The Dirac operator on space forms of positive curvature

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Abstract

The spectrum of the Dirac operator on spherical space forms is calculated. Manifolds with many Killing spinors are characterized. In the last section non-isometric space forms with the same Dirac spectrum are constructed.

1. Introduction

Riemannian spin manifolds carry an important natural operator, the *Dirac operator*. The Dirac operator is an elliptic differential operator of first order acting on spinor fields, hence its spectrum is discrete point spectrum if the underlying manifold is compact. An excellent introduction to the general theory of Dirac operators can be found in [15]. The relation between the spectrum and the geometry of the manifold is currently an object of intense research. Explicit calculation of the spectrum is possible only for very nice manifolds. For example, for homogeneous spaces the calculation can be reduced to representation theoretic computations which still can be very hard, see [2]. To the author's knowledge the first explicit calculation was done by Friedrich in [9] for the flat torus to demonstrate the dependence of the Dirac spectrum on the choice of spin structure.

In this paper we study the Dirac spectrum of the sphere and of its quotients. Ikeda obtained analogous results for the Laplace operator on spherical space forms in a series of papers [10]-[14]. In [10] he calculates the spectrum of the Laplace operator acting on functions, in [14] he does the same for the Laplace operator acting on p-forms. In [12] and [13] he constructs non-isometric examples with the same Laplace spectrum.

We begin with the calculation of the Dirac spectrum on the standard sphere. Sulanke already did this in her unplublished thesis [17] using the representation theoretic methods mentioned above. But the necessary computations in her work are lengthy and it seemed desirable to find a simpler way to do it. Our main tool is the use of Killing spinors. Killing spinors are spinor fields satisfying a certain highly over-determined differential equation. Generically, they don't exist, but on the standard sphere they can be used to trivialize the spinor bundle. In this trivialization the calculation can be carried out without too much pain. The eigenvalues on S^n turn out to have a very simple form, they are given by $\pm(\frac{n}{2}+k)$, $k \geq 0$ (Theorem 1).

In the third section we study quotients of spheres $\Gamma\backslash S^n$. Eigenspinors on the quotient correspond to Γ-invariant eigenspinors on the sphere. Therefore the quotient has the same eigenvalues as the sphere, but the multiplicities will in general be smaller. We define certain power series with such multiplicities as coefficients and express them in terms of Γ and the spin structure (Theorem 2). This way of encoding the spectrum of quotients had already been used by Ikeda for the Laplace operator.

As a direct consequence we obtain a formula for the dimension of the space of Killing spinors (Theorem 3). We show that a manifold with many Killing spinors in a sense to be made precise has to be either the sphere or in certain dimensions it can also be real projective space (Theorem 4). This improves a result by Franc [7, Thm. 2].

In the last section we construct non-isometric spherical space forms with metacyclic fundamental groups having the same Dirac spectrum. Therefore we see that the Dirac spectrum does not carry enough information to determine the isometry class of such a space form.

2. Dirac eigenvalues of S^n

Let S^n be the n-dimensional sphere carrying the standard metric of constant sectional curvature 1, $n \geq 2$. The classical Dirac operator acting on spinor fields over S^n is denoted by D and ∇ is the Levi-Civita connection acting on vector fields or on spinor fields. In this section we will calculate the spectrum of D. This can be performed by regarding $S^n = Spin(n+1)/Spin(n)$ as a homogeneous space and using representation theoretic methods, see S. Sulanke's thesis [17]. The necessary calculations however are lengthy and by now there is a much simpler way to do it using Killing spinors.

Let $\mu = \pm\frac{1}{2}$. A *Killing spinor* with Killing constant μ is a spinor field Ψ satisfying the equation

$$\tilde{\nabla}_X \Psi := \nabla_X \Psi - \mu \cdot X \cdot \Psi = 0 \tag{1}$$

for all tangent vectors X. Killing spinors are useful in this context because of the following well known lemma.

Lemma 1. *The spinor bundle ΣS^n can be trivialized by Killing spinors for $\mu = \frac{1}{2}$ as well as for $\mu = -\frac{1}{2}$.*

Proof. Since S^n is simply connected it is enough to show that the curvature of the connection $\tilde{\nabla}$ vanishes.

Let $p \in S^n$, let X, Y be vector fields near p, let Ψ be a spinor field near p.

