CCR- versus CAR-quantization on curved spacetimes

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Abstract. We provide a systematic construction of bosonic and fermionic locally covariant quantum field theories on curved backgrounds for large classes of free fields. It turns out that bosonic quantization is possible under much more general assumptions than fermionic quantization.

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1. Introduction

Classical fields on spacetime are mathematically modeled by sections of a vector bundle over a Lorentzian manifold. The field equations are usually partial differential equations. We introduce a class of differential operators, called Green-hyperbolic operators, which have good analytical solubility properties. This class includes wave operators as well as Dirac type operators but also the Proca and the Rarita-Schwinger operator.

In order to quantize such a classical field theory on a curved background, we need local algebras of observables. They come in two flavors, bosonic algebras encoding the canonical commutation relations and fermionic algebras encoding the canonical anti-commutation relations. We show how such algebras can be associated to manifolds equipped with suitable Green-hyperbolic operators. We prove that we obtain locally covariant quantum field theories in the sense of [12]. There is a large literature where such constructions are carried out for particular examples of fields, see e.g. [15, 16, 17, 22, 30]. In all these papers the well-posedness of the Cauchy problem plays an important role. We avoid using the Cauchy problem altogether and only make use of Green's operators. In this respect, our approach is similar to the one in [31]. This allows us to deal with larger classes of fields, see Section 3.7, and to treat them systematically. Much of the work on particular examples can be subsumed under this general approach.

It turns out that bosonic algebras can be obtained in much more general situations than fermionic algebras. For instance, for the classical Dirac field both constructions are possible. Hence, on the level of observable algebras, there is no spin-statistics theorem.

This is a condensed version of our paper [4] where full details are given. Here we confine ourselves to the results and the main arguments while we leave aside all technicalities. Moreover, [4] contains a discussion of states and the induced quantum fields.

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2. Algebras of canonical (anti-) commutation relations

We start with algebraic preparations and collect the necessary algebraic facts about CAR and CCR-algebras.

2.1. CAR algebras

The symbol "CAR" stands for "canonical anti-commutation relations". These algebras are related to pre-Hilbert spaces. We always assume the Hermitian inner product $(\cdot,\cdot)$ to be linear in the first argument and anti-linear in the second.

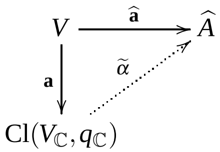
Definition 2.1. A CAR-*representation* of a complex pre-Hilbert space $(V,(\cdot,\cdot))$ is a pair $(\mathbf{a},A)$, where A is a unital C*-algebra and $\mathbf{a} : V \to A$ is an anti-linear map satisfying:

(i) $A = C^*(\mathbf{a}(V))$, that is, A is the C*-algebra generated by A,
(ii) $\{\mathbf{a}(v_1),\mathbf{a}(v_2)\} = 0$ and
(iii) $\{\mathbf{a}(v_1)^*,\mathbf{a}(v_2)\} = (v_1,v_2)\cdot 1$,

for all $v_1,v_2 \in V$.

As an example, for any complex pre-Hilbert vector space $(V,(\cdot,\cdot))$, the C*-completion $\mathrm{Cl}(V_{\mathbb{C}},q_{\mathbb{C}})$ of the algebraic Clifford algebra of the complexification $(V_{\mathbb{C}},q_{\mathbb{C}})$ of $(V,(\cdot,\cdot))$ is a CAR-representation of $(V,(\cdot,\cdot))$. See [4, App. A.1] for the details, in particular for the construction of the map $\mathbf{a} : V \to \mathrm{Cl}(V_{\mathbb{C}},q_{\mathbb{C}})$.

Theorem 2.2. *Let* $(V,(\cdot,\cdot))$ *be an arbitrary complex pre-Hilbert space. Let* $\widehat{A}$ *be any unital* C^**-algebra and* $\widehat{\mathbf{a}} : V \to \widehat{A}$ *be any anti-linear map satisfying Axioms* (ii) *and* (iii) *of Definition 2.1. Then there exists a unique* C^**-morphism* $\widetilde{\alpha} : \mathrm{Cl}(V_{\mathbb{C}},q_{\mathbb{C}}) \to \widehat{A}$ *such that*



commutes. Furthermore, $\widetilde{\alpha}$ is injective.
In particular, $(V,(\cdot\,,\cdot))$ has, up to C^*-isomorphism, a unique CAR*-representation.*

For an alternative description of the CAR-representation in terms of creation and annihilation operators on the fermionic Fock space we refer to [10, Prop. 5.2.2].

¿From now on, given a complex pre-Hilbert space $(V,(\cdot\,,\cdot))$, we denote the C*-algebra $\mathrm{Cl}(V_{\mathbb{C}},q_{\mathbb{C}})$ associated with the CAR-representation $(\mathbf{a},\mathrm{Cl}(V_{\mathbb{C}},q_{\mathbb{C}}))$ of $(V,(\cdot\,,\cdot))$ by $\mathrm{CAR}(V,(\cdot\,,\cdot))$. We list the properties of CAR-representations which are relevant for quantization, see also [10, Vol. II, Thm. 5.2.5, p. 15].

Proposition 2.3. *Let $(\mathbf{a},\mathrm{CAR}(V,(\cdot\,,\cdot)))$ be the* CAR*-representation of a complex pre-Hilbert space $(V,(\cdot\,,\cdot))$.*

(i) For every $v\in V$ one has $\|\mathbf{a}(v)\| = |v| = (v,v)^{\frac{1}{2}}$, where $\|\cdot\|$ denotes the C^*-norm on $\mathrm{CAR}(V,(\cdot\,,\cdot))$.
(ii) The C^*-algebra $\mathrm{CAR}(V,(\cdot\,,\cdot))$ is simple, i.e., it has no closed two-sided $*$-ideals other than $\{0\}$ and the algebra itself.
(iii) The algebra $\mathrm{CAR}(V,(\cdot\,,\cdot))$ is $\mathbb{Z}_2$-graded,

$$\mathrm{CAR}(V,(\cdot\,,\cdot)) = \mathrm{CAR}^{\mathrm{even}}(V,(\cdot\,,\cdot)) \oplus \mathrm{CAR}^{\mathrm{odd}}(V,(\cdot\,,\cdot)),$$

and $\mathbf{a}(V)\subset \mathrm{CAR}^{\mathrm{odd}}(V,(\cdot\,,\cdot))$.
(iv) Let $f : V \to V'$ be an isometric linear embedding, where $(V',(\cdot\,,\cdot)')$ is another complex pre-Hilbert space. Then there exists a unique injective C^*-morphism $\mathrm{CAR}(f) : \mathrm{CAR}(V,(\cdot\,,\cdot)) \to \mathrm{CAR}(V',(\cdot\,,\cdot)')$ such that

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & V' \\ \downarrow{\scriptstyle \mathbf{a}} & & \downarrow{\scriptstyle \mathbf{a}'} \\ \mathrm{CAR}(V,(\cdot\,,\cdot)) & \xrightarrow{\mathrm{CAR}(f)} & \mathrm{CAR}(V',(\cdot\,,\cdot)') \end{array}$$

commutes.

One easily sees that $\mathrm{CAR}(\mathrm{id}) = \mathrm{id}$ and that $\mathrm{CAR}(f'\circ f) = \mathrm{CAR}(f')\circ\mathrm{CAR}(f)$ for all isometric linear embeddings $V \xrightarrow{f} V' \xrightarrow{f'} V''$. Therefore we have constructed a covariant functor

$$\mathrm{CAR} : \mathsf{HILB} \longrightarrow \mathsf{C^*Alg},$$

where HILB denotes the category whose objects are the complex pre-Hilbert spaces and whose morphisms are the isometric linear embeddings and $\mathsf{C^*Alg}$ is the category whose objects are the unital C*-algebras and whose morphisms are the injective unit-preserving C*-morphisms.

For *real* pre-Hilbert spaces there is the concept of *self-dual* CAR-representations.

Definition 2.4. A *self-dual* CAR*-representation* of a real pre-Hilbert space $(V,(\cdot\,,\cdot))$ is a pair $(\mathbf{b},A)$, where A is a unital C*-algebra and $\mathbf{b} : V \to A$ is an $\mathbb{R}$-linear map satisfying:

(i) $A = C^*(\mathbf{b}(V))$,
(ii) $\mathbf{b}(v) = \mathbf{b}(v)^*$ and

(iii) $\{\mathbf{b}(v_1),\mathbf{b}(v_2)\} = (v_1,v_2)\cdot 1$,

for all $v, v_1, v_2 \in V$.

Note that a self-dual CAR-representation is not a CAR-representation in the sense of Definition 2.1. Given a self-dual CAR-representation, one can extend $\mathbf{b}$ to a $\mathbb{C}$-linear map from the complexification $V_{\mathbb{C}}$ to A. This extension $\mathbf{b} : V_{\mathbb{C}} \to A$ then satisfies $\mathbf{b}(\overline{v}) = \mathbf{b}(v)^*$ and $\{\mathbf{b}(v_1),\mathbf{b}(v_2)\} = (v_1,\overline{v_2})\cdot 1$ for all $v, v_1, v_2 \in V_{\mathbb{C}}$. These are the axioms of a self-dual CAR-representation as in [1, p. 386].

