Eigenvalues of Dirac Operators for Hyperbolic Degenerations

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We study the behaviour of the spectrum of the Dirac operator for sequences of compact hyperbolic spin manifolds whose limit is non-compact. If the spectrum of the limit manifold is discrete we show that this spectrum is approximated by the spectra of the compact manifolds.

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1 Introduction

The question under consideration in this article is whether the convergence of manifolds implies the convergence of the spectra of their corresponding natural differential operators. This question has been studied by many authors for the Laplace operator; in particular for collapsing manifolds and hyperbolic degenerations. For the collapse of manifolds the behaviour of the Dirac spectra has been studied by Ammann and Bär in [AB98], by Ammann in [A98] and in great generality by Lott in [L02].

Here we will consider the degenerations of compact hyperbolic manifolds M_i to a non-compact hyperbolic manifold M. Such sequences $(M_i)_i$ exist only in dimension 2 and 3, due to Teichmüller theory and Thurston's cusp closing theorem (see e.g. [G81]). The Laplace operator of the limit manifold has an essential spectrum $[\sigma, \infty)$ and possibly there are some discrete "small" eigenvalues below σ, where $\sigma = \frac{1}{4}$ in the 2-dimensional case and $\sigma = 1$ in the 3-dimensional case. Colbois and Courtois show in [CC89] and in [CC91] that the small eigenvalues of M are the limits of the smallest eigenvalues of the approximating manifolds M_i. The eigenvalues above σ get denser and denser during the degeneration; one has a clustering, and Ji (in [J93]) and Chavel and Dodziuk (in [CD94]) compute the accumulation rates.

For Dirac operators one also has to take the spin structure into account. In [B00] Bär obtains the following results: Depending on the spin structure the Dirac operator D of a complete non-compact hyperbolic manifold with finite volume has either a discrete spectrum or it holds $spec(D) = ess\,spec(D) = \mathbb{R}$. If one supposes degenerations $M_i \to M$ with compatible spin structures it turns out that for 3-dimensional degenerations M has to have a discrete Dirac spectrum, and there is no clustering. For 2-dimensional degenerations a continuous limit spectrum is possible, and Bär computes the accumulation rate in this case.

We will show the convergence of the Dirac eigenvalues in the case of a discrete limit spectrum. For this we will use the notion of (Λ, ε)-spectral closeness (compare [BD02]):

Definition 1.1. *Let $\varepsilon > 0$, $\Lambda > 0$. Two self-adjoint operators are called (Λ, ε)*-**spectral close**, *if*

1. *In the intervall $[-\Lambda, \Lambda]$ both operators have only discrete eigenvalues and no other spectrum, and $\pm\Lambda$ are not eigenvalues of either operator.*

2. *Both operators have the same total number m of eigenvalues in $(-\Lambda, +\Lambda)$.*

3. *If the eigenvalues in $(-\Lambda, +\Lambda)$ are denoted by $\lambda_1 \leq ... \leq \lambda_m$ and $\mu_1 \leq ... \leq \mu_m$ respectively (each eigenvalue repeated according to its multiplicity), then $|\lambda_j - \mu_j| < \varepsilon$ for $j = 1, ..., m$.*

Our main result is:

Theorem 1.2. *Let $(M_i)_{i\geq 1}$ be a hyperbolic degeneration in dimension 2 or 3 such that the limit manifold M has a discrete Dirac spectrum. Denote the Dirac operators on M_i and M by D_i and D respectively, $i \geq 1$, and let $\varepsilon > 0$ and $\Lambda > 0$ with $\pm\Lambda \notin spec(D)$.*
Then for all sufficiently large i the Dirac operators D_i are (Λ, ε)-spectral close to D.

This article is organized as follows: In section 2 we will study the identification of spinors for different Riemannian metrics. In section 3 we will modify Colbois' and Courtois' method of escaping sets (see [CC91]) for the square of the Dirac operator. This will provide a criterion for the convergence of the small eigenvalues: One has to check that the Dirichlet eigenvalues of the escaping sets get sufficiently large during the degeneration process. In section 4 the structure of hyperbolic degenerations is described, and in section 5 Bär's formula for Dirac operators on manifolds foliated by hypersurfaces is recalled. We will use this in section 6 to derive some lower bounds for Dirichlet eigenvalues of the square of Dirac operators on hyperbolic tubes, which enables us to prove Theorem 1.2 in section 7.

2 Identifying spinors for different metrics

In this section we will briefly describe the identification of spinors for different Riemannian metrics and we will compare the Rayleigh quotients for the corresponding Dirac operators.

One can define a spin structure for an oriented manifold M^n without using Riemannian metrics (see [BG92]): Let $GL^+(M)$ denote the $GL^+(n)$-principal bundle of oriented frames of the tangent spaces. A spin structure of M is a reduction $\pi : \widetilde{GL}^+(M) \to GL^+(M)$ to a $\widetilde{GL}^+(n)$-principal bundle, where $\widetilde{GL}^+(n)$ is the connected twofold covering group of $GL^+(n)$. Given a Riemannian metric g on M the $SO(n)$-principal bundle of oriented orthonormal frames is denoted by $SO(M, g)$. Then $\pi^{-1}(SO(M, g)) \to SO(M, g)$ gives a reduction to a $Spin(n)$-principal bundle, i.e. a spin structure in the usual sense ([LM89]). We consider two Riemannian metrics g and g' on a compact spin manifold M^n with (possibly empty) boundary. To describe the spinor identification given in [BG92] we will follow the presentation in [AD98, section 2.2]: There is a unique endomorphism of the tangent bundle B such that $g(BX, Y) = g'(X, Y)$ and $g(BX, Y) = g(X, BY)$ for all X, Y. Let $A : TM \to TM$ be the positive square root of B:

$$g(AX, AY) = g'(X, Y) \text{ and } g(X, AY) = g(AX, Y) \text{ for all } X, Y. \quad (1)$$

This induces an $SO(n)$-equivariant bundle isomorphism:

$$A : SO(M, g') \longrightarrow SO(M, g), \ (e_1, ..., e_n) \longmapsto (Ae_1, ..., Ae_n).$$

If one has a fixed spin structure on M one gets a lift to the $Spin(n)$-bundles

$$\begin{array}{ccc} Spin(M,g') & \xrightarrow{A} & Spin(M,g) \\ \downarrow & & \downarrow \\ SO(M,g') & \xrightarrow{A} & SO(M,g) \end{array}$$

This induces an isomorphism of the spinor bundles $A : \Sigma(M,g') \longrightarrow \Sigma(M,g)$ which is a fibrewise isometry and is compatible with Clifford multiplication: $A(X \cdot \psi) = A(X) \cdot A(\psi)$ for $X \in T_pM$ and $\psi \in \Sigma_p(M,g')$. Let ∇ and ∇' denote the Levi-Civita connections for g and g'. We introduce a third connection $\overline{\nabla}$ by setting

$$\overline{\nabla}_Z X := A\big(\nabla'_Z(A^{-1}X)\big).$$

$\overline{\nabla}$ is compatible with the metric g, and its torsion is:

$$\overline{T}(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] = \ (\nabla'_Y A)A^{-1}X - (\nabla'_X A)A^{-1}Y. \tag{2}$$

Using Koszul's formula one computes:

$$2g(\overline{\nabla}_X Y - \nabla_X Y, Z) = g(\overline{T}(X,Y),Z) - g(\overline{T}(X,Z),Y) - g(\overline{T}(Y,Z),X). \tag{3}$$

We choose a local g-orthonormal frame $(e_i)_i$, i.e. a local section of $SO(M,g)$. For ∇ and $\overline{\nabla}$ the corresponding connection 1-forms are given by $\omega_{ij} = g(\nabla e_i, e_j)$ and $\overline{\omega}_{ij} = g(\overline{\nabla} e_i, e_j)$. From (3) and (2) we derive:

$$\begin{aligned} |(\overline{\omega}_{ij} - \omega_{ij})(e_k)| &= \left|g\left(\overline{\nabla}_{e_k} e_i - \nabla_{e_k} e_i, e_j\right)\right| \\ &\leq \frac{1}{2}\left|g(\overline{T}(e_k,e_i),e_j) - g(\overline{T}(e_k,e_j),e_i) - g(\overline{T}(e_i,e_j),e_k)\right| \\ &\leq 3 \cdot \|\nabla' A\|_g \cdot \|A^{-1}\|_g \,, \end{aligned} \tag{4}$$

where $\|.\|_g$ denotes the maximum norm. We suppose $(e_i)_i$ lifts to a local section s of $Spin(M,g)$. An orthonormal basis $(\sigma_\alpha)_\alpha$ of $\Sigma_n = \mathbb{C}^{2^{[n/2]}}$ induces an orthonormal frame $(\psi_\alpha)_\alpha$ of the associated bundle $\Sigma(M,g) = Spin(M,g) \times_{Spin(n)} \Sigma_n$ by $\psi_\alpha = [s, \sigma_\alpha]$. Any spinor field φ can locally be written as $\varphi = \sum_\alpha \varphi^\alpha \psi_\alpha$ and [LM89, Chap.2, Thm.4.14] gives:

$$\nabla\varphi = \sum_\alpha d\varphi^\alpha \otimes \psi_\alpha + \frac{1}{2}\sum_{i<j} \omega_{ij} \otimes e_i \cdot e_j \cdot \varphi \text{ and } \overline{\nabla}\varphi = \sum_\alpha d\varphi^\alpha \otimes \psi_\alpha + \frac{1}{2}\sum_{i<j} \overline{\omega}_{ij} \otimes e_i \cdot e_j \cdot \varphi.$$

By (4) we get for the difference of the connections:

$$\left|\overline{\nabla}_X\varphi - \nabla_X\varphi\right|_g \leq \frac{1}{2}\sum_{i<j} |(\overline{\omega}_{ij} - \omega_{ij})(X)|\,|\varphi|_g \leq K \cdot \|\nabla' A\|_g \cdot \|A^{-1}\|_g \cdot |\varphi|_g \cdot |X|_g \,, \tag{5}$$

where K is a constant only depending on the dimension of M.
Next, we want to compare the Dirac operators D_g and $D_{g'}$ of (M,g) and (M,g'). For a spinor field $\varphi \in \Gamma\Sigma(M,g)$ we define:

$$D'_{g'}\varphi := A\left(D_{g'}(A^{-1}\varphi)\right).$$

For some g-orthonormal frame $(e_i)_i$ we get a g'-orthonormal frame $e'_i := A^{-1}e_i$. Locally, $D'_{g'}$ is given by

$$D'_{g'}\varphi = \sum_j A\left(e'_j \cdot \nabla'_{e'_j}(A^{-1}\varphi)\right) = \sum_j A\left(e'_j\right) \cdot A\left(\nabla'_{e'_j}(A^{-1}\varphi)\right)$$
$$= \sum_j e_j \cdot \left(\overline{\nabla}_{e'_j}\varphi\right) = \sum_j e_j \cdot \left(\overline{\nabla}_{A^{-1}e_j}\varphi\right).$$