For simplicity we assume $\nabla X(p) = \nabla Y(p) = 0$. We calculate at p

$$\begin{aligned}\tilde{\nabla}_X \tilde{\nabla}_Y \Psi &= (\nabla_X - \mu X)(\nabla_Y - \mu Y)\Psi \\ &= \nabla_X \nabla_Y \Psi - \mu Y \nabla_X \Psi - \mu X \nabla_Y \Psi + \frac{1}{4} XY\end{aligned}$$

from which we deduce

$$R^{\tilde{\nabla}}(X,Y)\Psi = R^{\Sigma}(X,Y)\Psi + \frac{1}{4}(XY - YX)\Psi. \tag{2}$$

The curvature R^Σ of the spinor bundle is related to the curvature R of the tangent bundle by the formula (see [15, p.110, Thm. 4.15])

$$R^{\Sigma}(X,Y) = \frac{1}{4}\sum_{i,j=1}^{n} \langle R(X,Y)e_i, e_j\rangle e_i e_j \tag{3}$$

where $e_1, \ldots, e_n$ is a local orthonormal basis of the tangent bundle.

Since the sectional curvature is constant 1, R is of the form

$$R(X,Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y. \tag{4}$$

Combining (3) and (4) yields

$$R^{\Sigma}(X,Y) = \frac{1}{4}(YX - XY) \tag{5}$$

which together with (2) gives $R^{\tilde{\nabla}} = 0$. $\square$

The following *Weitzenböck formula* relates the connection $\tilde{\nabla}$ over the sphere to the Dirac operator D.

Lemma 2. *On S^n with the standard metric of sectional curvature 1 the following formula holds:*

$$(D + \mu)^2 = \tilde{\nabla}^* \tilde{\nabla} + \frac{1}{4}(n-1)^2.$$

Proof. Let $p \in S^n$, let $e_1, \ldots, e_n$ be a local orthonormal frame near p such that $\nabla e_i(p) = 0$. At p we get

$$\begin{aligned}(D+\mu)^2 - \tilde{\nabla}^*\tilde{\nabla} &= (\sum_i e_i \nabla_{e_i} + \mu)(\sum_j e_j \nabla_{e_j} + \mu) + \sum_j \tilde{\nabla}_{e_j}\tilde{\nabla}_{e_j} \\ &= \sum_{i,j} e_i e_j \nabla_{e_i}\nabla_{e_j} + 2\mu D + \frac{1}{4} + \sum_j (\nabla_{e_j} - \mu e_j)(\nabla_{e_j} - \mu e_j)\end{aligned}$$

$$
\begin{aligned}
&= -\sum_j \nabla_{e_j}\nabla_{e_j} + \sum_{i<j} e_i e_j R^{\Sigma}(e_i,e_j) + 2\mu D + \frac{1}{4} \\
&\quad + \sum_j \nabla_{e_j}\nabla_{e_j} - 2\mu D - \frac{1}{4}n \\
&\overset{(5)}{=} \frac{1}{4}\sum_{i<j} e_i e_j (e_j e_i - e_i e_j) - \frac{1}{4}(n-1) \\
&= \frac{1}{4}n(n-1) - \frac{1}{4}(n-1) \\
&= \frac{1}{4}(n-1)^2. \square
\end{aligned}
$$

To proceed we choose an orthogonal basis $f_0 \equiv 1, f_1, f_2, \ldots$ of the L^2-functions on S^n, $L^2(S^n, \mathbb{R})$, consisting of eigenfunctions of the Laplace operator $\triangle = d^*d$, $\triangle f_i = \lambda_i f_i$. In view of Lemma 1 we see that $f_i\Psi_j$ form a basis of the L^2-spinor fields, $L^2(S^n, \Sigma S^n)$ where $\Psi_1, \ldots, \Psi_{2^{[n/2]}}$ are a trivialization of the spinor bundle by Killing spinors with Killing constant μ. The next lemma tells us that we found an eigenbasis for the operator $(D+\mu)^2$.

Lemma 3. $(D+\mu)^2(f_i\Psi_j) = (\lambda_i + \frac{1}{4}(n-1)^2) f_i \Psi_j$.

Proof. This follows directly from Lemma 2 and the fact that Ψ_j is $\tilde{\nabla}$-parallel.$\square$

The eigenvalues of the Laplace operator on S^n are well known, namely we have

Lemma 4. *The eigenvalues of the Laplace operator on S^n are*

$$k(n+k-1), k = 0, 1, 2, \ldots$$

with multiplicities

$$m_k = \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \frac{n+2k-1}{n+k-1}. \square$$

For a proof see [4, p. 159ff]. Combining Lemma 3 and Lemma 4 yields

Corollary. *$(D+\mu)^2$ has the eigenvalues $k(n+k-1) + \frac{1}{4}(n-1)^2, k = 0, 1, 2, \ldots$ with multiplicity $2^{[\frac{n}{2}]} \cdot m_k$.*$\square$

The next step is the calculation of the eigenvalues of $D+\mu$. First a general remark. If an operator A and a vector u satisfy

$$A^2u = \nu^2 u,$$

then we get for $v^\pm := \pm\nu u + Au$:

$$Av^\pm = \pm\nu v^\pm.$$

Hence if $v^\pm \neq 0$, then $\pm\nu$ is an eigenvalue of A.