Theorem 2.5. *For every real pre-Hilbert space $(V,(\cdot\,,\cdot))$, the C^*-Clifford algebra $\mathrm{Cl}(V_{\mathbb{C}},q_{\mathbb{C}})$ provides a self-dual CAR-representation of $(V,(\cdot\,,\cdot))$ via $\mathbf{b}(v) = \frac{i}{\sqrt{2}}v$.*

Moreover, self-dual CAR-representations have the following universal property: Let $\widehat{A}$ be any unital C^*-algebra and $\widehat{\mathbf{b}} : V \to \widehat{A}$ be any $\mathbb{R}$-linear map satisfying Axioms (ii) and (iii) of Definition 2.4. Then there exists a unique C^*-morphism $\widetilde{\beta} : \mathrm{Cl}(V_{\mathbb{C}},q_{\mathbb{C}}) \to \widehat{A}$ such that

$$\begin{array}{ccc} V & \xrightarrow{\ \widehat{\mathbf{b}}\ } & \widehat{A} \\ {\scriptstyle \mathbf{b}}\downarrow & \nearrow{\scriptstyle \widetilde{\beta}} & \\ \mathrm{Cl}(V_{\mathbb{C}},q_{\mathbb{C}}) & & \end{array}$$

commutes. Furthermore, $\widetilde{\beta}$ is injective.

In particular, $(V,(\cdot\,,\cdot))$ has, up to C^*-isomorphism, a unique self-dual CAR-representation.

¿From now on, given a real pre-Hilbert space $(V,(\cdot\,,\cdot))$, we denote the C^*-algebra $\mathrm{Cl}(V_{\mathbb{C}},q_{\mathbb{C}})$ associated with the self-dual CAR-representation $(\mathbf{b},\mathrm{Cl}(V_{\mathbb{C}},q_{\mathbb{C}}))$ of $(V,(\cdot\,,\cdot))$ by $\mathrm{CAR}_{\mathrm{sd}}(V,(\cdot\,,\cdot))$.

Proposition 2.6. *Let $(V,(\cdot\,,\cdot))$ be a real pre-Hilbert space and $(\mathbf{b},\mathrm{CAR}_{\mathrm{sd}}(V,(\cdot\,,\cdot)))$ its self-dual CAR-representation.*

(i) For every $v \in V$ one has $\|\mathbf{b}(v)\| = \frac{1}{\sqrt{2}}|v|$, where $\|\cdot\|$ denotes the C^*-norm on $\mathrm{CAR}_{\mathrm{sd}}(V,(\cdot\,,\cdot))$.

(ii) The C^*-algebra $\mathrm{CAR}_{\mathrm{sd}}(V,(\cdot\,,\cdot))$ is simple.

(iii) The algebra $\mathrm{CAR}_{\mathrm{sd}}(V,(\cdot\,,\cdot))$ is $\mathbb{Z}_2$-graded,

$$\mathrm{CAR}_{\mathrm{sd}}(V,(\cdot\,,\cdot)) = \mathrm{CAR}_{\mathrm{sd}}^{\mathrm{even}}(V,(\cdot\,,\cdot)) \oplus \mathrm{CAR}_{\mathrm{sd}}^{\mathrm{odd}}(V,(\cdot\,,\cdot)),$$

and $\mathbf{b}(V) \subset \mathrm{CAR}_{\mathrm{sd}}^{\mathrm{odd}}(V,(\cdot\,,\cdot))$.

(iv) Let $f : V \to V'$ be an isometric linear embedding, where $(V',(\cdot\,,\cdot)')$ is another real pre-Hilbert space. Then there exists a unique injective C^*-morphism $\mathrm{CAR}_{\mathrm{sd}}(f) : \mathrm{CAR}_{\mathrm{sd}}(V,(\cdot\,,\cdot)) \to \mathrm{CAR}_{\mathrm{sd}}(V',(\cdot\,,\cdot)')$ such that

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & V' \\ {\scriptstyle \mathbf{b}}\downarrow & & \downarrow{\scriptstyle \mathbf{b}'} \\ \mathrm{CAR}_{\mathrm{sd}}(V,(\cdot\,,\cdot)) & \xrightarrow{\ \mathrm{CAR}_{\mathrm{sd}}(f)\ } & \mathrm{CAR}_{\mathrm{sd}}(V',(\cdot\,,\cdot)') \end{array}$$

commutes.

The proofs are similar to the ones for CAR-representations of complex pre-Hilbert spaces as given in [4, App. A]. We have constructed a functor

$$\mathrm{CAR}_{\mathrm{sd}} : \mathsf{HILB}_{\mathbb{R}} \longrightarrow \mathsf{C^*Alg},$$

where $\mathsf{HILB}_{\mathbb{R}}$ denotes the category whose objects are the real pre-Hilbert spaces and whose morphisms are the isometric linear embeddings.

Remark 2.7. Let $(V,(\cdot\,,\cdot))$ be a complex pre-Hilbert space. If we consider V as a real vector space, then we have the real pre-Hilbert space $(V,\mathfrak{Re}(\cdot\,,\cdot))$. For the corresponding CAR-representations we have

$$\mathrm{CAR}(V,(\cdot\,,\cdot)) = \mathrm{CAR}_{\mathrm{sd}}(V,\mathfrak{Re}(\cdot\,,\cdot)) = \mathrm{Cl}(V_{\mathbb{C}},q_{\mathbb{C}})$$

and

$$\mathbf{b}(v) = \frac{i}{\sqrt{2}}(\mathbf{a}(v)-\mathbf{a}(v)^*).$$

2.2. CCR algebras

In this section, we recall the construction of the representation of any (real) symplectic vector space by the so-called canonical commutation relations (CCR). Proofs can be found in [5, Sec. 4.2].

Definition 2.8. A CCR-*representation* of a symplectic vector space (V,ω) is a pair (w,A), where A is a unital C*-algebra and w is a map $V \to A$ satisfying:

(i) $A = C^*(w(V))$,
(ii) $w(0) = 1$,
(iii) $w(-\varphi) = w(\varphi)^*$,
(iv) $w(\varphi+\psi) = e^{i\omega(\varphi,\psi)/2} w(\varphi)\cdot w(\psi)$,

for all $\varphi,\psi \in V$.

The map w is in general neither linear, nor any kind of group homomorphism, nor continuous as soon as V carries a topology which is different from the discrete one [5, Prop. 4.2.3].

Example 2.9. Given any symplectic vector space (V,ω), consider the Hilbert space $H := L^2(V,\mathbb{C})$, where V is endowed with the counting measure. Define the map w from V into the space $\mathscr{L}(H)$ of bounded endomorphisms of H by

$$(w(\varphi)F)(\psi) := e^{i\omega(\varphi,\psi)/2} F(\varphi+\psi),$$

for all $\varphi,\psi \in V$ and $F \in H$. It is well-known that $\mathscr{L}(H)$ is a C*-algebra with the operator norm as C*-norm, and that the map w satisfies the Axioms (ii)-(iv) from Definition 2.8, see e.g. [5, Ex. 4.2.2]. Hence setting $A := C^*(w(V))$, the pair (w,A) provides a CCR-representation of (V,ω).

This is essentially the only CCR-representation:

Theorem 2.10. *Let (V,ω) be a symplectic vector space and $(\hat{w},\widehat{A})$ be a pair satisfying the Axioms* (ii)-(iv) *of Definition 2.8. Then there exists a unique C*-morphism $\Phi : A \to \widehat{A}$ such that $\Phi\circ w = \hat{w}$, where (w,A) is the* CCR-*representation from Example 2.9. Moreover, Φ is injective.*

In particular, (V,ω) has a CCR-*representation, unique up to C*-isomorphism.*

We denote the C*-algebra associated to the CCR-representation of (V,ω) from Example 2.9 by $\mathrm{CCR}(V,\omega)$. As a consequence of Theorem 2.10, we obtain the following important corollary.

Corollary 2.11. *Let (V,ω) be a symplectic vector space and $(w,\mathrm{CCR}(V,\omega))$ its* CCR*-representation.*

(i) The C*-algebra $\mathrm{CCR}(V,\omega)$ is simple, i.e., it has no closed two-sided $*$-ideals other than $\{0\}$ and the algebra itself.

(ii) Let (V',ω') be another symplectic vector space and $f : V \to V'$ a symplectic linear map. Then there exists a unique injective C*-morphism $\mathrm{CCR}(f) : \mathrm{CCR}(V,\omega) \to \mathrm{CCR}(V',\omega')$ such that

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & V' \\ \Big\downarrow{\scriptstyle w} & & \Big\downarrow{\scriptstyle w'} \\ \mathrm{CCR}(V,\omega) & \xrightarrow{\mathrm{CCR}(f)} & \mathrm{CCR}(V',\omega') \end{array}$$

commutes.

Obviously $\mathrm{CCR}(\mathrm{id}) = \mathrm{id}$ and $\mathrm{CCR}(f' \circ f) = \mathrm{CCR}(f') \circ \mathrm{CCR}(f)$ for all symplectic linear maps $V \xrightarrow{f} V' \xrightarrow{f'} V''$, so that we have constructed a covariant functor

$$\mathrm{CCR} : \mathsf{Sympl} \longrightarrow \mathsf{C^*Alg}.$$

3. Field equations on Lorentzian manifolds

3.1. Globally hyperbolic manifolds

We begin by fixing notation and recalling general facts about Lorentzian manifolds, see e.g. [26] or [5] for more details. Unless mentioned otherwise, the pair (M,g) will stand for a smooth m-dimensional manifold M equipped with a smooth Lorentzian metric g, where our convention for Lorentzian signature is $(- + \cdots +)$. The associated volume element will be denoted by dV. We shall also assume our Lorentzian manifold (M,g) to be time-orientable, i.e., that there exists a smooth timelike vector field on M. Time-oriented Lorentzian manifolds will be also referred to as *spacetimes*. Note that in contrast to conventions found elsewhere, we do not assume that a spacetime be connected nor that its dimension be $m = 4$.