Let $(a_{ij})_{i,j}$ be the matrix of A with respect to the basis $(e_i)_i$: $Ae_i = \sum_j a_{ij}e_j$ for all i, and let $(a^{ij})_{i,j}$ be the matrix for A^{-1}, then we obtain:

$$\begin{aligned}\left(D_g - D'_{g'}\right)\varphi &= \sum_j e_j \cdot \left(\nabla_{e_j}\varphi - \overline{\nabla}_{A^{-1}e_j}\varphi\right) \\ &= \sum_{i,j}\left(\delta_{ij} - a^{ij}\right) e_j \cdot \nabla_{e_i}\varphi + \sum_{i,j} a^{ij} e_j \cdot \left(\nabla_{e_i} - \overline{\nabla}_{e_i}\right)\varphi\,.\end{aligned}$$

Using (5) we get the following pointwise estimate in any point of M:

$$\left|\left(D_g - D'_{g'}\right)\varphi\right|_g \leq K_1\|id - A^{-1}\|_g \cdot |\nabla\varphi|_g + K_2\|A^{-1}\|_g^2 \cdot \|\nabla' A\|_g \cdot |\varphi|_g\,, \tag{6}$$

where $|\nabla\varphi|_g^2 = \sum_j\langle\nabla_{e_j}\varphi, \nabla_{e_j}\varphi\rangle$, and where K_1 and K_2 are constants which depend only on the dimension of M. We define

$$d_g(g') := \left\|id - A^{-1}\right\|_g + \left\|\nabla' A\right\|_g \tag{7}$$

and as an immediate consequence of (6) we obtain:

Lemma 2.1. *For any integer $n \geq 1$ there exists a continuous function $\beta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\lim_{x\to 0}\beta(x) = 0$ such that the following holds:*
Let M be a compact n-dimensional spin manifold with boundary, and let g and g' be Riemannian metrics on M. Then for any spinor field φ on (M, g) this pointwise estimate holds:

$$\left|(D_g - D'_{g'})\varphi\right|_g^2 \leq \beta\left(d_g(g')\right) \cdot \left(|\nabla\varphi|_g^2 + |\varphi|_g^2\right)\,.$$

Without giving the proof we note:

Lemma 2.2. *Let $(g^m)_{m\geq 1}$ be a sequence of Riemannian metrics on M converging to the Riemannian metric g in the C^1-topology. Then one has*

$$d_g(g^m) \xrightarrow{m\to\infty} 0.$$

Next, we will compare the Rayleigh quotients for Dirac operators very explicitly.

In the expression $\beta(d_g(g'))$ we will omit the argument $d_g(g')$ and simply write β. We denote the scalar curvature of (M, g) by $scal_g$ and set

$$s^-_{(M,g)} = \min\left\{0, \sup_{p\in M}\left(-scal_g(p)\right)\right\}.$$

Proposition 2.3. *Let $\lambda \in \mathbb{R}$ and let g' be so close to g in the C^1-topology that one has $d_g(g') < 1$ and $\beta\left(d_g(g')\right) < 1$.*
For any smooth spinor field $\varphi \in \Gamma\Sigma(M,g)$ with $\varphi\,|_{\partial M} \equiv 0$ and $\eta_1 = \frac{\|(D_g-\lambda)\varphi\|_{L^2(g)}}{\|\varphi\|_{L^2(g)}}$ it holds:

$$\frac{\|(D_{g'}-\lambda)A^{-1}\varphi\|_{L^2(g')}}{\|A^{-1}\varphi\|_{L^2(g')}} \leq \left(\frac{1+d_g(g')}{1-d_g(g')}\right)^{\frac{n}{2}} \cdot \left(\eta_1 + \sqrt{\beta}\cdot\left(|\lambda| + \eta_1 + \tfrac{1}{2}\sqrt{s^-_{(M,g)}} + 1\right)\right) \tag{8}$$

and for any $\psi \in \Gamma\Sigma(M,g')$ with $\psi\,|_{\partial M} \equiv 0$ and $\eta_2 = \frac{\|(D_{g'}-\lambda)\psi\|_{L^2(g')}}{\|\psi\|_{L^2(g')}}$ we get

$$\frac{\|(D_g-\lambda)A\psi\|_{L^2(g)}}{\|A\psi\|_{L^2(g)}} \leq \left(\frac{1+d_g(g')}{1-d_g(g')}\right)^{\frac{n}{2}} \cdot \left(\eta_2 + \frac{\sqrt{\beta}}{1-\sqrt{\beta}}\left(|\lambda| + \eta_2\right)\right) \; + \; \frac{\sqrt{\beta}}{1-\sqrt{\beta}}\cdot\left(\tfrac{1}{2}\sqrt{s^-_{(M,g)}} + 1\right). \tag{9}$$

Proof. As $\varphi\,|_{\partial M} \equiv 0$ in the following partial integration the boundary terms vanish. We apply the Weitzenböck formula and obtain:

$$\|\nabla^g\varphi\|^2_{L^2(g)} \;=\; \langle(\nabla^g)^*\,\nabla^g\varphi,\varphi\rangle_{L^2(g)} \;\leq\; \|D_g\varphi\|^2_{L^2(g)} + \tfrac{1}{4}s^-_{(M,g)}\,\|\varphi\|^2_{L^2(g)}\,.$$

From this and from lemma 2.1 it follows:

$$\left|\|(D_g-\lambda)\varphi\|_{L^2(g)} - \left\|A(D_{g'}-\lambda)A^{-1}\varphi\right\|_{L^2(g)}\right| \leq \|(D_g - AD_{g'}A^{-1})\varphi\|_{L^2(g)}$$

$$\leq \sqrt{\beta}\cdot\left(\|\nabla^g\varphi\|^2_{L^2(g)} + \|\varphi\|^2_{L^2(g)}\right)^{\frac{1}{2}} \leq \sqrt{\beta}\left(\|D_g\varphi\|_{L^2(g)} + \left(\tfrac{1}{2}\sqrt{s^-_{(M,g)}} + 1\right)\|\varphi\|_{L^2(g)}\right) \tag{10}$$

For the volume elements of g and g' we have $\mathrm{dvol}_g = \det(A^{-1})\;\cdot\;\mathrm{dvol}_{g'}$.

By the definition (1) of $d_g(g')$ it is clear that the eigenvalues of A^{-1} are contained in the interval $[1-d_g(g'), 1+d_g(g')]$. Thus, $(1-d_g(g'))^n \leq \det(A^{-1}) \leq (1+d_g(g'))^n$, and one gets for the L^2-norms of a function $f : M \to \mathbb{R}$:

$$\left(1-d_g(g')\right)^{\frac{n}{2}}\cdot\|f\|_{L^2(g')} \leq \|f\|_{L^2(g)} \;\text{ and }\; \|f\|_{L^2(g)} \leq \left(1+d_g(g')\right)^{\frac{n}{2}}\cdot\|f\|_{L^2(g')}. \tag{11}$$

To prove (8) we deduce from the definition of η_1 that $\|D_g\varphi\|_{L^2(g)} \leq (|\lambda| + \eta_1)\,\|\varphi\|_{L^2(g)}$. From this inequality we derive by using (10) and the triangle inequality:

$$\left\|A(D_{g'}-\lambda)A^{-1}\varphi\right\|_{L^2(g)} \leq \left(\eta_1 + \sqrt{\beta}\,(|\lambda| + \eta_1) + \sqrt{\beta}\left(\tfrac{1}{2}\sqrt{s^-_{(M,g)}} + 1\right)\right)\cdot\|\varphi\|_{L^2(g)}.$$

Now, $A : \Sigma(M,g') \to \Sigma(M,g)$ is a fibrewise isometry. We apply (11) and have proved (8).

From (11) it follows for $\psi \in \Gamma\Sigma(M,g')$:

$$\frac{\|A(D_{g'}-\lambda)\psi\|_{L^2(g)}}{\|A\psi\|_{L^2(g)}} \leq \left(\frac{1+d_g(g')}{1-d_g(g')}\right)^{\frac{n}{2}}\cdot\eta_2\,. \tag{12}$$

Then, we set $\varphi = A\psi$. We see $\left\|D_{g'}\psi\right\|_{L^2(g')} \leq (|\lambda| + \eta_2)\,\|\psi\|_{L^2(g')}$ and apply (11) once more:

$$\|AD_{g'}\psi\|_{L^2(g)} \leq \left(1+d_g(g')\right)^{\frac{n}{2}}\|D_{g'}\psi\|_{L^2(g')} \leq \left(\frac{1+d_g(g')}{1-d_g(g')}\right)^{\frac{n}{2}}(|\lambda| + \eta_2)\,\|\varphi\|_{L^2(g)}.$$

Combined with (10) this yields:

$$\|D_g\varphi\|_{L^2(g)} \le \|AD_{g'}\psi\|_{L^2(g)} + \|\left(D_g - AD_{g'}A^{-1}\right)\varphi\|_{L^2(g)}$$

$$\le \left(\tfrac{1+d_g(g')}{1-d_g(g')}\right)^{\frac{n}{2}}(|\lambda|+\eta_2)\,\|\varphi\|_{L^2(g)} + \sqrt{\beta}\cdot\|D_g\varphi\|_{L^2(g)} + \sqrt{\beta}\left(\tfrac{1}{2}\sqrt{s^-_{(M,g)}}+1\right)\|\varphi\|_{L^2(g)}.$$

Taking $\beta < 1$ into account we conclude:

$$\|D_g\varphi\|_{L^2(g)} \le \tfrac{1}{1-\sqrt{\beta}}\left(\left(\tfrac{1+d_g(g')}{1-d_g(g')}\right)^{\frac{n}{2}}(|\lambda|+\eta_2) + \sqrt{\beta}\left(\tfrac{1}{2}\sqrt{s^-_{(M,g)}}+1\right)\right)\|\varphi\|_{L^2(g)}.$$

And using this and (10) we obtain:

$$\frac{\|(D_g - AD_{g'}A^{-1})\varphi\|_{L^2(g)}}{\|\varphi\|_{L^2(g)}} \le \sqrt{\beta}\frac{\|D_g\varphi\|_{L^2(g)}}{\|\varphi\|_{L^2(g)}} + \sqrt{\beta}\left(\tfrac{1}{2}\sqrt{s^-_{(M,g)}}+1\right)$$

$$\le \tfrac{\sqrt{\beta}}{1-\sqrt{\beta}}\left(\left(\tfrac{1+d_g(g')}{1-d_g(g')}\right)^{\frac{n}{2}}(|\lambda|+\eta_2)\right) + \tfrac{\sqrt{\beta}}{1-\sqrt{\beta}}\left(\tfrac{1}{2}\sqrt{s^-_{(M,g)}}+1\right). \tag{13}$$

Finally, by the triangle inequality we derive (9) from (12) and (13). □

We can derive from the previous proposition a uniform estimate for the deviation of the square of the Dirac operator for different metrics.