In our case $A = D+\mu$. Let us first look at the case $k=0$, i.e. $u = \Psi_j$ and $\nu = -\mu(n-1)$.

$$\begin{aligned} v^+ &= -\mu(n-1)\Psi_j + (D+\mu)\Psi_j \\ &= -2\mu(n-1)\Psi_j. \end{aligned}$$

Thus $-\mu(n-1)$ is an eigenvalue of $D+\mu$ of multiplicity at least $2^{[\frac{n}{2}]}$. Since the multiplicity of the eigenvalue $\frac{1}{4}(n-1)^2$ of $(D+\mu)^2$ is $2^{[\frac{n}{2}]}$, the eigenvalue $-\mu(n-1)$ of $D+\mu$ has also exactly multiplicity $2^{[\frac{n}{2}]}$.

Now the case $k \geq 1$, i.e. $u = f_i\Psi_j$, $i \geq 1$.

$$\begin{aligned} \nu &= \sqrt{k(n+k-1) + \frac{1}{4}(n-1)^2} \\ &= k + \frac{n-1}{2}. \end{aligned}$$

Now we know all the eigenvalues of D, namely $-\mu n$ is an eigenvalue with multiplicity $2^{[\frac{n}{2}]}$ and the other eigenvalues are $-\mu \pm (k + \frac{n-1}{2}), k = 1, 2, 3, \ldots$. It remains to determine the other multiplicities.

To do this let us recall that we may choose $\mu = +\frac{1}{2}$ or $\mu = -\frac{1}{2}$. We start with $\mu = -\frac{1}{2}$. We introduce the following notation for the eigenvalues of D.

$$\lambda_0^+ = \frac{n}{2},$$

$$\lambda_k^+ = \frac{n}{2} + k, k \geq 1,$$

$$\lambda_{-k}^+ = 1 - \frac{n}{2} - k, k \geq 1.$$

We know the multiplicity of λ_0^+, namely $m(\frac{n}{2}) = 2^{[\frac{n}{2}]}$, and from the above Corollary we know $m(\lambda_k^+) + m(\lambda_{-k}^+) = 2^{[\frac{n}{2}]} \cdot m_k$.

Using $\mu = +\frac{1}{2}$ and the notation

$$\lambda_0^- = -\frac{n}{2},$$

$$\lambda_k^- = -1 + \frac{n}{2} + k, k \geq 1,$$

$$\lambda^-_{-k} = -\frac{n}{2} - k, k \geq 1$$

we obtain $m(-\frac{n}{2}) = 2^{[\frac{n}{2}]}$ and $m(\lambda^-_k) + m(\lambda^-_{-k}) = 2^{[\frac{n}{2}]} \cdot m_k$.

Lemma 5.

$$m(\lambda^+_k) = m(\lambda^-_{-k}) = 2^{[\frac{n}{2}]} \cdot \left(\begin{array}{c} k+n-1 \\ k \end{array} \right), k \geq 0.$$

Proof by induction on k. We saw already that the claim is true for $k = 0$. Let us carry out the induction step $k \to k+1$.

$$\begin{array}{rcl} m(\lambda^+_{k+1}) & = & 2^{[\frac{n}{2}]} \cdot m_{k+1} - m(\lambda^+_{-k-1}) \\ & = & 2^{[\frac{n}{2}]} \cdot m_{k+1} - m(\lambda^-_{-k}) \\ & = & 2^{[\frac{n}{2}]} \cdot \left\{ \left(\begin{array}{c} n+k \\ k+1 \end{array} \right) \cdot \frac{n+2k+1}{n+k} - \left(\begin{array}{c} k+n-1 \\ k \end{array} \right) \right\} \\ & = & 2^{[\frac{n}{2}]} \cdot \left(\begin{array}{c} n+k \\ k+1 \end{array} \right) \square \end{array}$$

Summing up everything we get

Theorem 1. *The classical Dirac operator on the sphere S^n of constant sectional curvature 1 has the eigenvalues*

$$\pm(\frac{n}{2} + k), \qquad k \geq 0$$

with multiplicities

$$2^{[\frac{n}{2}]} \cdot \left(\begin{array}{c} k+n-1 \\ k \end{array} \right). \square$$

3. Space forms

The group of orientation preserving isometries of S^n is given by $\mathrm{Iso}^+(S^n) = SO(n+1)$, $SO(n+1)$ acting from the left by matrix multiplication on $S^n \subset \mathbb{R}^{n+1}$. Oriented compact connected manifolds of constant sectional curvature 1 are of the form $\Gamma \backslash S^n$ where Γ is a finite fixed point free subgroup of $SO(n+1)$. The special orthogonal group $SO(n+1)$ also forms the total space of the bundle of oriented orthonormal tangent frames, the projection onto S^n given by projection on the first column vector, say. The action of $SO(n+1)$ on S^n lifts to an action on $SO(n+1)$, simply given by matrix multiplication. $SO(n)$

acts from the right on $SO(n+1)$ leaving invariant the first column vector. The total space of the frame bundle of a quotient $\Gamma\backslash S^n$ is given by $\Gamma\backslash SO(n+1)$.

