^M_-(K) \subset J^M_-(K)$ and $\mathrm{supp}(\varphi_2) \subset I^M_+(K) \subset J^M_+(K)$. For $\psi := -P\varphi_1 = P\varphi_2$ we see that $\mathrm{supp}(\psi) \subset J^M_-(K) \cap J^M_+(K)$, hence $\psi \in \mathscr{D}(M,E)$.
We check that $G_+\psi = \varphi_2$. For all $\chi \in \mathscr{D}(M,E^*)$ we have

$$\int_M \chi \cdot (G_+P\varphi_2)\,\mathrm{dV} = \int_M (G^*_-\chi) \cdot (P\varphi_2)\,\mathrm{dV} = \int_M (P^*G^*_-\chi) \cdot \varphi_2\,\mathrm{dV} = \int_M \chi \cdot \varphi_2\,\mathrm{dV}$$

where G^*_- is the Green's operator for the adjoint operator P^* according to Lemma 7.3. Notice that for the second equation we use the fact that $\mathrm{supp}(\varphi_2) \cap \mathrm{supp}(G^*_-\chi) \subset J^M_+(K) \cap J^M_-(\mathrm{supp}(\chi))$ is compact. Similarly, one shows $G_-\psi = -\varphi_1$.
Now $G\psi = G_+\psi - G_-\psi = \varphi_2 + \varphi_1 = \varphi$, hence φ is in the image of G. $\square$

Proposition 7.7. *Let M be a globally hyperbolic Lorentzian manifold, let P be a wave operator acting on sections in a vector bundle E over M. Let G_+ and G_- be the advanced and retarded Green's operators for P respectively.*
Then $G_\pm : \mathscr{D}(M,E) \to C^\infty_{\mathrm{sc}}(M,E)$ and all maps in the complex (4) are continuous.

Proof. The maps $P : \mathscr{D}(M,E) \to \mathscr{D}(M,E)$ and $P : C^\infty_{\mathrm{sc}}(M,E) \to C^\infty_{\mathrm{sc}}(M,E)$ are continuous simply because P is a differential operator. It remains to show that $G : \mathscr{D}(M,E) \to C^\infty_{\mathrm{sc}}(M,E)$ is continuous.
Let $\varphi_j, \varphi \in \mathscr{D}(M,E)$ and $\varphi_j \to \varphi$ in $\mathscr{D}(M,E)$ for all j. Then there exists a compact subset $K \subset M$ such that $\mathrm{supp}(\varphi_j) \subset K$ for all j and $\mathrm{supp}(\varphi) \subset K$. Hence $\mathrm{supp}(G\varphi_j) \subset J^M(K)$ for all j and $\mathrm{supp}(G\varphi) \subset J^M(K)$. From the proof of Theorem 6.2 we know that $G_+\varphi$ coincides with the solution u to the Cauchy problem $Pu = \varphi$ with initial conditions $u|_{S_-} = (\nabla_\nu u)|_{S_-} = 0$ where $S_- \subset M$ is a spacelike Cauchy hypersurface such that $K \subset I^M_+(S_-)$. Theorem 5.6 tells us that if $\varphi_j \to \varphi$ in $\mathscr{D}(M,E)$, then the solutions $G_+\varphi_j \to G_+\varphi$ in $C^\infty(M,E)$. The proof for G_- is analogous and the statement for G follows. $\square$

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