For every subset A of a spacetime M we denote the causal future and past of A in M by $J_+(A)$ and $J_-(A)$, respectively. If we want to emphasize the ambient space M in which the causal future or past of A is considered, we write $J_\pm^M(A)$ instead of $J_\pm(A)$. Causal curves will always be implicitly assumed (future or past) oriented.

Definition 3.1. A *Cauchy hypersurface* in a spacetime (M,g) is a subset of M which is met exactly once by every inextensible timelike curve.

Cauchy hypersurfaces are always topological hypersurfaces but need not be smooth. All Cauchy hypersurfaces of a spacetime are homeomorphic.

Definition 3.2. A spacetime (M,g) is called *globally hyperbolic* if and only if it contains a Cauchy hypersurface.

A classical result of R. Geroch [18] says that a globally hyperbolic spacetime can be foliated by Cauchy hypersurfaces. It is a rather recent and very important result that this also holds in the smooth category: any globally hyperbolic spacetime is of the form $(\mathbb{R}\times\Sigma, -\beta dt^2 \oplus g_t)$, where each $\{t\}\times\Sigma$ is a smooth spacelike Cauchy hypersurface, β a smooth positive function and $(g_t)_t$ a smooth one-parameter family of Riemannian metrics on Σ [7, Thm. 1.1]. The hypersurface Σ can be even chosen such that $\{0\}\times\Sigma$ coincides with a given smooth spacelike Cauchy hypersurface [8, Thm. 1.2]. Moreover, any compact acausal smooth spacelike submanifold with boundary in a globally hyperbolic spacetime is contained in a smooth spacelike Cauchy hypersurface [8, Thm. 1.1].

Definition 3.3. A closed subset $A\subset M$ is called

- *spacelike compact* if there exists a compact subset $K\subset M$ such that $A\subset J^M(K) := J^M_-(K)\cup J^M_+(K)$,
- *future-compact* if $A\cap J_+(x)$ is compact for any $x\in M$,
- *past-compact* if $A\cap J_-(x)$ is compact for any $x\in M$.

A spacelike compact subset is in general not compact, but its intersection with any Cauchy hypersurface is compact, see e.g. [5, Cor. A.5.4].

Definition 3.4. A subset Ω of a spacetime M is called *causally compatible* if and only if $J^\Omega_\pm(x) = J^M_\pm(x)\cap\Omega$ for every $x\in\Omega$.

This means that every causal curve joining two points in Ω must be contained entirely in Ω.

3.2. Differential operators and Green's functions

A *differential operator* of order (at most) k on a vector bundle $S\to M$ over $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ is a linear map $P: C^\infty(M,S)\to C^\infty(M,S)$ which in local coordinates $x=(x^1,\dots,x^m)$ of M and with respect to a local trivialization looks like

$$P = \sum_{|\alpha|\le k} A_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha}.$$

Here $C^\infty(M,S)$ denotes the space of smooth sections of $S\to M$, $\alpha=(\alpha_1,\dots,\alpha_m)\in\mathbb{N}_0\times\cdots\times\mathbb{N}_0$ runs over multi-indices, $|\alpha|=\sum_{j=1}^m\alpha_j$ and $\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial(x^1)^{\alpha_1}\cdots\partial(x^m)^{\alpha_m}}$. The *principal symbol* σ_P of P associates to each covector $\xi\in T^*_xM$ a linear map $\sigma_P(\xi): S_x\to S_x$. Locally, it is given by

$$\sigma_P(\xi) = \sum_{|\alpha|=k} A_\alpha(x)\xi^\alpha,$$

where $\xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_m^{\alpha_m}$ and $\xi=\sum_j\xi_j dx^j$. If P and Q are two differential operators of order k and ℓ respectively, then $Q\circ P$ is a differential operator of order $k+\ell$ and

$$\sigma_{Q\circ P}(\xi) = \sigma_Q(\xi)\circ\sigma_P(\xi).$$

For any linear differential operator $P: C^\infty(M,S)\to C^\infty(M,S)$ there is a unique formally dual operator $P^*: C^\infty(M,S^*)\to C^\infty(M,S^*)$ of the same order characterized by

$$\int_M \langle\varphi, P\psi\rangle\,\mathrm{dV} = \int_M \langle P^*\varphi,\psi\rangle\,\mathrm{dV}$$

for all $\psi \in C^\infty(M,S)$ and $\varphi \in C^\infty(M,S^*)$ with $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\psi)$ compact. Here $\langle\cdot,\cdot\rangle : S^* \otimes S \to \mathbb{K}$ denotes the canonical pairing, i.e., the evaluation of a linear form in S_x^* on an element of S_x, where $x \in M$. We have $\sigma_{P^*}(\xi) = (-1)^k \sigma_P(\xi)^*$ where k is the order of P.

Definition 3.5. Let a vector bundle $S \to M$ be endowed with a non-degenerate inner product $\langle\cdot,\cdot\rangle$. A linear differential operator P on S is called *formally self-adjoint* if and only if

$$\int_M \langle P\varphi, \psi\rangle \,\mathrm{dV} = \int_M \langle \varphi, P\psi\rangle \,\mathrm{dV}$$

holds for all $\varphi, \psi \in C^\infty(M,S)$ with $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\psi)$ compact.

Similarly, we call P *formally skew-adjoint* if instead

$$\int_M \langle P\varphi, \psi\rangle \,\mathrm{dV} = -\int_M \langle \varphi, P\psi\rangle \,\mathrm{dV}.$$

We recall the definition of advanced and retarded Green's operators for a linear differential operator.

Definition 3.6. Let P be a linear differential operator acting on the sections of a vector bundle S over a Lorentzian manifold M. An *advanced Green's operator* for P on M is a linear map

$$G_+ : C_c^\infty(M,S) \to C^\infty(M,S)$$

satisfying:

(G_1) $P \circ G_+ = \mathrm{id}_{C_c^\infty(M,S)}$;
(G_2) $G_+ \circ P|_{C_c^\infty(M,S)} = \mathrm{id}_{C_c^\infty(M,S)}$;
(G_3^+) $\mathrm{supp}(G_+\varphi) \subset J_+^M(\mathrm{supp}(\varphi))$ for any $\varphi \in C_c^\infty(M,S)$.

A *retarded Green's operator* for P on M is a linear map $G_- : C_c^\infty(M,S) \to C^\infty(M,S)$ satisfying (G_1), (G_2), and

(G_3^-) $\mathrm{supp}(G_-\varphi) \subset J_-^M(\mathrm{supp}(\varphi))$ for any $\varphi \in C_c^\infty(M,S)$.

Here we denote by $C_c^\infty(M,S)$ the space of compactly supported smooth sections of S.

Definition 3.7. Let $P : C^\infty(M,S) \to C^\infty(M,S)$ be a linear differential operator. We call P *Green-hyperbolic* if the restriction of P to any globally hyperbolic subregion of M has advanced and retarded Green's operators.

The Green's operators for a given Green-hyperbolic operator P provide solutions φ of $P\varphi = 0$. More precisely, denoting by $C_{sc}^\infty(M,S)$ the set of smooth sections in S with spacelike compact support, we have the following

Theorem 3.8. *Let M be a Lorentzian manifold, let $S \to M$ be a vector bundle, and let P be a Green-hyperbolic operator acting on sections of S. Let $G_\pm$ be advanced and retarded Green's operators for P, respectively. Put*

$$G := G_+ - G_- : C_c^\infty(M,S) \to C_{sc}^\infty(M,S).$$

Then the following linear maps form a complex:

$$\{0\} \to C_c^\infty(M,S) \xrightarrow{P} C_c^\infty(M,S) \xrightarrow{G} C_{sc}^\infty(M,S) \xrightarrow{P} C_{sc}^\infty(M,S). \tag{1}$$

This complex is always exact at the first $C_c^\infty(M,S)$. If M is globally hyperbolic, then the complex is exact everywhere.

We refer to [4, Theorem 3.5] for the proof. Note that exactness at the first $C_c^\infty(M,S)$ in sequence (1) says that there are no non-trivial smooth solutions of $P\varphi = 0$ with compact support. Indeed, if M is globally hyperbolic, more is true. Namely, if $\varphi \in C^\infty(M,S)$ solves $P\varphi = 0$ and $\mathrm{supp}(\varphi)$ is future or past-compact, then $\varphi = 0$ (see e.g. [4, Remark 3.6] for a proof). As a straightforward consequence, the Green's operators for a Green-hyperbolic operator on a globally hyperbolic spacetime are unique [4, Remark 3.7].

3.3. Wave operators

The most prominent class of Green-hyperbolic operators are wave operators, sometimes also called normally hyperbolic operators.