Let $\Theta : Spin(n+1) \to SO(n+1)$ be the double covering mapping. Then $Spin(n+1)$ together with Θ is the spin structure of S^n. $Spin(n)$ acts as structure group from the right on $Spin(n+1)$ and $Spin(n+1)$ acts by group multiplication from the left on the total space of the spin structure. Spinor fields over S^n can be regarded as $Spin(n)$-equivariant mappings from $Spin(n+1)$ to the spinor space Σ_n. Spin structures of a quotient $\Gamma\backslash S^n$ are in 1-1 correspondence with homomorphisms $\epsilon : \Gamma \to Spin(n+1)$ such that $\Theta\circ\epsilon = id_\Gamma$. The total space of the spin structure is then given by $\epsilon(\Gamma)\backslash Spin(n+1)$ and spinor fields over the quotient correspond to $\epsilon(\Gamma)$-invariant spinor fields over S^n.

We only need to look at odd dimensional space forms because in even dimensions the only quotient of the sphere is real projective space which then is not even orientable, in particular not spin.

Let $M = \Gamma\backslash S^n$ be spin, $\Gamma \subset SO(n+1)$ a fixed point free subgroup, the spin structure of M being specified by $\epsilon : \Gamma \to Spin(n+1)$ such that $\Theta \circ \epsilon = id_\Gamma$, $n = 2m-1$ odd. The Dirac eigenvalues of S^n are of the form $\pm(\frac{n}{2}+k), k \geq 0$. The same holds for M but the multiplicities for M will in general be smaller than those for S^n. To know the Dirac spectrum of M means to know the multiplicities $m(\pm(\frac{n}{2}+k), D)$ for M. We encode this information into the following two power series

$$\begin{aligned} F_+(z) &= \sum_{k=0}^{\infty} m(\frac{n}{2}+k, D)z^k, \\ F_-(z) &= \sum_{k=0}^{\infty} m(-(\frac{n}{2}+k), D)z^k. \end{aligned}$$

Lemma 6. *$F_+(z)$ and $F_-(z)$ converge absolutely for $|z| < 1$.*

Proof. According to Theorem 1 $F_\pm(z)$ can be majorized by

$$2^{[\frac{n}{2}]} \cdot \sum_{k=0}^{\infty} \binom{k+n-1}{k} z^k.$$

This power series has radius of convergence $= 1$ because

$$\lim_{k\to\infty} \frac{\binom{k+n-1}{k}}{\binom{k+n}{k+1}} = 1.\Box$$

The aim of this section is to give formulas for $F_\pm(z)$ in terms of Γ and ϵ.

In even dimension $2m$ the complex spinor representation of $Spin(2m)$ on Σ_{2m} decomposes into two irreducible *half spin representations*

$$\begin{aligned} \rho^+ : Spin(2m) &\rightarrow Aut(\Sigma_{2m}^+), \\ \rho^- : Spin(2m) &\rightarrow Aut(\Sigma_{2m}^-). \end{aligned}$$

Let $\chi^\pm : Spin(2m) \rightarrow \mathbb{C}$ be the character of $\rho^\pm$. The main result of this section is

Theorem 2. *Let $\Gamma \backslash S^{2m-1}$ be a spherical space form with spin structure given by $\epsilon : \Gamma \rightarrow Spin(2m)$. Then the eigenvalues of the Dirac operator are $\pm(\frac{n}{2}+k), k \geq 0$, with multiplicities determined by*

$$\begin{aligned} F_+(z) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{\chi^-(\epsilon(\gamma)) - z\cdot\chi^+(\epsilon(\gamma))}{det(1_{2m} - z\cdot\gamma)}, \\ F_-(z) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{\chi^+(\epsilon(\gamma)) - z\cdot\chi^-(\epsilon(\gamma))}{det(1_{2m} - z\cdot\gamma)}. \end{aligned}$$

Before we prove the theorem let us draw a few conclusions. The formulas show that $F_\pm$ extend to meromorphic functions on the whole complex plane with finitely many poles.

Heat kernel asymptotics show that the volume of a closed Riemannian spin manifold is determined by its Dirac spectrum. The following argument, first used by Ikeda to study Laplace operators [10, Cor. 2.4], shows that for spherical space forms we need to know only half the spectrum.