For a self-adjoint operator L we introduce the following notation:
Let $a < b$ be real numbers such that $[a,b] \cap ess\,spec(L) = \emptyset$. For any eigenvalue λ we denote the corresponding eigenspace by $E_\lambda(L)$ and we set

$$E_{[a,b]}(L) = \bigoplus_{\substack{a\le\lambda\le b\\ \lambda\in spec(L)}} E_\lambda(L).$$

Corollary 2.4. *For given numbers $n \in \mathbb{N}$, $S \ge 0$, $\Lambda > 0$, $\varepsilon > 0$ und $\eta > 0$ there exists some $\delta > 0$ with the following property:*
Let (M,g) be a n-dimensional compact Riemannian spin manifold with boundary whose scalar curvature satisfies $scal_g \ge -S$. Let g' be a Riemannian metric on M with $d_g(g') < \delta$ and let $\mu \in [0,\Lambda]$. Then for the Dirichlet eigenspaces of D_g^2 and $D_{g'}^2$ it holds:

$$\begin{aligned}\dim E_{[\mu-\eta,\mu+\eta]}\left((D_g)^2\right) &\le \dim E_{[\mu-\eta-\varepsilon,\mu+\eta+\varepsilon]}\left((D_{g'})^2\right) \quad \textit{and}\\ \dim E_{[\mu-\eta,\mu+\eta]}\left((D_{g'})^2\right) &\le \dim E_{[\mu-\eta-\varepsilon,\mu+\eta+\varepsilon]}\left((D_g)^2\right).\end{aligned}$$

Proof. Supposed $scal_g \ge -S$ one has $s^-_{(M,g)} \le S$. As $\beta(d_g(g')) \to 0$ for $d_g(g') \to 0$ one can conclude from (8) and (9) for $\lambda = 0$ that there is $\delta > 0$ such that for any (M,g) with $scal_g \ge -S$ and for any $\varphi \in E_{[0,\mu+\eta]}(D_g^2)$ it holds for any Riemannian metric g' on M with $d_g(g') < \delta$:

$$\frac{\|D_g\varphi\|^2_{L^2(g)}}{\|\varphi\|^2_{L^2(g)}} - \varepsilon < \frac{\|D_{g'}A^{-1}\varphi\|^2_{L^2(g')}}{\|A^{-1}\varphi\|^2_{L^2(g')}} < \frac{\|D_g\varphi\|^2_{L^2(g)}}{\|\varphi\|^2_{L^2(g)}} + \varepsilon.$$

We use the variational characterisation of the eigenvalues of D_g^2 and $D_{g'}^2$. Then we obtain for the k-th eigenvalues $\mu_k(D_g^2)$ and $\mu_k(D_{g'}^2)$ if $\mu_k(D_g^2) < \Lambda + \eta$:

$$\mu_k(D_g^2) - \varepsilon < \mu_k(D_{g'}^2) < \mu_k(D_g^2) + \varepsilon$$

which provides the first inequality in the corollary. To verify the second inequality we proceed analoguously, possibly we have to take a smaller δ. $\square$

In a similar way one can use Proposition 2.3 to prove the already known fact that on a fixed closed spin manifold the convergence of Riemannian metrics in the C^1-topology implies the convergence of Dirac spectra:

Corollary 2.5 ([B96], Prop. 7.1). *Let (M,g) be a closed spin manifold and let $\varepsilon > 0$ and $\Lambda > 0$ with $\pm\Lambda \notin spec(D_g)$. Then there exists $\delta > 0$ such that for all Riemannian metrics g' with $d_g(g') < \delta$ the Dirac operators D_g and $D_{g'}$ are (Λ, ε)-spectral close.*

3 Method of escaping sets

In this section we will adapt Colbois' and Courtois' method of escaping sets developped for the Laplace operator in [CC91] to the case of the square of the Dirac operator.

We will consider sequences of closed n-dimensional Riemannian spin manifolds $(M_i, g_i)_i$ which converge to a complete non-compact Riemannian spin manifold (M,g) of the same dimension.

Definition 3.1. *The sequence (M_i, g_i) converges to (M,g) in the sense of* **local C^1-spin convergence**, *if in M there are n-dimensional compact submanifolds $(B_t)_t$ with boundary such that $B_s \subset B_t$ for $s < t$ and $\bigcup_t B_t = M$, and for all t and i there are maps $\Phi_{i,t} : B_t \to M_i$ which send B_t diffeomorphically to the image $C_{i,t} = \Phi_{i,t}(B_t)$ preserving orientation and spin structure such that the pull backs of the metrics converge in the C^1-topology*

$$\Phi_{i,t}^*\left(g_i\,|_{C_{i,t}}\right) \xrightarrow{i\to\infty} g\,|_{B_t} \quad in \ \ C^1.$$

Let $(K_i)_{i\in\mathbb{N}}$ be a sequence or n-dimensional compact submanifolds of M with boundary such that $K_i \subset K_{i+1}$ for all i and $\bigcup_i K_i = M$.
For any i the distance to the boundary defines a function

$$d_i : K_i \to \mathbb{R}, \quad x \longmapsto \operatorname{dist}(x, \partial K_i).$$

One can find $R_i > 0$ such that $\{d_i < R_i\} \subset K_i$ is open and d_i is differentiable on $\{d_i < R_i\}$.

Definition 3.2. *We call $(\Omega_i)_{i\geq 1}$ a* **sequence of escaping sets** *for $(M_i)_i$ if*

1. *There is a sequence of submanifolds $(K_i)_{i\geq 1}$ and a sequence $R_i > 0$ as above such that $R_i \to \infty$ for $i \to \infty$.*
2. *For any i there is $\Phi_i : K_i \to M_i$ which maps K_i diffeomorphically onto is image $L_i = \Phi_i(K_i)$ preserving orientation and spin structure such that for the quantity defined in (7) it holds:*
$$\lim_{i\to\infty} d_{g|_{K_i}}\left(\Phi_i^*(g_i\,|_{L_i})\right) = 0.$$
3. *For all i the set Ω_i is the closure of the complement of L_i, i.e. $\Omega_i = \overline{M_i \setminus L_i}$.*

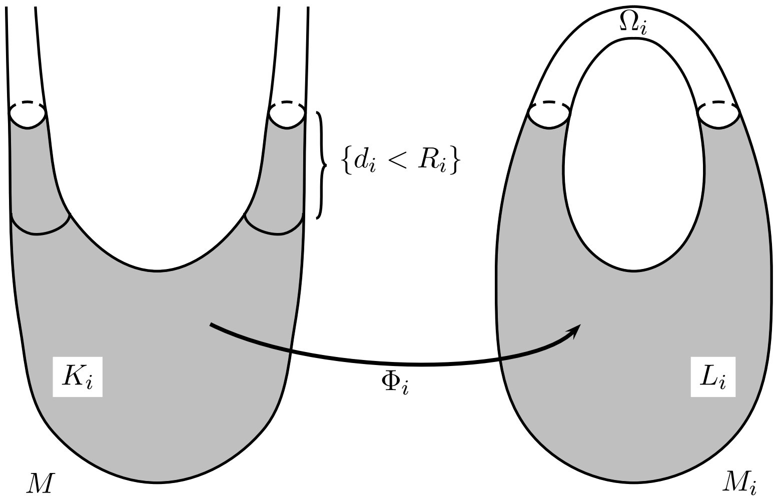

Figure 1: escaping set Ω_i.

Obviously, L_i and $\Omega_i \subset M_i$ are n-dimensional compact submanifolds with boundary.

Condition *1.* in Definition 3.2 is more restictive than it might seem at first glance. As the gradient of a distance function has norm 1 - whenever defined - one has $|\mathrm{grad}_g d_i|_g \equiv 1$ on $\{d_i < R_i\} \subset K_i$ for all i. Hence, $\{d_i < R_i\}$ is foliated by hypersurfaces which are all diffeomorphic to $d_i^{-1}(0) = \partial K_i$.

We denote the Dirac operators of (M_i, g_i) and (M, g) by D_i and D, respectively. As M is not compact we cannot assume that D has a discrete spectrum. We set

$$\sigma = \inf \mathit{ess\,spec}(D^2, M),$$

where $\inf \emptyset = \infty$, by definition.

The "small" eigenvalues of D^2 are those below σ, in the following we will always denote them:

$$0 \leq \mu_1 \leq ... \leq \mu_k \leq ... < \sigma$$

The only possible limit point of the small eigenvalues is σ. Therefore, for any $t < \sigma$ there is only a finite number of small eigenvalues below t.

Lemma 3.3. *Let $(\Omega_i)_i$ be a sequence of escaping sets for $(M_i)_i$.*
For $0 < t < \sigma$ with $t \notin \mathit{spec}(D^2, M)$ we denote the small eigenvalues of D^2 by

$$0 \leq \mu_1 \leq ... \leq \mu_N < t.$$

Let $0 \leq \mu_1^i \leq ... \leq \mu_{N(i)}^i < t$ denote the D^2-eigenvalues of K_i with Dirichlet boundary conditions below t. Then for sufficiently large i one has $N(i) = N$ and for $k = 1, ..., N$:

$$\lim_{i\to\infty} \mu_k^i = \mu_k.$$

Proof. For $\psi \in \Gamma(M,g)$ we put: $\|\psi\|^2_{H_1(M)} = \int_M \left\{|\psi|^2 + |D\psi|^2\right\}$. The sets $\widetilde{K_i} = K_i \setminus \{d_i < 1\}$ exhaust M, hence, we get for any L^2-spinor $\psi \in \Gamma(M,g)$:

$$\lim_{i\to\infty} \|\psi\|_{L^2(M\setminus\widetilde{K_i})} = 0. \tag{14}$$

For each i we choose some smooth function $u_i : M \to \mathbb{R}$ with $0 \leq u_i \leq 1$, $|\mathrm{grad}\,_g u_i|_g \leq 2$ and $u_i \equiv 1$ on $\widetilde{K_i}$ and $u_i \equiv 0$ on $M \setminus K_i$. By φ_k we denote the D^2-eigenspinors corresponding to μ_k, $k = 1, ..., N$. Then, there is a sequence δ_i with $\delta_i \to 0$ for $i \to \infty$ such that for $\varphi \in span\{\varphi_1, ..., \varphi_N\}$ with $\|\varphi\|_{L^2(M)} = 1$ it holds:

$$\begin{aligned}
\|\varphi - u_i\varphi\|^2_{H_1(M)} &\leq \int_M \left(|(1-u_i)\varphi|^2 + (|\mathrm{grad}\,_g(1-u_1)\cdot\varphi| + |(1-u_i)D\varphi|)^2\right) \\
&\leq N(5 + 2t + t^2) \cdot \max_{j=1,\ldots,N} \|\varphi_j\|^2_{L^2(M\setminus\widetilde{K_i})} \\
&\leq \delta_i.
\end{aligned}$$

Hence, for arbitrary $\varphi \in span\{\varphi_1, ..., \varphi_N\}$ we have shown:

$$\|(1-u_i)\varphi\|^2_{H_1(M)} \leq \delta_i \|\varphi\|^2_{L^2(M)}. \tag{15}$$

Now, $u_i\varphi$ fulfills Dirichlet conditions on K_i, and one has

$$\|u_i\varphi\|_{H_1(K_i)} = \|u_i\varphi\|_{H_1(M)} \leq \|\varphi\|_{H_1(M)} + \|(1-u_i)\varphi\|_{H_1(M)} < \infty.$$

Therefore, it makes sense to consider the Rayleigh quotient of $u_i\varphi$ where $\varphi \in span\{\varphi_1, ..., \varphi_k\}$ and $k \in \{1, ..., N\}$:

$$\frac{\|D(u_i\varphi)\|^2_{L^2(K_i)}}{\|u_i\varphi\|^2_{L^2(K_i)}} = \frac{\|D(u_i\varphi)\|^2_{L^2(M)}}{\|u_i\varphi\|^2_{L^2(M)}} \tag{16}$$

Using (15) and (16) we obain

$$\frac{\|D(u_i\varphi)\|^2_{L^2(K_i)}}{\|u_i\varphi\|^2_{L^2(K_i)}} \leq \frac{\mu_k + 2\sqrt{\delta_i}\sqrt{\mu_k} + \delta_i}{\left(1-\sqrt{\delta_i}\right)^2} \leq \mu_k + \rho_i \tag{17}$$

for some $\rho_i > 0$ with $\rho_i \to 0$. The min-max principle yields: $\mu_k^i \leq \mu_k + \rho_i$. By domain monotonicity we get $\mu_k^i \geq \mu_k$ and $N(i) \leq N$, from which the Lemma follows. □

Next, the Dirichlet eigenvalues or K_i and L_i are compared. We use property *2.* of escaping sets and apply Corollary 2.4. For this we have to assume a uniform lower bound for the scalar curvature of all K_i.