Definition 3.9. A linear differential operator of second order $P : C^\infty(M,S) \to C^\infty(M,S)$ is called a *wave operator* if its principal symbol is given by the Lorentzian metric, i.e., for all $\xi \in T^*M$ we have

$$\sigma_P(\xi) = -\langle \xi, \xi \rangle \cdot \mathrm{id}.$$

In other words, if we choose local coordinates $x^1, \dots, x^m$ on M and a local trivialization of S, then

$$P = -\sum_{i,j=1}^{m} g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^{m} A_j(x) \frac{\partial}{\partial x^j} + B(x),$$

where A_j and B are matrix-valued coefficients depending smoothly on x and (g^{ij}) is the inverse matrix of (g_{ij}) with $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$. If P is a wave operator, then so is its dual operator P^*. In [5, Cor. 3.4.3] it has been shown that wave operators are Green-hyperbolic.

Example 3.10 (d'Alembert operator). Let S be the trivial line bundle so that sections of S are just functions. The d'Alembert operator $P = \Box = -\mathrm{div} \circ \mathrm{grad}$ is a formally self-adjoint wave operator, see e.g. [5, p. 26].

Example 3.11 (connection-d'Alembert operator). More generally, let S be a vector bundle and let ∇ be a connection on S. This connection and the Levi-Civita connection on T^*M induce a connection on $T^*M \otimes S$, again denoted ∇. We define the connection-d'Alembert operator $\Box^\nabla$ to be the composition of the following three maps

$$C^\infty(M,S) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes S) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes T^*M \otimes S) \xrightarrow{-\mathrm{tr} \otimes \mathrm{id}_S} C^\infty(M,S),$$

where $\mathrm{tr} : T^*M \otimes T^*M \to \mathbb{R}$ denotes the metric trace, $\mathrm{tr}(\xi \otimes \eta) = \langle \xi, \eta \rangle$. We compute the principal symbol,

$$\sigma_{\Box^\nabla}(\xi)\varphi = -(\mathrm{tr} \otimes \mathrm{id}_S) \circ \sigma_\nabla(\xi) \circ \sigma_\nabla(\xi)(\varphi) = -(\mathrm{tr} \otimes \mathrm{id}_S)(\xi \otimes \xi \otimes \varphi) = -\langle \xi, \xi \rangle\, \varphi.$$

Hence $\Box^\nabla$ is a wave operator.

Example 3.12 (Hodge-d'Alembert operator). Let $S = \Lambda^k T^*M$ be the bundle of k-forms. Exterior differentiation $d : C^\infty(M,\Lambda^k T^*M) \to C^\infty(M,\Lambda^{k+1}T^*M)$ increases the degree by one while the codifferential $\delta = d^* : C^\infty(M,\Lambda^k T^*M) \to C^\infty(M,\Lambda^{k-1}T^*M)$ decreases the degree by one. While d is independent of the metric, the codifferential δ does depend on the Lorentzian metric. The operator $P = -d\delta - \delta d$ is a formally self-adjoint wave operator.

3.4. The Proca equation

The Proca operator is an example of a Green-hyperbolic operator of second order which is not a wave operator.

Example 3.13 (Proca operator). The discussion of this example follows [31, p. 116f]. The Proca equation describes massive vector bosons. We take $S = T^*M$ and let $m_0 > 0$. The Proca equation is

$$P\varphi := \delta d\varphi + m_0^2\varphi = 0\,, \tag{2}$$

where $\varphi \in C^\infty(M,S)$. Applying δ to (2) we obtain, using $\delta^2 = 0$ and $m_0 \neq 0$,

$$\delta\varphi = 0 \tag{3}$$

and hence

$$(d\delta + \delta d)\varphi + m_0^2\varphi = 0. \tag{4}$$

Conversely, (3) and (4) clearly imply (2).

Since $\tilde{P} := d\delta + \delta d + m_0^2$ is minus a wave operator, it has Green's operators $\tilde{G}_\pm$. We define

$$G_\pm : C_c^\infty(M,S) \to C_{sc}^\infty(M,S), \quad G_\pm := (m_0^{-2}d\delta + \mathrm{id}) \circ \tilde{G}_\pm = \tilde{G}_\pm \circ (m_0^{-2}d\delta + \mathrm{id})\,.$$

The last equality holds because d and δ commute with $\tilde{P}$, see [4, Lemma 2.16]. For $\varphi \in C_c^\infty(M,S)$ we compute

$$G_\pm P\varphi = \tilde{G}_\pm(m_0^{-2}d\delta + \mathrm{id})(\delta d + m_0^2)\varphi = \tilde{G}_\pm\tilde{P}\varphi = \varphi$$

and similarly $PG_\pm\varphi = \varphi$. Since the differential operator $m_0^{-2}d\delta + \mathrm{id}$ does not increase supports, the third axiom in the definition of advanced and retarded Green's operators holds as well.

This shows that G_+ and G_- are advanced and retarded Green's operators for P, respectively. Thus P is not a wave operator but Green-hyperbolic.

3.5. Dirac type operators

The most important Green-hyperbolic operators of first order are the so-called Dirac type operators.

Definition 3.14. A linear differential operator $D : C^\infty(M,S) \to C^\infty(M,S)$ of first order is called *of Dirac type* if $-D^2$ is a wave operator.

Remark 3.15. If D is of Dirac type, then i times its principal symbol satisfies the Clifford relations

$$(i\sigma_D(\xi))^2 = -\sigma_{D^2}(\xi) = -\langle\xi,\xi\rangle \cdot \mathrm{id},$$

hence by polarization

$$(i\sigma_D(\xi))(i\sigma_D(\eta)) + (i\sigma_D(\eta))(i\sigma_D(\xi)) = -2\langle\xi,\eta\rangle \cdot \mathrm{id}.$$

The bundle S thus becomes a module over the bundle of Clifford algebras $\mathrm{Cl}(TM)$ associated with $(TM, \langle\cdot,\cdot\rangle)$. See [6, Sec. 1.1] or [23, Ch. I] for the definition and properties of the Clifford algebra $\mathrm{Cl}(V)$ associated with a vector space V with inner product.

Remark 3.16. If D is of Dirac type, then so is its dual operator D^*. On a globally hyperbolic region let G_+ be the advanced Green's operator for D^2, which exists since $-D^2$ is a wave operator. Then it is not hard to check that $D \circ G_+$ is an advanced Green's operator for D, see [25, Thm. 3.2]. The same discussion applies to the retarded Green's operator. Hence any Dirac type operator is Green-hyperbolic.

Example 3.17 (Classical Dirac operator). If the spacetime M carries a spin structure, then one can define the spinor bundle $S = \Sigma M$ and the classical Dirac operator

$$D : C^\infty(M,\Sigma M) \to C^\infty(M,\Sigma M), \quad D\varphi := i\sum_{j=1}^{m} \varepsilon_j e_j \cdot \nabla_{e_j}\varphi.$$

Here $(e_j)_{1\le j\le m}$ is a local orthonormal basis of the tangent bundle, $\varepsilon_j = \langle e_j, e_j\rangle = \pm 1$ and "$\cdot$" denotes the Clifford multiplication, see e.g. [6] or [3, Sec. 2]. The principal symbol of D is given by

$$\sigma_D(\xi)\psi = i\xi^\sharp \cdot \psi.$$

Here $\xi^\sharp$ denotes the tangent vector dual to the 1-form ξ via the Lorentzian metric, i.e., $\langle \xi^\sharp, Y\rangle = \xi(Y)$ for all tangent vectors Y over the same point of the manifold. Hence

$$\sigma_{D^2}(\xi)\psi = \sigma_D(\xi)\sigma_D(\xi)\psi = -\xi^\sharp \cdot \xi^\sharp \cdot \psi = \langle \xi, \xi\rangle\,\psi.$$

Thus $P = -D^2$ is a wave operator. Moreover, D is formally self-adjoint, see e.g. [3, p. 552].

Example 3.18 (Twisted Dirac operators). More generally, let $E \to M$ be a complex vector bundle equipped with a non-degenerate Hermitian inner product and a metric connection ∇^E over a spin spacetime M. In the notation of Example 3.17, one may define the Dirac operator of M twisted with E by

$$D^E := i\sum_{j=1}^{m} \varepsilon_j e_j \cdot \nabla^{\Sigma M\otimes E}_{e_j} : C^\infty(M, \Sigma M \otimes E) \to C^\infty(M, \Sigma M \otimes E),$$

where $\nabla^{\Sigma M\otimes E}$ is the tensor product connection on $\Sigma M \otimes E$. Again, D^E is a formally self-adjoint Dirac type operator.

Example 3.19 (Euler operator). In Example 3.12, replacing $\Lambda^k T^*M$ by $S := \Lambda T^*M \otimes \mathbb{C} = \oplus_{k=0}^m \Lambda^k T^*M \otimes \mathbb{C}$, the Euler operator $D = i(d-\delta)$ defines a formally self-adjoint Dirac type operator. In case M is spin, the Euler operator coincides with the Dirac operator of M twisted with ΣM if m is even and with $\Sigma M \oplus \Sigma M$ if m is odd.

Example 3.20 (Buchdahl operators). On a 4-dimensional spin spacetime M, consider the standard orthogonal and parallel splitting $\Sigma M = \Sigma_+ M \oplus \Sigma_- M$ of the complex spinor bundle of M into spinors of positive and negative chirality. The finite dimensional irreducible representations of the simply-connected Lie group $\mathrm{Spin}^0(3,1)$ are given by $\Sigma_+^{(k/2)} \otimes \Sigma_-^{(\ell/2)}$ where $k, \ell \in \mathbb{N}$. Here $\Sigma_+^{(k/2)} = \Sigma_+^{\odot k}$ is the k-th symmetric tensor product of the positive half-spinor representation Σ_+ and similarly for $\Sigma_-^{(\ell/2)}$. Let the associated vector bundles $\Sigma_\pm^{(k/2)}M$ carry the induced inner product and connection.