Corollary 1. *If $\Gamma_1\backslash S^{2m-1}$ and $\Gamma_2\backslash S^{2m-1}$ have the same positive or the same negative Dirac spectrum, then*

$$|\Gamma_1| = |\Gamma_2|.$$

Proof. The power series $F_+(z)$ has a pole of order n at $z=1$ and

$$\lim_{z\rightarrow 1}(1-z)^n F_+(z) = \frac{2^{m-1}}{|\Gamma|}.$$

Thus $|\Gamma|$ is determined by F_+, hence by the positive Dirac spectrum. The same argument applies to F_-.□

Corollary 2. *Let Γ_1 and Γ_2 be two finite fixed point free subgroups of $SO(2m)$, let $\epsilon_i : \Gamma_i \rightarrow Spin(2m)$ be two homomorphisms such that $\Theta\circ\epsilon_i = id_{\Gamma_i}$.*

If there exists a bijective mapping $\Phi : \Gamma_1 \to \Gamma_2$ such that for every $\gamma \in \Gamma_1$ the two elements $\epsilon_1(\gamma)$ and $\epsilon_2(\Phi(\gamma))$ are conjugate in $Spin(2m)$, then the two space forms $\Gamma_1 \backslash S^{2m-1}$ and $\Gamma_2 \backslash S^{2m-1}$ are Dirac isospectral.

Proof. The power series $F_\pm$ coincide for the two groups because all ingredients are invariant under conjugation.□

Proof of Theorem 2. As in the previous section we first look at the operators $(D \pm \frac{1}{2})^2$. Put

$$G_\pm(z) := \sum_{k=0}^{\infty} m((\frac{n-1}{2} + k)^2, (D \pm \frac{1}{2})^2) z^k.$$

Over S^n the eigenspace E^k for the eigenvalue $(\frac{n-1}{2} + k)^2$ of $(D + \frac{1}{2})^2$ is spanned by products $f \cdot \Psi$ where Ψ is a Killing spinor with Killing constant $\mu = \frac{1}{2}$ and $f \in \mathcal{H}^k = \{$harmonic homogeneous polynomials of degree k on $\mathbb{R}^{2m}$, restricted to $S^{2m-1}\}$.

$Spin(2m)$ acts on the spinor fields over S^n and leaves invariant the eigenspaces E^k. We want to determine the dimension of the $\epsilon(\Gamma)$-invariant subspace of E^k because this is exactly the multiplicity $m((\frac{n-1}{2} + k)^2, (D + \frac{1}{2})^2)$.

How does the action of $Spin(2m)$ on the $\frac{1}{2}$-Killing spinors look like?
As mappings $\Psi : Spin(2m) \to \Sigma_n$ the $\frac{1}{2}$-Killing spinors are of the form

$$\Psi(g) = \rho^+(g^{-1}) \cdot \sigma, \sigma = \Psi(1) \in \Sigma_n,$$

see [6, Prop. 12]. Hence an element $g_0 \in Spin(2m)$ acts on Ψ by

$$\begin{aligned} (g_0\Psi)(g) &= \Psi(g_0^{-1}g) \\ &= \rho^+(g^{-1}g_0) \cdot \sigma \\ &= \rho^+(g^{-1})\rho^+(g_0)\sigma. \end{aligned}$$

By identifying Ψ with σ we see that the action of $Spin(2m)$ on the space of $\frac{1}{2}$-Killing spinors is equivalent to ρ^+.

Now let ρ_k be the representation of $SO(2m)$ on $\mathcal{H}^k$ with character χ_k. We have just seen that the representation of $Spin(2m)$ on E^k is equivalent to $(\rho_k \circ \Theta) \otimes \rho^+$. The dimension of the $\epsilon(\Gamma)$-invariant subspace is given by

$$\begin{aligned} dim(E^k)^{\epsilon(\Gamma)} &= \langle \chi_k \cdot \chi_{\rho^+ \circ \epsilon}, 1 \rangle \\ &= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_k(\gamma)\chi^+(\epsilon(\gamma)). \end{aligned} \tag{6}$$

Ikeda [10, p. 81] calculated

$$\sum_{k=0}^{\infty} \chi_k(\gamma) z^k = \frac{1 - z^2}{det(1_{2m} - z \cdot \gamma)}. \tag{7}$$

From (6) and (7) we obtain

$$G_+(z) = \frac{1-z^2}{|\Gamma|} \sum_{\gamma\in\Gamma} \frac{\chi^+(\epsilon(\gamma))}{det(1_{2m} - z\cdot\gamma)}.$$

In the same way we get

$$G_-(z) = \frac{1-z^2}{|\Gamma|} \sum_{\gamma\in\Gamma} \frac{\chi^-(\epsilon(\gamma))}{det(1_{2m} - z\cdot\gamma)}.$$

As in the previous section we have

$$m(\frac{n}{2}+k, D) + m(-\frac{n}{2}-k+1, D) = m((\frac{n}{2}+k)^2, (D-\frac{1}{2})^2)$$

which means for the power series

$$F_+(z) + z\cdot F_-(z) = G_-(z). \tag{8}$$

Similarly,

$$z\cdot F_+(z) + F_-(z) = G_+(z). \tag{9}$$

Solving (8) and (9) for F_+ and F_- finishes the proof. □

4. Killing spinors

In the second section we have seen that on a spherical space form of curvature 1 the Killing spinors with Killing constant $\mu = \pm\frac{1}{2}$ are exactly the eigenspinors of the Dirac operator for the eigenvalue $-\mu n$. To get the dimension of the space of Killing spinors we only need to plug in $z = 0$ into the series $F_\mp(z)$.