Lemma 3.4. *Let $(K_i)_i$ and $(L_i)_i$ be as in Definition 3.2. Let the scalar curvature of M be bounded from below, i.e. $scal_g \geq -S$ for some $S \geq 0$. For $0 < t < \sigma$ with $t \notin spec(D^2, M)$ let the D^2-eigenvalues of K_i with Dirichlet boundary conditions be denoted by*

$$0 \leq \mu_1^i \leq ... \leq \mu_{N(i)}^i < t,$$

and let the D_i^2-eigenvalues of L_i with Dirichlet boundary conditions be denoted by

$$0 \leq \nu_1^i \leq ... \leq \nu_{K(i)}^i < t.$$

Then for sufficiently large i it follows $K(i) = N(i) = N$ and for all $k = 1, ..., N$:

$$\lim_{i\to\infty} \left(\mu_k^i - \nu_k^i\right) = 0.$$

Proof. Let the distinct D^2-eigenvalues of M which are smaller than t be denoted by $0 \leq \mu_{j_1} < ... < \mu_{j_r} < t$. For $\varepsilon > 0$ sufficiently small the intervals $[\mu_{j_k} - 3\varepsilon, \mu_{j_k} + 3\varepsilon]$ are disjoint and contained in $(-\infty, t)$ for all $k = 1, ..., r$. For large i Lemma 3.3 gives

$$\dim E_{\mu_{j_k}}(D^2, M) = \dim E^{\mathit{Dirichlet}}_{[\mu_{j_k}-\varepsilon, \mu_{j_k}+\varepsilon]}(D^2, K_i) = \dim E^{\mathit{Dirichlet}}_{[\mu_{j_k}-3\varepsilon, \mu_{j_k}+3\varepsilon]}(D^2, K_i). \tag{18}$$

Condition *2.* of Definition 3.2 says that on K_i the Riemannian metrics $g\,|_{K_i}$ and $\Phi_i^*(g_i\,|_{L_i})$ satisfy: $d_{g|_{K_i}}(\Phi_i^*(g_i\,|_{L_i})) \to 0$ and the associated spin structures are equivalent. As on K_i the scalar curvature is bounded from below by $-S$ Corollary 2.4 gives for large i:

$$\dim E^{\mathit{Dirichlet}}_{[\mu_{j_k}-\varepsilon, \mu_{j_k}+\varepsilon]}(D^2, K_i) \leq \dim E^{\mathit{Dirichlet}}_{[\mu_{j_k}-2\varepsilon, \mu_{j_k}+2\varepsilon]}(D_i^2, L_i) \leq \dim E^{\mathit{Dirichlet}}_{[\mu_{j_k}-3\varepsilon, \mu_{j_k}+3\varepsilon]}(D^2, K_i). \tag{19}$$

For large i Corollary 2.4 also yields:

$$\dim E^{\mathit{Dirichlet}}_{[\mu_{j_k}+2\varepsilon, \mu_{j_{k+1}}-2\varepsilon]}(D_i^2, L_i) \leq \dim E^{\mathit{Dirichlet}}_{[\mu_{j_k}+\varepsilon, \mu_{j_{k+1}}-\varepsilon]}(D^2, K_i) = 0.$$

Hence, for large i the operator D_i^2 on L_i has no Dirichlet eigenvalues in $[0, t)\backslash \bigcup_k [\mu_{j_k} - 2\varepsilon, \mu_{j_k} + 2\varepsilon]$. For s ufficiently large i we get by (18) and (19): $K(i) = N(i) = N$ and $|\mu_k^i - \nu_k^i| < 3\varepsilon$, from which the Lemma follows. ☐

Theorem 3.5. *Let $(M_i, g_i) \to (M, g)$ in the sense of local C^1-spin convergence and let the scalar curvature of (M, g) be bounded from below. Suppose for all k for which there is a k-th small eigenvalue $\mu_k < \sigma$ one has $R(i) \geq k$ for sufficiently large i and $\lim_{i\to\infty} \lambda_k^i = \mu_k$. Then for any sequence of escaping sets $(\Omega_i)_i$ for $(M_i)_i$ it follows:*

$$\liminf_{i\to\infty} \lambda_1^{\mathit{Dirichlet}}(\Omega_i, D_i^2) \geq \sigma.$$

Proof. We assume for some $\tau < \sigma$:

$$\liminf_{i\to\infty} \lambda_1^{\mathit{Dirichlet}}(\Omega_i, D_i^2) = \tau. \tag{20}$$

We choose some $t \geq 0$ with $\tau < t < \sigma$, $t \notin spec(D^2, M)$. Keeping the notions of Lemma 3.4 we denote the Dirichlet eigenvalues of D^2 on K_i by

$$0 \leq \mu_1^i \leq ... \leq \mu_{N(i)}^i < t,$$

and the the Dirichlet eigenvalues of D_i^2 on L_i by

$$0 \leq \nu_1^i \leq ... \leq \nu_{K(i)}^i < t.$$

Lemma 3.3 and Lemma 3.4 give for large i: $K(i) = N(i) = N$, where N denotes the number of D^2-eigenvalues of M being smaller than t, and $\lim_{i\to\infty} \mu_k^i = \mu_k$ and $\lim_{i\to\infty} (\mu_k^i - \nu_k^i) = 0$ for $k \leq N$. Hence,

$$\lim_{i\to\infty} (\lambda_k^i - \nu_k^i) = \lim_{i\to\infty} (\lambda_k^i - \mu_k) + \lim_{i\to\infty} (\mu_k - \mu_k^i) + \lim_{i\to\infty} (\mu_k^i - \nu_k^i) = 0.$$

On L_i we choose orthonormal eigenspinors $\varphi_1^i, ..., \varphi_N^i$ associated to the eigenvalues $\nu_1^i, ..., \nu_N^i$, and on Ω_i we choose some eigenspinor ψ^i for the smallest Dirichlet eigenvalue ν_0^i. We extend $\varphi_1^i, ..., \varphi_N^i$ and ψ^i by zero to get piecewise smooth spinors on M^i which have finite H_1-norms and which we will denote again by $\varphi_1^i, ..., \varphi_N^i$ and ψ^i. Now, $supp(\varphi_k^i) \cap supp(\psi^i) \subset \partial L_k$ is a zero set for any k. Hence, the spinors $\varphi_1^i, ..., \varphi_N^i, \psi^i$ are mutually perpendicular with respect to the L^2-scalar product, in particular they are linearly independent. By assumption (20) for infinitely many i one has $\nu_0^i < t$. Hence, for infinitely many i on the $(N+1)$-dimensional vector space $V^i = span\{\varphi_1^i, ..., \varphi_N^i, \psi^i\}$ the Rayleigh quotient bounded by t, i.e. $\|D_i\sigma\|_{L^2}^2 < t \cdot \|\sigma\|_{L^2}^2$ for all $\sigma \in V^i$. By the min-max principle one has at least $N+1$ eigenvalues below t, which contradicts the fact that $N(i) = N$ for large i. □

One also has a reverse of this.

Theorem 3.6. *Let $(M_i, g_i) \to (M, g)$ in the sense of local C^1-spin convergence and let the scalar curvature of (M, g) be bounded from below.*
Let $(\Omega_i)_i$ be a sequence of escaping sets for $(M_i)_i$ such that $\widetilde{\Omega}_i = \Omega_i \cup \{d_i \circ \Phi_i^{-1} \leq \frac{R_i}{2}\}$ also gives a sequence of escaping sets for $(M_i)_i$ with the property:

$$\liminf_{i\to\infty} \lambda_1^{Dirichlet}(\widetilde{\Omega}_i, D_i^2) \geq \sigma. \tag{21}$$

Then for all k for which there exists a k-th small eigenvalue $\mu_k < \sigma$ one has for sufficiently large i: $R(i) \geq k$ and $\lim_{i\to\infty} \lambda_k^i = \mu_k$.

Proof of Theorem 3.6. Consider $t < \sigma$, $t \notin spec(D^2, M)$. Let $M(i)$ denote the number of $(D_i)^2$-eigenvalues of M_i being smaller than t:

$$0 \leq \lambda_1^i \leq ... \leq \lambda_{M(i)}^i < t.$$

By Lemma 3.3 and Lemma 3.4 we get $K(i) = N(i) = N$ for large i and $\lim_{i\to\infty} \nu_k^i = \mu_k$.
We will show $M(i) = N$ for large i and $\lim_{i\to\infty} (\lambda_k^i - \nu_k^i) = 0$ for $k = 1, ..., N$.
First, we get rom domain monotonicity: $\lambda_k^i \leq \nu_k^i$ for all k, and hence $M(i) \geq K(i) = N$. Next, we consider $k_0 \geq N$ such that $M(i) \geq k_0$ for infinitely many i. W.l.o.g. we can assume $M(i) \geq k_0$ for all i. We choose eigenspinors ψ_k^i with $\|\psi_k^i\|_{L^2(M_i)} = 1$ associated to the eigenvalues $\lambda_1^i, ..., \lambda_{k_0}^i$. For any spinor φ on Ω_i we set

$$\|\varphi\|_{H_1(\Omega_i)}^2 = \int_{\Omega_i} \left(|\varphi|^2 + |D_i\varphi|^2\right) \mathrm{dvol}_{g_i}.$$

Proposition 3.7. *Let $k \in \mathbb{N}$ such that $\lambda_k^i < t$ for all i. Then, for the associated normed eigenspinors it holds*

$$\lim_{i\to\infty} \|\psi_k^i\|_{H_1(\Omega_i)} = 0.$$

Before proving this proposition we will finish the proof of Theorem 3.6. We choose some smooth functions $v_i : M_i \to [0,1]$ with $v_i \mid_{M_i \setminus \widetilde{\Omega}_i} \equiv 1$, $v_i \mid_{\Omega_i} \equiv 0$ and $|\mathrm{grad}_{\,g_i} v_i|_{g_i} \leq 2$, which is possible for large i because $\frac{R_i}{2} > 2$. Then, we get for $\varphi \in span\{\psi_1^i, ..., \psi_k^i\}$ with $\|\varphi\|_{L^2(M_i)} = 1$:

$$\|\varphi - v_i\varphi\|^2_{H_1(M_i)} \leq 2 \cdot \|\varphi\|^2_{H_1(\widetilde{\Omega}_i)} \leq 2 \cdot \sum_{j=1}^{k} \|\psi_j^i\|^2_{H_1(\widetilde{\Omega}_i)}.$$

Proposition 3.7 shows that this tends to zero as $i \to \infty$. Therefore, $v_i\varphi$ has finite $H_1(M_i)$-norm and Dirichlet boundary conditions on L_i are fulfilled. The same Rayleigh quotient argument as in (17) yields some null sequence $(\rho_i)_{i\geq 1}$ such that for all $k = 1, ..., k_0$:

$$\nu_k^i \leq \lambda_k^i + \rho_i. \tag{22}$$

It follows that all $\nu_1^i, ..., \nu_{k_0}^i$ are smaller than t, and therefore $k_0 \leq N$ and $M(i) = N$ for any sufficiently large i. Furthermore, (22) gives $\lim\limits_{i\to\infty}(\lambda_k^i - \nu_k^i) = 0$ for all $k = 1, ..., N$. □

Next, we will prove Proposition 3.7.