For $s \in \mathbb{N}$, $s \geq 1$, consider the twisted Dirac operator $D^{(s)}$ acting on sections of $\Sigma M \otimes \Sigma_+^{((s-1)/2)}M$. In the induced splitting

$$\Sigma M \otimes \Sigma_+^{((s-1)/2)}M = \Sigma_+ M \otimes \Sigma_+^{((s-1)/2)}M \oplus \Sigma_- M \otimes \Sigma_+^{((s-1)/2)}M$$

the operator $D^{(s)}$ is of the form

$$\begin{pmatrix} 0 & D_-^{(s)} \\ D_+^{(s)} & 0 \end{pmatrix}$$

because Clifford multiplication by vectors exchanges the chiralities. The Clebsch-Gordan formulas [11, Prop. II.5.5] tell us that the representation $\Sigma_+ \otimes \Sigma_+^{(\frac{s-1}{2})}$ splits as

$$\Sigma_+ \otimes \Sigma_+^{(\frac{s-1}{2})} = \Sigma_+^{(\frac{s}{2})} \oplus \Sigma_+^{(\frac{s}{2}-1)}.$$

Hence we have the corresponding parallel orthogonal projections

$$\pi_s : \Sigma_+ M \otimes \Sigma_+^{(\frac{s-1}{2})}M \to \Sigma_+^{(\frac{s}{2})}M \quad \text{and} \quad \pi'_s : \Sigma_+ M \otimes \Sigma_+^{(\frac{s-1}{2})}M \to \Sigma_+^{(\frac{s}{2}-1)}M.$$

On the other hand, the representation $\Sigma_- \otimes \Sigma_+^{(\frac{s-1}{2})}$ is irreducible. Now *Buchdahl operators* are the operators of the form

$$B^{(s)}_{\mu_1,\mu_2,\mu_3} := \begin{pmatrix} \mu_1 \cdot \pi_s + \mu_2 \cdot \pi'_s & D_-^{(s)} \\ D_+^{(s)} & \mu_3 \cdot \mathrm{id} \end{pmatrix},$$

where $\mu_1, \mu_2, \mu_3 \in \mathbb{C}$ are constants. By definition, $B^{(s)}_{\mu_1,\mu_2,\mu_3}$ is of the form $D^{(s)} + b$, where b is of order zero. In particular, $B^{(s)}_{\mu_1,\mu_2,\mu_3}$ is a Dirac-type operator, hence it is Green-hyperbolic. For a definition of Buchdahl operators using indices we refer to [13, 14, 35] and to [24, Def. 8.1.4, p. 104].

3.6. The Rarita-Schwinger operator

For the Rarita-Schwinger operator on Riemannian manifolds, we refer to [34, Sec. 2], see also [9, Sec. 2]. In this section let the spacetime M be spin and consider the Clifford-multiplication $\gamma : T^*M \otimes \Sigma M \to \Sigma M$, $\theta \otimes \psi \mapsto \theta^\sharp \cdot \psi$, where ΣM is the complex spinor bundle of M. Then there is the representation-theoretic splitting of $T^*M \otimes \Sigma M$ into the orthogonal and parallel sum

$$T^*M \otimes \Sigma M = \iota(\Sigma M) \oplus \Sigma^{3/2}M,$$

where $\Sigma^{3/2}M := \ker(\gamma)$ and $\iota(\psi) := -\frac{1}{m}\sum_{j=1}^m e_j^* \otimes e_j \cdot \psi$. Here again $(e_j)_{1\leq j\leq m}$ is a local orthonormal basis of the tangent bundle. Let $\mathscr{D}$ be the twisted Dirac operator on $T^*M \otimes \Sigma M$, that is, $\mathscr{D} := i \cdot (\mathrm{id} \otimes \gamma) \circ \nabla$, where ∇ denotes the induced covariant derivative on $T^*M \otimes \Sigma M$.

Definition 3.21. The *Rarita-Schwinger operator* on the spin spacetime M is defined by $\mathscr{Q} := (\mathrm{id} - \iota \circ \gamma) \circ \mathscr{D} : C^\infty(M, \Sigma^{3/2}M) \to C^\infty(M, \Sigma^{3/2}M)$.

By definition, the Rarita-Schwinger operator is pointwise obtained as the orthogonal projection onto $\Sigma^{3/2}M$ of the twisted Dirac operator $\mathscr{D}$ restricted to a section of $\Sigma^{3/2}M$. As for the Dirac operator, its characteristic variety coincides with the set of lightlike covectors, at least when $m \geq 3$, see [4, Lemma 2.26]. In particular, [21, Thms. 23.2.4 & 23.2.7] imply that the Cauchy problem for $\mathscr{Q}$ is well-posed in case M is globally hyperbolic. Since the well-posedness of the Cauchy problem implies the existence of advanced and retarded Green's operators (compare e.g. [4, Theorem 3.3.1 & Prop. 3.4.2] for wave operators), the operator $\mathscr{Q}$ has advanced and retarded Green's operators. Hence $\mathscr{Q}$ is not of Dirac type but is Green-hyperbolic.

Remark 3.22. The equations originally considered by Rarita and Schwinger in [28] correspond to the twisted Dirac operator $\mathscr{D}$ restricted to $\Sigma^{3/2}M$ but not projected back to $\Sigma^{3/2}M$. In other words, they considered the operator

$$\mathscr{D}|_{C^\infty(M,\Sigma^{3/2}M)} : C^\infty(M,\Sigma^{3/2}M) \to C^\infty(M, T^*M \otimes \Sigma M).$$

These equations are over-determined. Therefore it is not a surprise that non-trivial solutions restrict the geometry of the underlying manifold as observed by Gibbons [19] and that this operator has no Green's operators.

3.7. Combining given operators into a new one

Given two Green-hyperbolic operators we can form the direct sum and obtain a new operator in a trivial fashion. Namely, let $S_1, S_2 \to M$ be two vector bundles over a globally hyperbolic manifold M and let P_1 and P_2 be two Green-hyperbolic operators acting on sections of S_1 and S_2 respectively. Then

$$P_1 \oplus P_2 := \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} : C^\infty(M, S_1 \oplus S_2) \to C^\infty(M, S_1 \oplus S_2)$$

is Green-hyperbolic [5, Lemma 2.27]. Note that the two operators need not have the same order. Hence Green-hyperbolic operators need not be hyperbolic in the usual sense.

4. Algebras of observables

Our next aim is to quantize the classical fields governed by Green-hyperbolic differential operators. We construct local algebras of observables and we prove that we obtain locally covariant quantum field theories in the sense of [12].

4.1. Bosonic quantization

In this section we show how a quantization process based on canonical commutation relations (CCR) can be carried out for formally self-adjoint Green-hyperbolic operators. This is a functorial procedure. We define the first category involved in the quantization process.

Definition 4.1. The category $\mathsf{GlobHypGreen}$ consists of the following objects and morphisms:

- An object in $\mathsf{GlobHypGreen}$ is a triple (M,S,P), where
  - ► M is a globally hyperbolic spacetime,

  - ► S is a real vector bundle over M endowed with a non-degenerate inner product $\langle\cdot,\cdot\rangle$ and
  - ► P is a formally self-adjoint Green-hyperbolic operator acting on sections of S.
- A morphism between objects (M_1,S_1,P_1) and (M_2,S_2,P_2) of GlobHypGreen is a pair (f,F), where
  - ► f is a time-orientation preserving isometric embedding $M_1 \to M_2$ with $f(M_1)$ causally compatible and open in M_2,
  - ► F is a fiberwise isometric vector bundle isomorphism over f such that the following diagram commutes:

$$\begin{array}{ccc} C^\infty(M_2,S_2) & \xrightarrow{P_2} & C^\infty(M_2,S_2) \\ \Big\downarrow{\scriptstyle \mathrm{res}} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\ C^\infty(M_1,S_1) & \xrightarrow{P_1} & C^\infty(M_1,S_1), \end{array} \tag{5}$$

  where $\mathrm{res}(\varphi) := F^{-1}\circ\varphi\circ f$ for every $\varphi\in C^\infty(M_2,S_2)$.

Note that morphisms exist only if the manifolds have equal dimension and the vector bundles have the same rank. Note, furthermore, that the inner product $\langle\cdot,\cdot\rangle$ on S is not required to be positive or negative definite.

The causal compatibility condition, which is not automatically satisfied (see e.g. [5, Fig. 33]), ensures the commutation of the extension and restriction maps with the Green's operators. Namely, if (f,F) be a morphism between two objects (M_1,S_1,P_1) and (M_2,S_2,P_2) in the category GlobHypGreen, and if $(G_1)_\pm$ and $(G_2)_\pm$ denote the respective Green's operators for P_1 and P_2, then we have

$$\mathrm{res}\circ(G_2)_\pm\circ\mathrm{ext} = (G_1)_\pm .$$

Here $\mathrm{ext}(\varphi)\in C^\infty_{\mathrm{c}}(M_2,S_2)$ is the extension by 0 of $F\circ\varphi\circ f^{-1} : f(M_1)\to S_2$ to M_2, for every $\varphi\in C^\infty_{\mathrm{c}}(M_1,S_1)$, see [4, Lemma 3.2].