Theorem 3. *Let $\Gamma\backslash S^{2m-1}$ be a spherical space form with spin structure given by $\epsilon : \Gamma \to Spin(2m)$. Then the dimension of the space of Killing spinors with Killing constant $\mu = \frac{1}{2}$ is given by*

$$\frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \chi^+(\epsilon(\gamma))$$

whereas for the Killing constant $\mu = -\frac{1}{2}$ the dimension is

$$\frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \chi^-(\epsilon(\gamma)).\square$$

Killing spinors on 3-dimensional spherical space forms have been studied by Friedrich in [8] and the 5-dimensional case was done by Sulanke in [18].

Example. Let us look at the case $\Gamma = \{\pm 1_{2m}\}$, i.e. $\Gamma \backslash S^{2m-1} = \mathbb{RP}^{2m-1}$ is real projective space. It is not hard to see that the two preimages $\pm\omega \in Spin(2m)$ of $-1_{2m} \in SO(2m)$ satisfy

$$(\pm\omega)^2 = (-1)^m.$$

In fact, if we view $Spin(2m)$ as sitting in the Clifford algebra $Cl(\mathbb{R}^{2m})$, then ω is just $\omega = e_1 \cdot e_2 \cdot \ldots \cdot e_{2m}$.

If m is odd, then there is no homomorphism $\epsilon : \Gamma \to Spin(2m)$ with $\Theta \circ \epsilon = id_\Gamma$ because we need to have

$$1 = \epsilon(1_{2m}) = \epsilon((-1_{2m})^2) = \epsilon(-1_{2m})^2 = (\pm\omega)^2 = (-1)^m.$$

In other words, $\mathbb{RP}^{2m-1}$ is not spin if m is odd.

If m is even, then $\mathbb{RP}^{2m-1}$ carries two spin structures given by $\epsilon_\pm(-1_{2m}) = \pm\omega$. The decomposition $\Sigma_{2m} = \Sigma^+_{2m} \oplus \Sigma^-_{2m}$ is nothing but the eigenspace decompsition for ω [15, p. 129]. Therefore $\chi^\pm(\omega) = \pm 2^{m-1}$. For the spin structure given by ϵ_+ we get from Theorem 3 that there are 2^{m-1} linearly independent Killing spinors with Killing constant $\mu = \frac{1}{2}$ whereas there are no nontrivial Killing spinors with $\mu = -\frac{1}{2}$. If we change to ϵ_- then we have to interchange μ and $-\mu$.

We have seen that in dimension $n \equiv 3(4)$ there is another manifold, namely real projective space, besides the sphere having the maximal number of linearly independent Killing spinors at least for one of the two possible Killing constants $\pm\frac{1}{2}$. Franc showed in [7, Thm. 2] that there are no further examples in the class of lens spaces. But there are actually no further such examples at all.

Theorem 4. *Let M be a closed connected Riemannian spin manifold of dimension n having $2^{[\frac{n}{2}]}$ linearly independent Killing spinors with the same Killing constant $\mu = \pm\frac{1}{2}$. Then either M is isometric to the standard sphere S^n or $n \equiv 3(4)$ and M is isometric to $\mathbb{RP}^n$.*

Proof. From the classification of simply connected manifolds with Killing spinors [3] it follows that the universal covering of M is isometric to S^n. Hence M is a spherical space form, $M = \Gamma\backslash S^n$. If n is even, the only space form is real projective space which is not spin in this case.

We may therefore assume $n = 2m-1$ odd. Let's say M carries 2^{m-1} Killing spinors with Killing constant $\mu = +\frac{1}{2}$. For $\gamma \in \Gamma$ the automorphism $\rho^+(\epsilon(\gamma))$ is unitary, thus all eigenvalues have absolute value 1. Triangle inequality applied to the formula of Theorem 3 shows that the dimension of the space of Killing spinors with a fixed Killing constant is bounded by 2^{m-1}. In our case we have equality, hence all eigenvalues of all $\rho^+(\epsilon(\gamma))$ must be one. In other words, Γ acts trivially via $\rho^+ \circ \epsilon$.