We recall the notions from Definition 3.2: A non-compact manifold M is exhausted by compact submanifolds with boundary $(K_i)_i$. The distance to the boundary of K_i is denoted $d_i : K_i \to \mathbb{R}$ and it is assumed that d_i is differentiable on $\{d_i < R_i\}$. Using the diffeomorphisms $\Phi_i : K_i \to L_i$ we define some modified distance functions:

$$f_i : L_i \to \mathbb{R}, \quad f_i = d_i \circ \Phi_i^{-1}.$$

Then, f_i is differentiable on $\{f_i < R_i\}$. For $0 \leq r < s \leq R_i$ we define

$$C_i(r,s) = f_i^{-1}\big([r,s)\big).$$

As f_i has only regular points in $\{f_i < R_i\}$, for $0 < r < R_i$ the set $f_i^{-1}(r) \subset M_i$ is a hypersurface.

Lemma 3.8. *Let $t > 0$. Then one can find some null sequence $(\delta_i)_i$ s.t. the following holds: Let $\lambda < t$ be a $(D_i)^2$-eigenvalue and let ψ be some associated normed eigenspinor on M_i. Then there is a $r \in [1, \frac{R_i}{2}]$ such that:*

1. $\|\psi\|^2_{H_1(C_i(r-1,r))} < \delta_i$ *and*
2. $\|D_i\psi\|^2_{L^2(f_i^{-1}(r))} + \|\psi\|^2_{L^2(f_i^{-1}(r))} < \delta_i.$

Proof. Given ψ and i, we set $F = |\psi|^2 + |D_i\psi|^2$. From $\lambda < t$ it follows:

$$\int_{M_i} F \,\mathrm{dvol}_{g_i} < 1 + t.$$

Let l_i denote the smallest integer below $(\frac{R_i}{4}-1)$. Then, for $l=0,...,l_i$ the sets $C_i(2l,2l+2)$ are well defined. For l we set $B_l = \int\limits_{C_i(2l,2l+2)} F\,\mathrm{dvol}_{g_i}$ and get:

$$\sum_{l=0}^{l_i} B_l \le \int_{C_i(0,\frac{R_i}{2})} F\,\mathrm{dvol}_{g_i} \le \int_{M_i} F\,\mathrm{dvol}_{g_i} < 1+t.$$

Therefore, there exists some $m\in\{0,...,l_i\}$ satisfying $B_m < \frac{1+t}{l_i}$. As f_i is a differentiable function on $C_i(2m,2m+2)$ the coarea formula (see [C84, Chap. IV.1]) yields:

$$B_m = \int_{C_i(2m,2m+2)} F\,\mathrm{dvol}_{g_i} = \int_{2m}^{2m+2} \mathrm{dr} \int_{f_i^{-1}(r)} \frac{F}{|\mathrm{grad}_{\,g_i} f_i|_{g_i}}\ \mathrm{d\,area}_r.$$

From this we conclude that for some $r\in[2m+1,2m+2)\subset[1,\frac{R_i}{2}]$ one has

$$\int_{f_i^{-1}(r)} \frac{F}{|\mathrm{grad}_{\,g_i} f_i|_{g_i}}\ \mathrm{d\,area}_r < \frac{1+t}{l_i}\,. \tag{23}$$

Now, d_i is the Riemannian distance with respect to g, it is differentiable on $\{d_i<R_i\}$, and therefore we get: $|\mathrm{grad}_{\,g} d_i|_g \equiv 1$. We consider the endomorphisms A_i of TK_i as in (1), being self adjoint with respect to g and satisfying $g(A_iX,A_iY)=\Phi_i^*g(X,Y)$. Then, we get:

$$\left|A_i\,\mathrm{grad}_{\,\Phi_i^*g_i} d_i\right|_g = \left|\mathrm{grad}_{\,\Phi_i^*g_i} d_i\right|_{\Phi_i^*g_i} = |\mathrm{grad}_{\,g_i} f_i|_{g_i} \quad \text{and}$$

$$\mathrm{grad}_{\,g} d_i = (A_i)^2\,\mathrm{grad}_{\,\Phi_i^*g_i} d_i.$$

This gives the following pointwise estimate on $C_i(0,R_i)$:

$$\begin{aligned}\left|1-|\mathrm{grad}_{\,g_i} f_i|_{g_i}\right| &= \left|\left|(A_i)^2\,\mathrm{grad}_{\,\Phi_i^*g_i} d_i\right|_g - \left|A_i\,\mathrm{grad}_{\,\Phi_i^*g_i} d_i\right|_g\right| \\ &\le \left|\left(id-A_i^{-1}\right)\cdot(A_i)^2\,\mathrm{grad}_{\,\Phi_i^*g_i} d_i\right|_g \\ &\le \left\|id-A_i^{-1}\right\|_{g|_{K_i}} \cdot \left|(A_i)^2\,\mathrm{grad}_{\,\Phi_i^*g_i} d_i\right|_g \\ &\le d_{g|_{K_i}}\left(\Phi_i^*(g_i\,|_{L_i})\right)\,.\end{aligned}$$

Therefore, there is some null sequence $(\varepsilon_i)_i$ such that $\sup\limits_{C_i(0,R_i)}\left|1-\frac{1}{|\mathrm{grad}_{\,g_i} f_i|_{g_i}}\right| < \varepsilon_i$ for all i.

We define $\delta_i = \frac{1+t}{l_i}\cdot\frac{1}{1-\varepsilon_i}$. For $i\to\infty$ this converges to 0 because R_i and l_i $i\to\infty$ tend to ∞. One gets

$$\int\limits_{C_i(r-1,r)} F\,\mathrm{dvol}_{g_i} \le B_m < \tfrac{1+t}{l_i} < \delta_i.$$

The inverse triangle inequality gives $\frac{1}{|\mathrm{grad}_{\,g_i} f_i|_{g_i}} \ge 1-\varepsilon_i$. Combined with (23) this yields:

$$\int_{f_i^{-1}(r)} F\ \ \mathrm{d\,area}_r\ \le\ \tfrac{1}{1-\varepsilon_i}\ \int_{f_i^{-1}(r)} \frac{F}{|\mathrm{grad}_{\,g_i} f_i|_{g_i}}\ \mathrm{d\,area}_r < \delta_i.$$

□

Proof of Proposition 3.7. For a normed eigenspinor ψ_k^i there is some $r_k^i \in [1, \frac{R_i}{2}]$ with the properties *1.* and 2. in Lemma 3.8. The set $X_k^i = \Omega_i \cup C_i(0, r_k^i) \subset M_i$ is a smooth submanifold with boundary $f_i^{-1}(r_k^i)$. Let ν denote the outward unit vector field. Then, by Green's formula (see e.g. [B90, p.5]) we get:

$$\int_{X_k^i} |D_i\psi_k^i|^2 = \int_{X_k^i} \langle (D_i)^2\psi_k^i, \psi_k^i\rangle + \int_{f_i^{-1}(r_k^i)} \langle \nu \cdot \psi_k^i, D_i\psi_k^i\rangle\,,$$

where scalar product, norm, volume, integral and Clifford multiplication are taken with respect to g_i. The Cauchy-Schwarz inequality and property *2.* of Lemma 3.8 yield:

$$\begin{aligned} \left| \int_{X_k^i} |D_i\psi_k^i|^2 - \lambda_k^i \int_{X_k^i} |\psi_k^i|^2 \right| &= \left| \int_{f_i^{-1}(r_k^i)} \langle \nu \cdot \psi_k^i, D_i\psi_k^i\rangle \right| \\ &\leq \left(\int_{f_i^{-1}(r_k^i)} |D_i\psi_k^i|^2 \right)^{\frac{1}{2}} \cdot \left(\int_{f_i^{-1}(r_k^i)} |\psi_k^i|^2 \right)^{\frac{1}{2}} < \delta_i. \end{aligned}$$

We assume that there exists some positive constant c and a subsequence $(\psi_k^j)_j$ such that

$$\int_{X_k^j} |\psi_k^j|^2 \geq c > 0. \tag{24}$$

Setting $\rho_i = \frac{1}{c} \cdot \delta_i$ we get: $\frac{\int_{X_k^j} |D_j\psi_k^j|^2}{\int_{X_k^j} |\psi_k^j|^2} \leq \lambda_k^j + \rho_j$ for all j.

We choose a smooth function $u_j : M_j \to [0,1]$ vanishing on $M_j \setminus X_k^j$ and being constantly 1 on $\Omega_j \cup C_i(0, r_k^i - 1)$ with $|\mathrm{grad}_{g_j} u_j|_{g_j} \leq 2$. Then, $\varphi_k^i = u_j\psi_k^i$ satisfies Dirichlet boundary conditions on X_k^i and its H_1-norm is finite. Using property *1.* in Lemma 3.8 we find a null sequence $(\eta_j)_j$ such that

$$\frac{\int_{X_k^j} |D_j\varphi_k^j|^2}{\int_{X_k^j} |\varphi_k^j|^2} \leq \frac{\int_{X_k^j} |D_j\psi_k^j|^2}{\int_{X_k^j} |\psi_k^j|^2} + \eta_j \leq \lambda_k^j + \eta_j + \rho_j.$$

It follows that for large j the first Dirichlet eigenvalue of X_k^j is smaller than t. On the other hand, $X_k^i \subset \widetilde{\Omega}_j$ and by domain monotonicity we get:

$$\liminf_{j\to\infty} \lambda_1^{Dirichlet}\left(X_k^j, (D_j)^2\right) \geq \liminf_{j\to\infty} \lambda_1^{Dirichlet}\left(\widetilde{\Omega}_j, (D_j)^2\right) \geq \sigma > t,$$

which yields a contradiction. Hence, (24) is false, and we have proved the desired result:

$$\|\psi_k^i\|^2_{H_1(\Omega_i)} < (1+t) \cdot \|\psi_k^i\|^2_{L^2(X_k^i)} \xrightarrow{i\to\infty} 0.$$

□

The next theorem gives a criterion for the convergence of Dirac operators in the sense of (Λ, ε)-spectral closeness. We will apply this to hyperbolic degenerations in the following sections.

Theorem 3.9. *Let $(M_i, g_i) \to (M, g)$ in the sense of local C^1-spin convergence and let the scalar curvature of (M, g) be bounded from below.*
Let $(\Omega_i)_i$ be a sequence of escaping sets for $(M_i)_i$ such that $\widetilde{\Omega}_i = \Omega_i \cup \{d_i \circ \Phi_i^{-1} \leq \frac{R_i}{2}\}$ also gives a sequence of escaping sets for $(M_i)_i$ with the property:

$$\liminf_{i\to\infty} \lambda_1^{Dirichlet}(\widetilde{\Omega}_i, D_i^2) \geq \sigma.$$

Then for all $\varepsilon > 0$ and $\Lambda > 0$ with $\pm\Lambda \notin spec(D, M)$ and $\Lambda^2 < \sigma$ and for all sufficiently large i the Dirac operators D_i and D are (Λ, ε)-spectral close.