What is most important for our purpose is that the Green's operators for a formally self-adjoint Green-hyperbolic operator provide a symplectic vector space in a canonical way. First recall how the Green's operators of an operator and of its formally dual operator are related: if M is a globally hyperbolic spacetime, G_+,G_- are the advanced and retarded Green's operators for a Green-hyperbolic operator P acting on sections of $S\to M$ and G_+^*,G_-^* denote the advanced and retarded Green's operators for P^*, then

$$\int_M \langle G_\pm^*\varphi,\psi\rangle\,\mathrm{dV} = \int_M \langle\varphi, G_\mp\psi\rangle\,\mathrm{dV} \tag{6}$$

for all $\varphi\in C^\infty_{\mathrm{c}}(M,S^*)$ and $\psi\in C^\infty_{\mathrm{c}}(M,S)$, see e.g. [4, Lemma 3.3]. This implies:

Proposition 4.2. *Let (M,S,P) be an object in the category* GlobHypGreen. *Set $G := G_+ - G_-$, where G_+,G_- are the advanced and retarded Green's operator for P, respectively.*

Then the pair $(\mathrm{SYMPL}(M,S,P),\omega)$ *is a symplectic vector space, where*

$$\mathrm{SYMPL}(M,S,P) := C_c^\infty(M,S)/\ker(G) \quad \textit{and} \quad \omega([\varphi],[\psi]) := \int_M \langle G\varphi,\psi\rangle\,\mathrm{dV}.$$

Here the square brackets $[\cdot]$ *denote residue classes modulo* $\ker(G)$.

Proof. The bilinear form $(\varphi,\psi)\mapsto\int_M\langle G\varphi,\psi\rangle\,\mathrm{dV}$ on $C_c^\infty(M,S)$ is skew-symmetric as a consequence of (6) because P is formally self-adjoint. Its null space is exactly $\ker(G)$. Therefore the induced bilinear form ω on the quotient space $\mathrm{SYMPL}(M,S,P)$ is non-degenerate and hence a symplectic form. □

Theorem 3.8 shows that $G(C_c^\infty(M,S))$ coincides with the space of smooth solutions of the equation $P\varphi = 0$ which have spacelike compact support. In particular, given an object (M,S,P) in GlobHypGreen, the map G induces an isomorphism

$$\mathrm{SYMPL}(M,S,P) = C_c^\infty(M,S)/\ker(G) \xrightarrow{\cong} \ker(P)\cap C_{sc}^\infty(M,S).$$

Hence we may think of $\mathrm{SYMPL}(M,S,P)$ as the space of classical solutions of the equation $P\varphi = 0$ with spacelike compact support.

Now, let (f,F) be a morphism between objects (M_1,S_1,P_1) and (M_2,S_2,P_2) in the category GlobHypGreen. Then the extension by zero induces a symplectic linear map $\mathrm{SYMPL}(f,F):\mathrm{SYMPL}(M_1,S_1,P_1)\to\mathrm{SYMPL}(M_2,S_2,P_2)$ with

$$\mathrm{SYMPL}(\mathrm{id}_M,\mathrm{id}_S) = \mathrm{id}_{\mathrm{SYMPL}(M,S,P)} \tag{7}$$

and, for any further morphism $(f',F'):(M_2,S_2,P_2)\to(M_3,S_3,P_3)$,

$$\mathrm{SYMPL}((f',F')\circ(f,F)) = \mathrm{SYMPL}(f',F')\circ\mathrm{SYMPL}(f,F). \tag{8}$$

Remark 4.3. Under the isomorphism $\mathrm{SYMPL}(M,S,P)\to\ker(P)\cap C_{sc}^\infty(M,S)$ induced by G, the extension by zero corresponds to an extension as a smooth solution of $P\varphi=0$ with spacelike compact support. In other words, for any morphism (f,F) from (M_1,S_1,P_1) to (M_2,S_2,P_2) in GlobHypGreen we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{SYMPL}(M_1,S_1,P_1) & \xrightarrow{\mathrm{SYMPL}(f,F)} & \mathrm{SYMPL}(M_2,S_2,P_2) \\
\cong\downarrow & & \downarrow\cong \\
\ker(P_1)\cap C_{sc}^\infty(M_1,S_1) & \xrightarrow[\text{a solution}]{\text{extension as}} & \ker(P_2)\cap C_{sc}^\infty(M_2,S_2).
\end{array}$$

Summarizing, we have constructed a covariant functor

$$\mathrm{SYMPL}:\mathsf{GlobHypGreen}\longrightarrow\mathsf{Sympl},$$

where Sympl denotes the category of real symplectic vector spaces with symplectic linear maps as morphisms. In order to obtain an algebra-valued functor, we compose SYMPL with the functor CCR which associates to any symplectic vector space its Weyl algebra. Here "CCR" stands for "canonical commutation relations". This is a general algebraic construction which is independent of the context of Green-hyperbolic operators and which is carried out in Section 2.2. As a result, we obtain the functor

$$\mathfrak{A}_{\mathrm{bos}} := \mathrm{CCR}\circ\mathrm{SYMPL}:\mathsf{GlobHypGreen}\longrightarrow\mathsf{C^*Alg},$$

where $\mathsf{C^*Alg}$ is the category whose objects are the unital C^*-algebras and whose morphisms are the injective unit-preserving C^*-morphisms.

In the remainder of this section we show that the functor $\mathfrak{A}_{\mathrm{bos}}$ is a bosonic locally covariant quantum field theory. We call two subregions M_1 and M_2 of a spacetime M *causally disjoint* if and only if $J^M(M_1) \cap M_2 = \emptyset$. In other words, there are no causal curves joining M_1 and M_2.

Theorem 4.4. *The functor* $\mathfrak{A}_{\mathrm{bos}} : \mathsf{GlobHypGreen} \longrightarrow \mathsf{C^*Alg}$ *is a bosonic locally covariant quantum field theory, i.e., the following axioms hold:*

(i) ***(Quantum causality)*** *Let* (M_j, S_j, P_j) *be objects in* $\mathsf{GlobHypGreen}$*,* $j = 1,2,3$*, and* (f_j, F_j) *morphisms from* (M_j, S_j, P_j) *to* (M_3, S_3, P_3)*,* $j = 1,2$*, such that* $f_1(M_1)$ *and* $f_2(M_2)$ *are causally disjoint regions in* M_3*. Then the subalgebras* $\mathfrak{A}_{\mathrm{bos}}(f_1, F_1)(\mathfrak{A}_{\mathrm{bos}}(M_1, S_1, P_1))$ *and* $\mathfrak{A}_{\mathrm{bos}}(f_2, F_2)(\mathfrak{A}_{\mathrm{bos}}(M_2, S_2, P_2))$ *of* $\mathfrak{A}_{\mathrm{bos}}(M_3, S_3, P_3)$ *commute.*

(ii) ***(Time slice axiom)*** *Let* (M_j, S_j, P_j) *be objects in* $\mathsf{GlobHypGreen}$*,* $j = 1,2$*, and* (f, F) *a morphism from* (M_1, S_1, P_1) *to* (M_2, S_2, P_2) *such that there is a Cauchy hypersurface* $\Sigma \subset M_1$ *for which* $f(\Sigma)$ *is a Cauchy hypersurface of* M_2*. Then*

$$\mathfrak{A}_{\mathrm{bos}}(f, F) : \mathfrak{A}_{\mathrm{bos}}(M_1, S_1, P_1) \to \mathfrak{A}_{\mathrm{bos}}(M_2, S_2, P_2)$$

is an isomorphism.

Proof. We first show (i). For notational simplicity we assume without loss of generality that f_j and F_j are inclusions, $j = 1,2$. Let $\varphi_j \in C_c^\infty(M_j, S_j)$. Since M_1 and M_2 are causally disjoint, the sections $G\varphi_1$ and φ_2 have disjoint support, thus

$$\omega([\varphi_1],[\varphi_2]) = \int_M \langle G\varphi_1, \varphi_2 \rangle \, \mathrm{dV} = 0.$$

Now relation (iv) in Definition 2.8 tells us

$$w([\varphi_1]) \cdot w([\varphi_2]) = w([\varphi_1] + [\varphi_2]) = w([\varphi_2]) \cdot w([\varphi_1]).$$

Since $\mathfrak{A}_{\mathrm{bos}}(f_1, F_1)(\mathfrak{A}_{\mathrm{bos}}(M_1, S_1, P_1))$ is generated by elements of the form $w([\varphi_1])$ and $\mathfrak{A}_{\mathrm{bos}}(f_2, F_2)(\mathfrak{A}_{\mathrm{bos}}(M_2, S_2, P_2))$ by elements of the form $w([\varphi_2])$, the assertion follows.

In order to prove (ii) we show that $\mathrm{SYMPL}(f, F)$ is an isomorphism of symplectic vector spaces provided f maps a Cauchy hypersurface of M_1 onto a Cauchy hypersurface of M_2. Since symplectic linear maps are always injective, we only need to show surjectivity of $\mathrm{SYMPL}(f, F)$. This is most easily seen by replacing $\mathrm{SYMPL}(M_j, S_j, P_j)$ by $\ker(P_j) \cap C_{\mathrm{sc}}^\infty(M_j, S_j)$ as in Remark 4.3. Again we assume without loss of generality that f and F are inclusions.