If m is odd, then the tensor product $\rho^+ \otimes \rho^+$ contains the complexification of the standard representation on $\mathbb{R}^{2m}$ given by $\Theta : Spin(2m) \to SO(2m)$, see [5, p. 280]. The only element of $SO(2m)$ acting trivially on $\mathbb{C}^{2m}$ is the neutral element 1_{2m}. Thus Γ must be trivial and M is isometric to S^n.

If m is even the tensor product $\rho^+ \otimes \rho^+$ contains the complexification of the representation on $\Lambda^2\mathbb{R}^{2m}$, see again [5, p. 280]. The only matrices in $SO(2m)$ acting trivially on $\Lambda^2\mathbb{C}^{2m}$ are $\pm 1_{2m}$. Hence $\Gamma = \{\pm 1_{2m}\}$ or Γ is trivial. $\square$

5. Isospectral examples

In this section we will construct examples of spherical space forms which are not isometric but which have the same Dirac spectra. Therefore the Dirac spectrum does not determine the isometry class of a spherical space form. In this respect the Dirac operator behaves similarly to the Laplace operator, compare Ikeda's papers [12] and [13].

One might expect that the simplest class to look for isospectral examples are lens spaces but we want to use Corollary 2 to Theorem 2 and we need two fixed point free subgroups of $Spin(2m)$ such that there is a bijection between them under which the corresponding elements are conjugate in $Spin(2m)$. If the groups in question are cyclic, then this implies that the two groups are conjugate (conjugation by the element in $Spin(2m)$ which sends a generator of the first group to something in the second) and hence the corresponding space forms are isometric. Therefore we have to deal with more complicated fundamental groups. This does not mean that isospectral lens spaces are isometric but the argument would have to be different.

Let a and b be two positive odd integers, let r be a positive integer such that $r^b \equiv 1(a)$ and $((r-1)b, a) = 1$. We denote by $\Gamma(a, b, r)$ the group generated by the two elements A and B satisfying the relations $A^a = B^b = 1$ and $BAB^{-1} = A^r$. Such groups are called *metacyclic.*

Lemma 7. *A spherical space form with fundamental group isomorphic to $\Gamma(a, b, r)$ as above has exactly one spin structure.*

Proof. Let $\pm\tilde{A}, \pm\tilde{B} \in Spin(2m)$ be the preimages of $A, B \in \Gamma(a, b, r) \subset SO(2m)$. For the spin structure corresponding to $\epsilon : \Gamma(a, b, r) \to Spin(2m)$ we need to have

$$\tilde{A}^a = \epsilon(A^a) = 1. \tag{10}$$

Since a is odd we can arrange (10) by passing from $\tilde{A}$ to $-\tilde{A}$ if necessary. Similarly, by choosing the correct preimage $\tilde{B}$ of B we have $\tilde{B}^b = 1$ and we can put $\epsilon(B) = \tilde{B}$.

We have already seen uniqueness of the spin structure. It remains to check

$$\tilde{B}\tilde{A}\tilde{B}^{-1} = \tilde{A}^r. \tag{11}$$

We know

$$\tilde{B}\tilde{A}\tilde{B}^{-1} = \delta\tilde{A}^r, \delta = \pm 1. \tag{12}$$

By taking (12) to the power a we get $1 = \delta^a$, hence $\delta = 1$ which is (11).□

Lemma 8. *Let Γ_1 and Γ_2 be two fixed point free subgroups of $SO(2m)$ isomorphic to $\Gamma(a,b,r)$. Let $\epsilon_i : \Gamma_i \to \tilde{\Gamma}_i \subset Spin(2m)$ be the spin structures from Lemma 7. If $X \in \Gamma_1$ and $Y \in \Gamma_2$ are conjugate in $SO(2m)$, then $\epsilon_1(X)$ and $\epsilon_2(Y)$ are conjugate in $Spin(2m)$.*

Proof. Let $S \in SO(2m)$ such that $SXS^{-1} = Y$. Choose a preimage $\tilde{S}$ of S in $Spin(2m)$. Then we know

$$\tilde{S}\epsilon_1(X)\tilde{S}^{-1} = \delta\epsilon_2(Y), \delta = \pm 1. \tag{13}$$

Taking (13) to the power ab yields $1 = \delta^{ab} = \delta$. □

Now we construct two different embeddings $\Gamma(a,b,r) \to SO(2m)$. Let d be the smallest positive integer such that $r^d \equiv 1(a)$. Hence d divides b, i.e. $b = db'$ for some integer b'. Put

$$R(\theta) = \begin{pmatrix} \cos(2\pi\theta) & \sin(2\pi\theta) \\ -\sin(2\pi\theta) & \cos(2\pi\theta) \end{pmatrix} \in SO(2)$$

and define

$$\pi_1(A) = \pi_2(A) = \begin{pmatrix} R(\frac{1}{a}) & & & \\ & R(\frac{r}{a}) & & \\ & & \ddots & \\ & & & R(\frac{r^{d-1}}{a}) \end{pmatrix} \in SO(2d),$$

$$\pi_1(B) = \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ R(\frac{1}{b'}) & 0 & \cdots & 0 \end{pmatrix} \in SO(2d).$$