Proof. By Theorem 3.6, for large i the total multiplicities of eigenvalues in $[-\Lambda, +\Lambda]$ is the same for D and D_i. The Dirac operator D on M is defined as the closure of its restriction to smooth spinors with compact support. Therefore, for each eigenvalue λ one can find a smooth compactly supported spinor ψ on M such that

$$\frac{\|(D-\lambda)\psi\|_{L^2(M)}}{\|\psi\|_{L^2(M)}} < \frac{\varepsilon}{2}.$$

Then one identifies ψ with a compactly supported spinor φ_i on M_i. Condition *2.* in Definition 3.2 and Proposition 2.4 imply that for large i one has

$$\frac{\|(D_i-\lambda)\varphi_i\|_{L^2(M_i)}}{\|\varphi\|_{L^2(M_i)}} < \varepsilon,$$

from which the claim follows. □

4 Hyperbolic manifolds of finite volume

In this section we will recall the structure of complete hyperbolic manifolds with finite volume and hyperbolic degenerations. A thorough treatment of this subject can be found in [T80], a shorter description is given in [CD94] and in [B00].

Let (M, g) be a complete hyperbolic manifold, i.e. M has constant negative curvature -1. For $\delta > 0$ we define the *δ-thin part* of M:

$$M_\delta = \{x \in M \mid injrad(x) \leq \delta\}$$

where *injrad* denotes the injectivity radius. The complement $M_{0,\delta} = M \setminus M_\delta$ is called the *δ-thick part* of M.

For small $\delta > 0$ the δ-thin part of an n-dimensional complete oriented hyperbolic manifold M of finite volume is a disjoint union of a finite number of cusps:

$$M_\delta = \dot{\bigcup_{j=1,\dots,k}} \mathcal{E}_j$$

such that for any j one has a compact connected manifold N_j carrying a flat metric g_{N_j} such that the cusp is $\mathcal{E}_j = N_j \times [0, \infty)$ with a warped product metric $g_{\mathcal{E}_j} = e^{-2t} \cdot g_{N_j} + dt^2$.

In dimensions 2 und 3 non-compact complete oriented hyperbolic manifolds (M, g) of finite volume can be approximated by compact hyperbolic manifolds (M_i, g_i). In dimension 2 this is

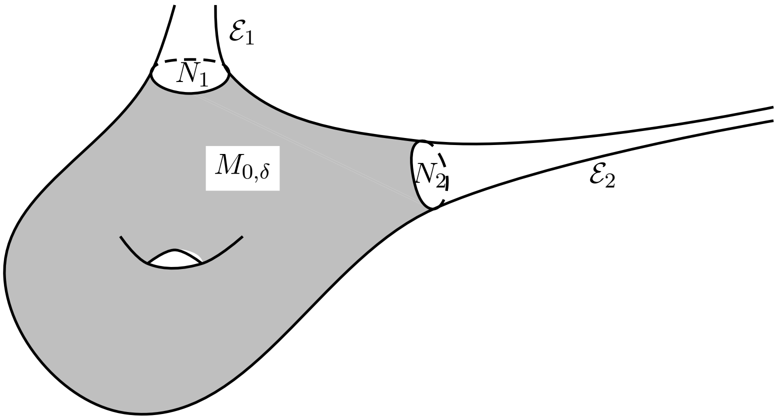

Figure 2: decomposition into δ-thin and δ-thick part.

due to Teichmüller theory and in three dimensions this is true because of Thurston's cusp closing theorem. Such sequences of compact hyperbolic manifolds are called *hyperbolic degenerations*. The structure of the approximation $(M_i, g_i) \to (M, g)$ is the following.

First, we describe the 3-dimensional case: For small $\delta > 0$ let $M_0^\delta = M_{0,\delta}$ denote the δ-thick part of M. The δ-thin part consists of cusps $\mathcal{E}_1^\delta, ..., \mathcal{E}_k^\delta$. For the compact manifold M_i we get the decomposition into δ-thick and δ-thin part:

$$M_i = M_{i,0}^\delta \,\dot\cup\, \dot{\bigcup_j} T_{i,j}^\delta,$$

where each $T_{i,j}^\delta$ is a closed tubular neighbourhood of radius $R_{i,j}^\delta$ about a simply closed geodesic $\gamma_{i,j}$ in M_i whose length is $l_{i,j} = L[\gamma_{i,j}]$. The boundary $N_{i,j}^\delta = \partial T_{i,j}^\delta$ is a flat torus.
There are diffeomorphisms $\Phi_i^\delta : \overline{M_0^\delta} \to \overline{M_{i,0}^\delta}$ between compact manifolds with boundary such that the pull back metrics converge in the C^1-topology:

$$(\Phi_i^\delta)^*(g_i|_{\overline{M_{i,0}^\delta}}) \to g|_{\overline{M_0^\delta}} \quad \text{for } i \to \infty\,.$$

For $i \to \infty$ the lengths of the geodesics tend to zero and the radii of the tubes tend to infinity: $l_{i,j} \to 0$ and $R_{i,j}^\delta \to \infty$. Each tube degenerates into one cusp.

In two dimensions it is possible to find even continuous degenerations: We consider the Teichmüller space of a closed surface of genus at least two. Hyperbolic degenerations correspond to paths in the Te ichmüller space converging to the boundary. For each sequence $(M_i, g_i)_i$ on such a path with limit point (M, g) the structure of approximation is essentially the same as in the 3-dimensional case: For each small $\delta > 0$ we get a decomposition of M_i:

$$M_i = M_{i,0}^\delta \,\dot\cup \dot{\bigcup_{j=1,\dots,k}} T_{i,j}^\delta$$

where again the $T^\delta_{i,j}$ are closed tubular neighbourhood around simply closed geodesics $\gamma_{i,j}$ with length $l_{i,j} = L[\gamma_{i,j}]$. The boundary of such a tube consists of two disjoint circles: $\partial T^\delta_{i,j} = S^1 \dot\cup S^1$. For $i \to \infty$ the lengths of the geodesics tend to zero and the radii of the tubes tend to infinity: $l_{i,j} \to 0$ and $R^\delta_{i,j} \to \infty$. Here each tube degenerates into two cusps. The thin-thick decomposition of M for small δ is:

$$M = M^\delta_0 \,\dot\cup \dot{\bigcup_{j=1,\ldots,2k}} \mathcal{E}^\delta_j .$$

And again there are diffeomorphisms $\Phi^\delta_i : \overline{M^\delta_0} \to \overline{M^\delta_{i,0}}$ such that for $i \to \infty$ one obtains

$$(\Phi^\delta_i)^*(g_i|_{\overline{M^\delta_{i,0}}}) \to g|_{\overline{M^\delta_0}} \quad \text{in the } C^1\text{-topology.}$$

For any degeneration there is $\delta_0 > 0$ such that for each $\delta < \delta_0$ one can find such diffeomorphisms $(\Phi^\delta_i)_i$.

As we want to compare the associated Dirac operators we will from now on assume that all manifolds M_i and M are spin and that all diffeomorphisms Φ^δ_i respect the spin structures. This means we consider the case that the hyperbolic degeneration converge in the sense of local C^1-spin convergence.

One observes that then there exists a sequence of escaping sets such that the ecsaping sets are finally contained in the δ-thin parts (see [P03]):

Lemma 4.1. *Let $(M_i, g_i)_i \to (M, g)$ be a hyperbolic degeneration in dimension 2 or 3. Then there is a sequence of escaping sets $(\Omega_i)_i$ such that $\widetilde{\Omega}_i = \Omega_i \cup \{d_i \circ \Phi_i^{-1} \leq \frac{R_i}{2}\}$ gives again a sequence of escaping sets and for any $\delta < \delta_0$ one gets for sufficiently large i:*

$$\widetilde{\Omega}_i \subset \dot{\bigcup_j} T^\delta_{i,j} \subset M_i \, .$$

In order to apply theorem 3.9 we need lower bounds for the Dirichlet eigenvalues of the square of the Dirac operator on $\widetilde{\Omega}_i$. We will derive estimates for the tube $T^\delta_{i,j}$ and use domain monotonicity.

5 Dirac operators on manifolds foliated by hypersurfaces

As described above the δ-thin parts consist of tubes and cusps. All cusps and all 2-dimensional tubes are warped products. After removing the central geodesic any 3-dimensional tube is foliated by tori. This is the reason why one is interested in foliations by hypersurfaces in this context.

Let M be a Riemannian spin manifold of dimension n foliated by oriented hypersurfaces $\{N\}$. Any spin structure on M induces one on N in a natural way: We denote the normal unit vector field of the foliation by ν, the associated form operator by B, i.e. $B(X) = -\nabla_X \nu$, and the mean curvature by $H = \frac{1}{n-1}\,\mathrm{tr}(B)$.

We restrict the spinor bundle ΣM to a hypersurface N, we have to distinguish two cases: For n odd $\Sigma M|_N$ is just the spinor bundle of N, and for n even $\Sigma M|_N$ is isomorphic to $\Sigma N \oplus \Sigma N$, the sum of two copies of the spinor bundle of N. The Clifford multiplication with respect to N is given by

$$X \otimes \varphi \mapsto X \cdot \nu \cdot \varphi,$$

where "$\cdot$" denotes the Clifford multiplication with respect to M.

The spinorial Levi-Civita connections ∇^M and ∇^N of M and N are related by the following formula: Let X be a tangential vector of N and let φ be a section of $\Sigma M|_N$, then one has ([B96, Prop.2.1]):

$$\nabla^M_X \varphi = \nabla^N_X \varphi + \tfrac{1}{2} B(X) \cdot \nu \cdot \varphi. \tag{25}$$

Let D^M be the Dirac operator of M. For odd n let D^N denote the Dirac operator of N and for even n let D^N denote the direct sum of the Dirac operator of N and its negative. Then D^N acts on sections in $\Sigma M|_N$ and one gets the following relation between D^M and D^N (see [B96, Prop.2.2]):

$$D^M \varphi = \nu \cdot D^N \varphi - \tfrac{n-1}{2} H \varphi + \nabla^M_\nu \varphi \qquad \text{for} \quad \varphi \in \Gamma\left(\Sigma M|_N\right). \tag{26}$$

Furthermore, we define the operator $\mathfrak{D}^B$ acting on sections φ of $\Sigma M|_N$ as follows: For some orthonormal frame $e_1, ..., e_{n-1}$ of $T_p N$ we set

$$\mathfrak{D}^B \varphi|_p = \sum_{i=1}^{n-1} e_i \cdot \nu \cdot \nabla^N_{B(e_i)} \varphi = \sum_{i=1}^{n-1} B(e_i) \cdot \nu \cdot \nabla^N_{e_i} \varphi \,.$$

Applying the hypersurface formula (26) twice and identifying all occuring terms one gets the following formula for foliations:

Proposition 5.1 ([B00], Prop. 4). *Let M be an n-dimensional Riemannian spin manifold foliated by oriented hypersurfaces $\{N\}$. Then it holds:*

$$\begin{aligned}(D^M)^2 \;\; &= \;\; (D^N)^2 - (\nabla^M_\nu)^2 - \mathfrak{D}^B + (n-1) H \, \nabla^M_\nu + \nabla^M_{\nabla_\nu \nu} \\ &\quad - \tfrac{n-1}{2} \left(\operatorname{grad}_N H\right) \cdot \nu - \tfrac{(n-1)^2}{4} H^2 + \tfrac{1}{2} |B|^2 - \tfrac{1}{2} \nu \cdot Ric(\nu).\end{aligned}$$

Here $\operatorname{grad}_N H$ *denotes the gradient of H along N, Ric is the Ricci tensor of M and $|B|$ denotes the Hilbert-Schmidt norm of B, i.e. $|B|^2 = \sum_j \lambda_j^2$, where $\lambda_1, ..., \lambda_{n-1}$ are the eigenvalues of B.*

Next, we will state Bär's results on the Dirac spectra of hyperbolic degenerations. The dependence on the spin structure is crucial.