Let $\psi \in C_{\mathrm{sc}}^\infty(M_2, S_2)$ be a solution of $P_2\psi = 0$. Let φ be the restriction of ψ to M_1. Then φ solves $P_1\varphi = 0$ and has spacelike compact support in M_1, see [4, Lemma 3.11]. We will show that there is only one solution in M_2 with spacelike compact support extending φ. It will then follow that ψ is the image of φ under the extension map corresponding to $\mathrm{SYMPL}(f, F)$ and surjectivity will be shown.

To prove uniqueness of the extension, we may, by linearity, assume that $\varphi = 0$. Then ψ_+ defined by

$$\psi_+(x) := \begin{cases} \psi(x), & \text{if } x \in J_+^{M_2}(\Sigma), \\ 0, & \text{otherwise,} \end{cases}$$

is smooth since ψ vanishes in an open neighborhood of Σ. Now ψ_+ solves $P_2\psi_+ = 0$ and has past-compact support. As noticed just below Theorem 3.8, this implies $\psi_+ \equiv 0$, i.e., ψ vanishes on $J_+^{M_2}(\Sigma)$. One shows similarly that ψ vanishes on $J_-^{M_2}(\Sigma)$, hence $\psi = 0$. □

The quantization process described in this subsection applies in particular to formally self-adjoint wave and Dirac-type operators.

4.2. Fermionic quantization

Next we construct a fermionic quantization. For this we need a functorial construction of Hilbert spaces rather than symplectic vector spaces. As we shall see this seems to be possible only under much more restrictive assumptions. The underlying Lorentzian manifold M is assumed to be a globally hyperbolic spacetime as before. The vector bundle S is assumed to be complex with Hermitian inner product $\langle\cdot\,,\cdot\rangle$ which may be indefinite. The formally self-adjoint Green-hyperbolic operator P is assumed to be of first order.

Definition 4.5. A formally self-adjoint Green-hyperbolic operator P of first order acting on sections of a complex vector bundle S over a spacetime M is of *definite type* if and only if for any $x \in M$ and any future-directed timelike tangent vector $\mathfrak{n} \in T_xM$, the bilinear map

$$S_x \times S_x \to \mathbb{C}, \qquad (\varphi, \psi) \mapsto \langle i\sigma_P(\mathfrak{n}^\flat)\cdot\varphi, \psi\rangle,$$

yields a positive definite Hermitian scalar product on S_x.

Example 4.6. The classical Dirac operator P from Example 3.17 is, when defined with the correct sign, of definite type, see e.g. [6, Sec. 1.1.5] or [3, Sec. 2].

Example 4.7. If $E \to M$ is a semi-Riemannian or semi-Hermitian vector bundle endowed with a metric connection over a spin spacetime M, then the twisted Dirac operator from Example 3.18 is of definite type if and only if the metric on E is positive definite. This can be seen by evaluating the tensorized inner product on elements of the form $\sigma \otimes v$, where $v \in E_x$ is null.

Example 4.8. The operator $P = i(d - \delta)$ on $S = \Lambda T^*M \otimes \mathbb{C}$ is of Dirac type but not of definite type. This follows from Example 4.7 applied to Example 3.19, since the natural inner product on ΣM is not positive definite. An alternative elementary proof is the following: for any timelike tangent vector $\mathfrak{n}$ on M and the corresponding covector $\mathfrak{n}^\flat$, one has

$$\langle i\sigma_P(\mathfrak{n}^\flat)\mathfrak{n}^\flat, \mathfrak{n}^\flat\rangle = -\langle \mathfrak{n}^\flat \wedge \mathfrak{n}^\flat - \mathfrak{n} \lrcorner \mathfrak{n}^\flat, \mathfrak{n}^\flat\rangle = \langle \mathfrak{n}, \mathfrak{n}\rangle\langle 1, \mathfrak{n}^\flat\rangle = 0.$$

Example 4.9. An elementary computation shows that the Rarita-Schwinger operator defined in Section 3.6 is not of definite type if $m \geq 3$, see [4, Ex. 3.16].

We define the category GlobHypDef, whose objects are triples (M,S,P), where M is a globally hyperbolic spacetime, S is a complex vector bundle equipped with a complex inner product $\langle\cdot,\cdot\rangle$, and P is a formally self-adjoint Green-hyperbolic operator of definite type acting on sections of S. The morphisms are the same as in the category GlobHypGreen.

We construct a covariant functor from GlobHypDef to HILB, where HILB denotes the category whose objects are complex pre-Hilbert spaces and whose morphisms are isometric linear embeddings. As in Section 4.1, the underlying vector space is the space of classical solutions to the equation $P\varphi = 0$ with spacelike compact support. We put

$$\mathrm{SOL}(M,S,P) := \ker(P) \cap C^\infty_{\mathrm{sc}}(M,S).$$

Here "SOL" stands for classical solutions of the equation $P\varphi = 0$ with spacelike compact support. We endow $\mathrm{SOL}(M,S,P)$ with a positive definite Hermitian scalar product as follows: consider a smooth spacelike Cauchy hypersurface $\Sigma \subset M$ with its future-oriented unit normal vector field $\mathfrak{n}$ and its induced volume element dA and set

$$(\varphi,\psi) := \int_\Sigma \langle i\sigma_P(\mathfrak{n}^\flat)\cdot\varphi|_\Sigma, \psi|_\Sigma\rangle\,\mathrm{dA}, \tag{9}$$

for all $\varphi,\psi \in C^\infty_{\mathrm{sc}}(M,S)$. The Green's formula for formally self-adjoint first-order differential operators [32, p. 160, Prop. 9.1] (see also [4, Lemma 3.17]) implies that $(\cdot,\cdot)$ does not depend on the choice of Σ. Of course, it is positive definite because of the assumption that P is of definite type. In case P is not of definite type, the sesquilinear form $(\cdot,\cdot)$ is still independent of the choice of Σ but may be degenerate, see [4, Remark 3.18].

For any object (M,S,P) in GlobHypDef we equip $\mathrm{SOL}(M,S,P)$ with the Hermitian scalar product in (9) and thus turn $\mathrm{SOL}(M,S,P)$ into a pre-Hilbert space.

Given a morphism (f,F) from (M_1,S_1,P_1) to (M_2,S_2,P_2) in GlobHypDef, then this is also a morphism in GlobHypGreen and hence induces a homomorphism $\mathrm{SYMPL}(f,F) : \mathrm{SYMPL}(M_1,S_1,P_1) \to \mathrm{SYMPL}(M_2,S_2,P_2)$. As explained in Remark 4.3, there is a corresponding extension homomorphism $\mathrm{SOL}(f,F)$: $\mathrm{SOL}(M_1,S_1,P_1) \to \mathrm{SOL}(M_2,S_2,P_2)$. In other words, $\mathrm{SOL}(f,F)$ is defined such that the diagram

$$\begin{array}{ccc}
\mathrm{SYMPL}(M_1,S_1,P_1) & \xrightarrow{\mathrm{SYMPL}(f,F)} & \mathrm{SYMPL}(M_2,S_2,P_2) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{SOL}(M_1,S_1,P_1) & \xrightarrow{\mathrm{SOL}(f,F)} & \mathrm{SOL}(M_2,S_2,P_2)
\end{array} \tag{10}$$

commutes. The vertical arrows are the vector space isomorphisms induced be the Green's propagators G_1 and G_2, respectively.

Lemma 4.10. *The vector space homomorphism* $\mathrm{SOL}(f,F) : \mathrm{SOL}(M_1,S_1,P_1) \to \mathrm{SOL}(M_2,S_2,P_2)$ *preserves the scalar products, i.e., it is an isometric linear embedding of pre-Hilbert spaces.*

We refer to [4, Lemma 3.19] for a proof. The functoriality of SYMPL and diagram (10) show that SOL is a functor from GlobHypDef to HILB, the category of pre-Hilbert spaces with isometric linear embeddings. Composing with the functor CAR (see Section 2.1), we obtain the covariant functor

$$\mathfrak{A}_{\mathrm{ferm}} := \mathsf{CAR} \circ \mathsf{SOL} : \mathsf{GlobHypDef} \longrightarrow \mathsf{C^*Alg}.$$

The fermionic algebras $\mathfrak{A}_{\mathrm{ferm}}(M,S,P)$ are actually $\mathbb{Z}_2$-graded algebras, see Proposition 2.3 (iii).

Theorem 4.11. *The functor* $\mathfrak{A}_{\mathrm{ferm}} : \mathsf{GlobHypDef} \longrightarrow \mathsf{C^*Alg}$ *is a fermionic locally covariant quantum field theory, i.e., the following axioms hold:*

(i) ***(Quantum causality)*** *Let* (M_j,S_j,P_j) *be objects in* $\mathsf{GlobHypDef}$, $j = 1,2,3$, *and* (f_j,F_j) *morphisms from* (M_j,S_j,P_j) *to* (M_3,S_3,P_3), $j = 1,2$, *such that* $f_1(M_1)$ *and* $f_2(M_2)$ *are causally disjoint regions in* M_3. *Then the subalgebras* $\mathfrak{A}_{\mathrm{ferm}}(f_1,F_1)(\mathfrak{A}_{\mathrm{ferm}}(M_1,S_1,P_1))$ *and* $\mathfrak{A}_{\mathrm{ferm}}(f_2,F_2)(\mathfrak{A}_{\mathrm{ferm}}(M_2,S_2,P_2))$ *of* $\mathfrak{A}_{\mathrm{ferm}}(M_3,S_3,P_3)$ *super-commute*[1].