For a positive integer l with $(l,b) = 1$ we set

$$\pi_2(B) = \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ R(\frac{l}{b'}) & 0 & \cdots & 0 \end{pmatrix} \in SO(2d).$$

Lemma 9. *For any integers s and t the matrices $\pi_1(A^sB^{lt})$ and $\pi_2(A^sB^t)$ are conjugate in $O(2d)$.*

Proof. Elementary calculation [13, p.443] shows that $\pi_1(A^sB^{lt})$ and $\pi_2(A^sB^t)$ have the same characteristic polynomial. Since two matrices in $SO(2d)$ have the same characteristic polynomial if and only if they are conjugate in $O(2d)$ the Lemma is proved. $\square$

Now we set $m = 2d$ and define two embeddings $i_1, i_2 : \Gamma(a, b, r) \to SO(2m)$ by

$$i_1(X) = (\pi_1 \oplus \pi_1)(X) = \begin{pmatrix} \pi_1(X) & 0 \\ 0 & \pi_1(X) \end{pmatrix},$$

$$i_2(X) = (\pi_2 \oplus \pi_2)(X) = \begin{pmatrix} \pi_2(X) & 0 \\ 0 & \pi_2(X) \end{pmatrix},$$

$$\Gamma_1 = i_1(\Gamma(a, b, r)), \Gamma_2 = i_2(\Gamma(a, b, r)).$$

Lemma 10. *The two space forms $\Gamma_1\backslash S^{2m-1}$ and $\Gamma_2\backslash S^{2m-1}$ are Dirac isospectral.*

Proof. Define a bijective map $\Phi : \Gamma_1 \to \Gamma_2$ by

$$\Phi(i_1(A^sB^{lt})) = i_2(A^sB^t).$$

From Lemma 9 we know that for each s and t there is an $S \in O(2d)$ such that $\pi_2(A^sB^t) = S\pi_1(A^sB^{lt})S^{-1}$. Thus

$$\Phi(i_1(A^sB^{lt})) = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \cdot i_1(A^sB^{lt}) \cdot \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}^{-1}.$$

Since $\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \in SO(2m)$ Lemma 8 says that $\epsilon(i_1(A^sB^{lt}))$ and $\epsilon(\Phi(i_1(A^sB^{lt})))$ are conjugate in $Spin(2m)$ and the Lemma follows from Corollary 2 in Section 3.$\square$

It remains to find conditions under which $\Gamma_1\backslash S^{2m-1}$ and $\Gamma_2\backslash S^{2m-1}$ are not isometric. Let us recall the conditions on the positive integers a, b, b', r, d and l. a and b are odd, r satisfies $r^b \equiv 1(a)$ and $((r-1)b, a) = 1$. d is the smallest number such that $r^d \equiv 1(a)$ and $b = db'$. Finally, $(l, b) = 1$ and $m = 2d$.

From the classification of spherical space forms [19, p. 171] it is known that if the two space forms $\Gamma_1 \backslash S^{2m-1}$ and $\Gamma_2 \backslash S^{2m-1}$ are isometric, then there is an integer t such that

$$(t, b) = 1, t \equiv 1(d) \text{ and } l \equiv \pm t(b').$$

Now we need to arrange everything so that the latter is not possible.

We start with a positive odd number $d \geq 5$. By Dirichlet's prime number theorem [16, p. 73, Thm. 2] we can choose a prime number a of the form $a = 1 + kd, k \geq 1$. We put $b = d^2, b' = d$ and $m = 2d$. From $(d, a) = 1$ we obtain $(b, a) = 1$. The multiplicative group $(\mathbb{Z}/a\mathbb{Z})^*$ of the field $\mathbb{Z}/a\mathbb{Z}$ is cyclic of order kd [16, p. 4, Thm. 2]. Hence there is an element $r \in (\mathbb{Z}/a\mathbb{Z})^*$ of order d. Finally, we set $l = 2$. Now all necessary conditions are fulfilled.

Assume there is a t as above. Then we get $2 = l \equiv \pm t \equiv \pm 1(d)$ which implies $1 \equiv 0(d)$ or $3 \equiv 0(d)$, a contradiction in either case. We summarize

Theorem 5. *Let $d \geq 5$ be odd. Then there exist two non-isometric spherical space forms of dimension $4d - 1$ having the same Dirac spectrum. Their fundamental groups are isomorphic to $\Gamma(a, b, r)$ where a, b, and r are chosen as above.*□

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