Definition 5.2. *Let M be a complete non-compact hyperbolic manifold of finite volume, and let $\mathcal{E} = N \times [0, \infty)$ be a cusp of M. A spin structure on M is called* **trivial** *along $\mathcal{E}$ if the induced Dirac operator D^N on N has a non-trivial kernel.*

Theorem 5.3 ([B00], Thm. 1). *Let M be a complete non-compact hyperbolic manifold of finite volume equipped with a spin structure.*
If the spin structure is trivial along some cusp of M the spectrum of the Dirac operator is

$$spec(D) = ess\, spec(D) = \mathbb{R}.$$

Otherwise, if the spin structure is non-trivial along all cusps the Dirac operator has a discrete spectrum.

For any self-adjoint operator A with discrete spectrum we define the eigenvalue counting function

$$\mathcal{N}_A(-x, x) = \#\big((-x, x) \cap spec(A)\big) \qquad \text{for} \;\; x > 0.$$

Now, we consider hyperbolic degenerations (whose identification maps Φ_i^δ respect the spin structures). If the limit manifold has Dirac spectrum $\mathbb{R}$ one expects a clustering which can occur in dimension 2. Whereas in the 3-dimensional case there is no clustering:

Theorem 5.4 ([B00], Thm. 2 and 3). *1. Let $(M_i)_{i\geq 1}$ be a 2-dimensional hyperbolic degeneration such that the spin structure of the limit manifold is trivial along all cups. Then, for the associated Dirac operators D_i the following holds for any $x > 0$:*

$$\mathcal{N}_{D_i}(-x,x) = \frac{4x}{\pi}\sum_{j=1}^{k}\log\left(\tfrac{1}{l_{ij}}\right) + O_x(1) \quad \text{for } i \to \infty$$

where $l_{i,j}$ denotes the length of the closed geodesic inside the tube $T^{\delta}_{i,j}$.

2. For any 3-dimensional hyperbolic degeneration the spin structure of the limit manifold in non-trivial along all cusps. The Dirac spectrum of the limit manifold is discrete and one gets for the Dirac operators D_i for any $x > 0$: $\mathcal{N}_{D_i}(-x,x) = O_x(1)$ for $i \to \infty$.

We will cover the case of a discrete limit spectrum and we will show that the spectra converge in the sense of (Λ,ε)-spectral closeness.

6 Lower eigenvalue bounds for hyperbolic tubes

The geometry of distance tubes of closed geodesics in hyperbolic manifolds of dimension 2 and 3 is well understood. In this section we will derive estimates for the Dirichlet eigenvalues of the square of the Dirac operator on such tubes.

6.1 The 2-dimensional case

In dimension 2 any distance tube is a warped product $T[0,R] = [-R,R]\times S^1$ with the metric $ds^2 = dr^2 + l^2\cosh^2(r)d\theta^2$ where $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$ and l is the length of the simply closed geodesic γ.

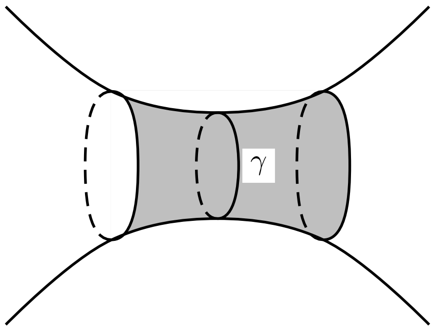

Figure 3: distance tube $T[0,R]$ for a simply closed geodesic γ.

Hence $T[0,R]$ is foliated by circles $N = S^1$. The normal unit vector field is $\nu = \frac{\partial}{\partial r}$ and for the corresponding shape operator B one gets $B(\frac{\partial}{\partial\theta}) = -\tanh(r)\frac{\partial}{\partial\theta}$. Therefore, we have $|B|^2 = \tanh^2(r)$ and $H = -\tanh(r)$ and, in particular, $\mathrm{grad}_N H \equiv 0$. Furthermore, $\nabla^M_\nu \nu \equiv 0$ and $Ric = -id$. Plugging all this into the formula of proposition 5.1 we obtain:

$$(D^M)^2 = (D^N)^2 - \mathfrak{D}^B - (\nabla^M_\nu)^2 - \tanh(r)\nabla^M_\nu + \tfrac{1}{4}\tanh^2(r) - \tfrac{1}{2}. \tag{27}$$

For spinors φ on $T[0,R] \subset M$ we compute using the Leibniz rule:

$$\nabla_\nu^M \nabla_\nu^M \left(\sqrt{\cosh(r)}\,\varphi\right) \quad = \quad \sqrt{\cosh(r)}\left(\left(\tfrac{1}{2} - \tfrac{1}{4}\tanh^2(r)\right)\varphi + \tanh(r)\nabla_\nu^M \varphi + \nabla_\nu^M \nabla_\nu^M \varphi\right).$$

Defining $\mathfrak{B}\varphi = -\frac{1}{\sqrt{\cosh(r)}}\left(\nabla_\nu^M \nabla_\nu^M \left(\sqrt{\cosh(r)}\,\varphi\right)\right)$ we get by (27):

$$(D^M)^2 = \left((D^N)^2 - \mathfrak{D}^B\right) + \mathfrak{B}.$$

Let φ be a spinor field on $T[0,R]$ satisfying $\varphi\,|_{\partial T[0,R]} \equiv 0$. The volume element of $T[0,R]$ is given by $d\mathrm{vol} = l\cosh(r)drd\theta$. Then we ge by performing a partial integration with respect to r and by noticing that the term $|\nabla_\nu^M \sqrt{\cosh(r)}\varphi|^2$ is non-negative:

$$\begin{aligned}
\langle \varphi, \mathfrak{B}\varphi\rangle_{L^2(T[0,R])} &= -l\int_{S^1} d\theta \int_{-R}^{+R} \langle \nabla_\nu^M \nabla_\nu^M \sqrt{\cosh(r)}\varphi, \sqrt{\cosh(r)}\varphi\rangle dr \\
&\geq l\int_{S^1} d\theta \int_{-R}^{+R} \left(-\tfrac{\partial}{\partial r}\langle \nabla_\nu^M \sqrt{\cosh(r)}\varphi, \sqrt{\cosh(r)}\varphi\rangle\right) dr \\
&= -l\int_{S^1} d\theta \left(\langle \nabla_\nu^M \sqrt{\cosh(r)}\varphi, \sqrt{\cosh(r)}\varphi\rangle\big|_{-R}^{+R}\right) = 0\,.
\end{aligned} \tag{28}$$

Hence,

$$\left\|D^M\varphi\right\|^2_{L^2(T[0,R])} = \langle \varphi, (D^M)^2\varphi\rangle_{L^2(T[0,R])} \geq \langle \varphi, \left((D^N)^2 - \mathfrak{D}^B\right)\varphi\rangle_{L^2(T[0,R])}\,. \tag{29}$$

Denoting the circle $\{r\}\times S^1$ by T_r we notice $(D^{T_r})^2 = (\nabla^{T_r})^*\nabla^{T_r}$ as the induced metric on T_r is flat. For T_r we have $|B| = |\tanh(r)| \leq 1$ and therefore $\left\|\mathfrak{D}^B\varphi\right\|^2_{L^2(T_r)} \leq \left\|\nabla^{T_r}\varphi\right\|^2_{L^2(T_r)} = \left\|D^{T_r}\varphi\right\|^2_{L^2(T_r)}$.
We denote the smallest $(D^{T_r})^2$-eigenvalue by μ_r and get:

$$\begin{aligned}
\langle \left((D^{T_r})^2 - \mathfrak{D}^B\right)\varphi, \varphi\rangle_{L^2(T_r)} &\geq \|D^{T_r}\varphi\|^2_{L^2(T_r)} - \|D^{T_r}\varphi\|_{L^2(T_r)}\cdot\|\varphi\|_{L^2(T_r)} \\
&\geq (\mu_r - \sqrt{\mu_r})\|\varphi\|^2_{L^2(T_r)}\,.
\end{aligned}$$

As $r\mapsto\mu_r$ is decreasing on $[0,R]$ and increasing on $[-R,0]$ integration with respect to r yields

$$\langle \left((D^{T_r})^2 - \mathfrak{D}^B\right)\varphi, \varphi\rangle_{L^2(T[0,R])} \geq (\mu_R - \sqrt{\mu_R})\|\varphi\|^2_{L^2(T[0,R])}\,,$$

where we have used that $x\mapsto x-\sqrt{x}$ gives an increasing function on $[1,\infty)$. Using (29) we have proved the following:

Proposition 6.1. *Let T be a distance tube of a simply closed geodesic in a 2-dimensional hyperbolic spin manifold M. Let μ denote the smallest eigenvalue of the square of the induced Dirac operator on the boundary ∂T. Then for any smooth spinor field φ on T with $\varphi\,|_{\partial T}\equiv 0$ it holds:*

$$\|D^M\varphi\|^2_{L^2(T)} \geq (\mu - \sqrt{\mu})\cdot\|\varphi\|^2_{L^2(T)}. \tag{30}$$

The proof given above works only for $\mu = \mu_R \geq 1$. But we notice that (30) is trivially true for $\mu_R \leq 1$ because the right hand side is non-positive in this case.

6.2 The 3-dimensional case

Let $T[0,R]$ denote a distance tube of radius R for a simply closed geodesic γ in a hyperbolic 3-manifold M. For $0 \le r_1 < r_2 \le R$ we set $T[r_1,r_2] = \{x \in M | r_1 \le \mathrm{dist}(x,\gamma) \le r_2\}$. The geometry of $T[0,R]$ is well understood (see [CD94, section 2]): We consider a geodesic $\widetilde{\gamma}$ in the 3-dimensional hyperbolic space $\mathbb{H}^3$ with $|\frac{d}{dt}\widetilde{\gamma}| \equiv 1$. Let $\widetilde{T}$ denote the distance tube of radius R around $\widetilde{\gamma}$. We choose a parallel vector field V along $\widetilde{\gamma}$ which has constant length 1 and is perpendicular to $\widetilde{\gamma}$. The corresponding Fermi coordinates are given as follows: Let r denote the distance to $\widetilde{\gamma}$, and let t denote the arc length along $\widetilde{\gamma}$. Define $\theta \in S^1$ as the angle taken with respect to V in the unit circle perpendicular to $\widetilde{\gamma}$. In these coordinates the hyperbolic metric is given by: $g_{\mathbb{H}^3} = dr^2 + \cosh(r)^2\, dt^2 + \sinh(r)^2\, d\theta^2$.