(ii) ***(Time slice axiom)*** *Let* (M_j,S_j,P_j) *be objects in* $\mathsf{GlobHypDef}$, $j = 1,2$, *and* (f,F) *a morphism from* (M_1,S_1,P_1) *to* (M_2,S_2,P_2) *such that there is a Cauchy hypersurface* $\Sigma \subset M_1$ *for which* $f(\Sigma)$ *is a Cauchy hypersurface of* M_2. *Then*

$$\mathfrak{A}_{\mathrm{ferm}}(f,F) : \mathfrak{A}_{\mathrm{ferm}}(M_1,S_1,P_1) \to \mathfrak{A}_{\mathrm{ferm}}(M_2,S_2,P_2)$$

is an isomorphism.

Proof. To show (i), we assume without loss of generality that f_j and F_j are inclusions. Let $\varphi_1 \in \mathrm{SOL}(M_1,S_1,P_1)$ and $\psi_1 \in \mathrm{SOL}(M_2,S_2,P_2)$. Denote the extensions to M_3 by $\varphi_2 := \mathrm{SOL}(f_1,F_1)(\varphi_1)$ and $\psi_2 := \mathrm{SOL}(f_2,F_2)(\psi_1)$. Choose a compact submanifold K_1 (with boundary) in a spacelike Cauchy hypersurface Σ_1 of M_1 such that $\mathrm{supp}(\varphi_1) \cap \Sigma_1 \subset K_1$ and similarly K_2 for ψ_1. Since M_1 and M_2 are causally disjoint, $K_1 \cup K_2$ is acausal. Hence, by [8, Thm. 1.1], there exists a Cauchy hypersurface Σ_3 of M_3 containing K_1 and K_2. As in the proof of Lemma 4.10 one sees that $\mathrm{supp}(\varphi_2) \cap \Sigma_3 = \mathrm{supp}(\varphi_1) \cap \Sigma_1$ and similarly for ψ_2. Thus, when restricted to Σ_3, φ_2 and ψ_2 have disjoint support. Hence $(\varphi_2,\psi_2) = 0$. This shows that the subspaces $\mathrm{SOL}(f_1,F_1)(\mathrm{SOL}(M_1,S_1,P_1))$ and $\mathrm{SOL}(f_2,F_2)(\mathrm{SOL}(M_2,S_2,P_2))$ of $\mathrm{SOL}(M_3,S_3,P_3)$ are perpendicular. Since the even (resp. odd) part of the Clifford algebra of a vector space V with quadratic form is linearly spanned by the even (resp. odd) products of vectors in V, Definition 2.1 shows that the corresponding CAR-algebras must super-commute.

To see (ii) we recall that (f,F) is also a morphism in $\mathsf{GlobHypGreen}$ and that we know from Theorem 4.4 that $\mathrm{SYMPL}(f,F)$ is an isomorphism. ¿From diagram (10) we see that $\mathrm{SOL}(f,F)$ is an isomorphism. Hence $\mathfrak{A}_{\mathrm{ferm}}(f,F)$ is also an isomorphism. □

Remark 4.12. Since causally disjoint regions should lead to commuting observables also in the fermionic case, one usually considers only the even part $\mathfrak{A}_{\mathrm{ferm}}^{\mathrm{even}}(M,S,P)$ as the observable algebra while the full algebra $\mathfrak{A}_{\mathrm{ferm}}(M,S,P)$ is called the *field algebra*.

[1] This means that the odd parts of the algebras anti-commute while the even parts commute with everything.

There is a slightly different description of the functor $\mathfrak{A}_{\mathrm{ferm}}$. Let $\mathsf{HILB}_{\mathbb{R}}$ denote the category whose objects are the real pre-Hilbert spaces and whose morphisms are the isometric linear embeddings. We have the functor REAL : $\mathsf{HILB} \to \mathsf{HILB}_{\mathbb{R}}$ which associates to each complex pre-Hilbert space $(V,(\cdot\,,\cdot))$ its underlying real pre-Hilbert space $(V,\mathfrak{Re}(\cdot\,,\cdot))$. By Remark 2.7,

$$\mathfrak{A}_{\mathrm{ferm}} = \mathrm{CAR}_{\mathrm{sd}} \circ \mathrm{REAL} \circ \mathrm{SOL}.$$

Since the self-dual CAR-algebra of a real pre-Hilbert space is the Clifford algebra of its complexification and since for any complex pre-Hilbert space V we have

$$\mathrm{REAL}(V) \otimes_{\mathbb{R}} \mathbb{C} = V \oplus V^*,$$

$\mathfrak{A}_{\mathrm{ferm}}(M,S,P)$ is also the Clifford algebra of $\mathrm{SOL}(M,S,P) \oplus \mathrm{SOL}(M,S,P)^* = \mathrm{SOL}(M,S\oplus S^*,P\oplus P^*)$. This is the way this functor is often described in the physics literature, see e.g. [31, p. 115f].

Self-dual CAR-representations are more natural for real fields. Let M be globally hyperbolic and let $S \to M$ be a *real* vector bundle equipped with a real inner product $\langle\cdot\,,\cdot\rangle$. A formally skew-adjoint[2] differential operator P acting on sections of S is called of *definite type* if and only if for any $x \in M$ and any future-directed timelike tangent vector $\mathfrak{n} \in T_xM$, the bilinear map

$$S_x \times S_x \to \mathbb{R}, \qquad (\varphi,\psi) \mapsto \langle \sigma_P(\mathfrak{n}^\flat)\cdot\varphi,\psi\rangle,$$

yields a positive definite Euclidean scalar product on S_x. An example is given by the real Dirac operator

$$D := \sum_{j=1}^{m} \varepsilon_j e_j \cdot \nabla_{e_j}$$

acting on sections of the real spinor bundle $\Sigma^{\mathbb{R}}M$.

Given a smooth spacelike Cauchy hypersurface $\Sigma \subset M$ with future-directed timelike unit normal field $\mathfrak{n}$, we define a scalar product on $\mathrm{SOL}(M,S,P) = \ker(P) \cap C^\infty_{\mathrm{sc}}(M,S,P)$ by

$$(\varphi,\psi) := \int_\Sigma \langle \sigma_P(\mathfrak{n}^\flat)\cdot\varphi|_\Sigma,\psi|_\Sigma\rangle\,\mathrm{dA}.$$

With essentially the same proofs as before, one sees that this scalar product does not depend on the choice of Cauchy hypersurface Σ and that a morphism $(f,F) : (M_1,S_1,P_1) \to (M_2,S_2,P_2)$ gives rise to an extension operator $\mathrm{SOL}(f,F) : \mathrm{SOL}(M_1,S_1,P_1) \to \mathrm{SOL}(M_2,S_2,P_2)$ preserving the scalar product. We have constructed a functor

$$\mathrm{SOL} : \mathsf{GlobHypSkewDef} \longrightarrow \mathsf{HILB}_{\mathbb{R}},$$

where $\mathsf{GlobHypSkewDef}$ denotes the category whose objects are triples (M,S,P) with M globally hyperbolic, $S \to M$ a real vector bundle with real inner product and P a formally skew-adjoint, Green-hyperbolic differential operator of definite type acting on sections of S. The morphisms are the same as before.

Now the functor

$$\mathfrak{A}^{\mathrm{sd}}_{\mathrm{ferm}} := \mathrm{CAR}_{\mathrm{sd}} \circ \mathrm{SOL} : \mathsf{GlobHypSkewDef} \longrightarrow \mathsf{C^*Alg}$$

[2] instead of self-adjoint!

is a locally covariant quantum field theory in the sense that Theorem 4.11 holds with $\mathfrak{A}_{\mathrm{ferm}}$ replaced by $\mathfrak{A}^{\mathrm{sd}}_{\mathrm{ferm}}$.

5. Conclusion

We have constructed three functors,

$$\mathfrak{A}_{\mathrm{bos}} : \mathsf{GlobHypGreen} \longrightarrow \mathsf{C^*Alg},$$
$$\mathfrak{A}_{\mathrm{ferm}} : \mathsf{GlobHypDef} \longrightarrow \mathsf{C^*Alg},$$
$$\mathfrak{A}^{\mathrm{sd}}_{\mathrm{ferm}} : \mathsf{GlobHypSkewDef} \longrightarrow \mathsf{C^*Alg}.$$

The first functor turns out to be a bosonic locally covariant quantum field theory while the second and third are fermionic locally covariant quantum field theories.

The category GlobHypGreen seems to contain basically all physically relevant free fields such as fields governed by wave equations, Dirac equations, the Proca equation and the Rarita-Schwinger equation. It contains operators of all orders. Bosonic quantization of Dirac fields might be considered unphysical but the discussion shows that there is no spin-statistics theorem on the level of observable algebras. In order to obtain results like Theorem 5.1 in [33] one needs more structure, namely representations of the observable algebras with good properties.

The categories GlobHypDef and GlobHypSkewDef are much smaller. They contain only operators of first order with Dirac operators as main examples. But even certain twisted Dirac operators such as the Euler operator do not belong to this class. The category GlobHypSkewDef is essentially the real analogue of GlobHypDef.

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