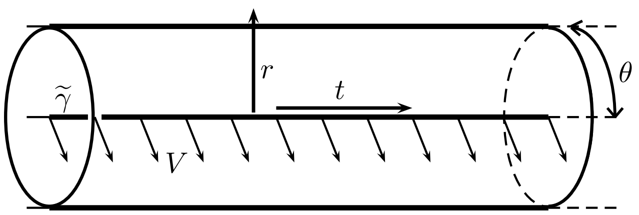

Figure 4: Fermi coordinates for $\widetilde{T} \subset \mathbb{H}^3$.

Now, $A : (r,t,\theta) \mapsto (r, t + L[\gamma], \theta + \alpha)$ defines a decktransformation for $T[0,R]$ where $L[\gamma]$ is the length of γ and α is a fixed angle. Then we get $T[0,R]$ as quotient $\widetilde{T}/\langle A\rangle$.

For $0 < r \le R$ the distance tori $T_r = \{x \in M \,|\, dist(x,\gamma)\}$ define a foliation of $T[0,R] \setminus \{\gamma\}$. The associated normal unit vector field ν is given by $\frac{\partial}{\partial r}$. For the shape operator B we compute: $B(\frac{\partial}{\partial\theta}) = -\coth(r)\frac{\partial}{\partial\theta}$ and $B(\frac{\partial}{\partial t}) = -\tanh(r)\frac{\partial}{\partial t}$. Hence, the mean curvature of T_r is given by $H = -\frac{1}{2}(\tanh(r) + \coth(r)) = -\coth(2r)$ and consequently $\mathrm{grad}_N H \equiv 0$. Furthermore, we get $|B|^2 = \tanh(r)^2 + \coth(r)^2 = 2\left(2\coth(2r)^2 - 1\right)$ and $Ric = -2 \cdot id$. We plug all this into the formula of proposition 5.1 and obtain:

$$(D^M)^2 = (D^N)^2 - \mathfrak{D}^B - (\nabla^M_\nu)^2 - 2\coth(2r)\nabla^M_\nu + \coth(2r)^2 - 2\,.$$

For any spinor field ψ on $\{r > 0\}$ the Leibniz rule yields:

$$\nabla^M_\nu\nabla^M_\nu\left(\sqrt{\sinh(2r)}\,\psi\right) = \sqrt{\sinh(2r)}\,\left(\left(2 - \coth(2r)^2\right)\psi + 2\coth(2r)\nabla^M_\nu\psi + \nabla^M_\nu\nabla^M_\nu\psi\right)$$

Setting $\mathfrak{B}\psi = -\frac{1}{\sqrt{\sinh(2r)}}\left(\nabla^M_\nu\nabla^M_\nu\left(\sqrt{\sinh(2r)}\,\psi\right)\right)$ for spinors ψ on $\{r > 0\}$ we get:

$$(D^M)^2 = \left((D^N)^2 - \mathfrak{D}^B\right) + \mathfrak{B}. \tag{31}$$

Now, we consider a spinor φ on $T[0,R]$ which vanishes on $\partial T[0,R] = T_R$. We fix $K \in (0,R)$ and write:

$$\|D^M\varphi\|^2_{L^2(T[0,R])} = \langle\varphi,(D^M)^2\varphi\rangle_{L^2(T[0,K])} + \langle\varphi,(D^M)^2\varphi\rangle_{L^2(T[K,R])}.$$

Next, we want to provide a lower bound for $\langle\varphi,(D^M)^2\varphi\rangle_{L^2(T[K,R])}$.

By the definition of $\mathfrak{D}^B$ we have $|\mathfrak{D}^B\varphi|\le|B|\cdot|\nabla^N\varphi|$. By the Weitzenböck formula we obtain $\|\nabla^N\varphi\|^2_{L^2(T_r)}=\|D^N\varphi\|^2_{L^2(T_r)}$ as $N=T_r$ is a flat torus. Consequently, for $r\in[K,R]$ one gets $|B|^2=2(2\coth(2r)^2-1)\le 4\coth(2K)^2$ and hence:

$$\|\mathfrak{D}^B\varphi\|^2_{L^2(T_r)}\le 4\coth(2K)^2\,\|D^N\varphi\|^2_{L^2(T_r)}.$$

Let μ_r denote the smallest eigenvalue of the square of the induced Dirac operator on T_r. It follows:

$$\langle\varphi,\left((D^N)^2-\mathfrak{D}^B\right)\varphi\rangle_{L^2(T_r)}\ge\left(\mu_r-2\sqrt{\mu_r}\,\coth(2K)\right)\|\varphi\|^2_{L^2(T_r)}\,.$$

As the Dirac eigenvalues of flat tori are explicitely known (see [Fr84]) one can show that $r\mapsto\mu_r$ is monotonic decreasing on $[K,R]$ (see [B00, Lemma 2]). Assuming $\mu_R\ge 4\coth(2K)^2$ we have $\mu_r-2\sqrt{\mu_r}\,\coth(2K)\ge\mu_R-2\sqrt{\mu_R}\,\coth(2K)$ for all $r\in[K,R]$ and therefore:

$$\langle\varphi,\left((D^N)^2-\mathfrak{D}^B\right)\varphi\rangle_{L^2(T[K,R])}\ge\left(\mu_R-2\sqrt{\mu_R}\,\coth(2K)\right)\|\varphi\|^2_{L^2(T[K,R])}\,. \tag{32}$$

Then, we consider $\langle\varphi,\mathfrak{B}\varphi\rangle_{L^2(T[K,R])}$.

In Fermi coordinates the volume element is $\mathrm{dvol}_M=\frac{1}{2}\sinh(2r)dr\,dt\,d\theta$. We perform a partial integration with respect to r, notice that the term $|\nabla^M_\nu\sqrt{\sinh(2r)}\varphi|^2$ is non-negative and get

$$\begin{aligned}
\langle\varphi,\mathfrak{B}\varphi\rangle_{L^2(T[K,R])} &= \tfrac{1}{2}\int_{(t,\theta)}d(t,\theta)\int_K^R\langle\nabla^M_\nu\nabla^M_\nu\sqrt{\sinh(2r)}\varphi,\sqrt{\sinh(2r)}\varphi\rangle dr\\
&\ge -\tfrac{1}{2}\int_{(t,\theta)}d(t,\theta)\left[\langle\nabla^M_\nu\sqrt{\sinh(2r)}\varphi,\sqrt{\sinh(2r)}\varphi\rangle\right]_K^R\\
&= \tfrac{1}{2}\int_{(t,\theta)}d(t,\theta)\;\langle\nabla^M_\nu\sqrt{\sinh(2r)}\varphi,\sqrt{\sinh(2r)}\varphi\rangle\Big|_{r=K}\\
&= \tfrac{1}{2}\int_{(t,\theta)}d(t,\theta)\;\left(\sinh(2K)\langle\nabla^M_\nu\varphi,\varphi\rangle+\cosh(2K)|\varphi|^2\right)\\
&= \int_{T_K}\langle\nabla^M_\nu\varphi,\varphi\rangle\;+\;\coth(2K)\|\varphi\|^2_{L^2(T_K)}
\end{aligned} \tag{33}$$

as $\mathrm{dvol}_{T_K}=\frac{1}{2}\sinh(2K)dt\,d\theta$ in these coordinates.

Combining (32) and (33) and assuming again $\mu_R\ge 4\coth(2K)^2$ we get by (31):

$$\begin{aligned}
\langle\varphi,(D^M)^2\varphi\rangle_{L^2(T[K,R])} &\ge \left(\mu_R-2\sqrt{\mu_R}\,\coth(2K)\right)\|\varphi\|^2_{L^2(T[K,R])}\\
&\quad+\int_{T_K}\langle\nabla^M_\nu\varphi,\varphi\rangle+\coth(2K)\|\varphi\|^2_{L^2(T_K)}
\end{aligned} \tag{34}$$

To treat the term $\langle\varphi,(D^M)^2\varphi\rangle_{L^2(T[0,K])}$ we use the Weitzenböck formula once more, here the scalar curvature is $scal_M\equiv -6$, and we get:

$$\begin{aligned}
\langle\varphi,(D^M)^2\varphi\rangle_{L^2(T[0,K])} &= \langle\varphi,(\nabla^M)^*\nabla^M\varphi\rangle_{L^2(T[0,K])}-\tfrac{3}{2}\|\varphi\|^2_{L^2(T[0,K])}\\
&= \|\nabla^M\varphi\|^2_{L^2(T[0,K])}-\int_{T_K}\langle\nabla^M_\nu\varphi,\varphi\rangle-\tfrac{3}{2}\|\varphi\|^2_{L^2(T[0,K])}\,.
\end{aligned}$$

This and (34) implies the following, again under the assumption $\mu_R \geq 4\coth(2K)^2$:

$$\langle\varphi,(D^M)^2\varphi\rangle_{L^2(T[0,R])} \geq (\mu_R - 2\sqrt{\mu_R}\,\coth(2K))\,\|\varphi\|^2_{L^2(T[K,R])} - \tfrac{3}{2}\|\varphi\|^2_{L^2(T[0,K])} + \|\nabla^M\varphi\|^2_{L^2(T[0,K])} + \coth(2K)\|\varphi\|^2_{L^2(T_K)}\,. \tag{35}$$

In the next step we will will find a lower bound for the last two terms in (35) by means of $\|\varphi\|^2_{L^2(T[0,K])}$. For this we consider $\rho \in (0,K)$, write $\varphi\|^2_{L^2(T[\rho,K])}$ in Fermi coordinates and perform a partial integration:

$$\begin{aligned}\int_{T[\rho,K]} |\varphi|^2 &= \tfrac{1}{2}\int_{(t,\theta)} d(t,\theta)\int_\rho^K dr\,|\varphi|^2\sinh(2r)\\
&= \tfrac{1}{4}\int_{(t,\theta)} d(t,\theta)\Big(\big(|\varphi|^2(K)\cosh(2K) - |\varphi|^2(\rho)\cosh(2\rho)\big)\\
&\qquad - \int_\rho^K dr\partial_\nu|\varphi|^2\cosh(2r)\Big)\\
&= -\tfrac{1}{2}\int_{T[\rho,K]}\partial_\nu|\varphi|^2\coth(2r) + \tfrac{1}{4}\int_{(t,\theta)} d(t,\theta)\int_\rho^K \partial_\nu|\varphi|^2\,\cosh(2\rho)\\
&\qquad + \tfrac{1}{4}\int_{(t,\theta)} d(t,\theta)\,|\varphi|^2(K)\,(\cosh(2K) - \cosh(2\rho))\,.\end{aligned}$$

Defining $f_\rho(r) = \frac{\cosh(2r)-\cosh(2\rho)}{\sinh(2r)}$ for $\rho \geq 0$ this can be rewritten as

$$\int_{T[\rho,K]} |\varphi|^2 = \tfrac{1}{2}\,f_\rho(K)\int_{T_K}|\varphi|^2 - \tfrac{1}{2}\int_{T[\rho,K]} f_\rho(r)\,\partial_\nu|\varphi|^2. \tag{36}$$

We observe that $f_\rho(r)$ is monotonic increasing in r and monotonic decreasing in ρ, which gives $\lim\limits_{r\to 0} f_0(r) = 0$. As $\lim\limits_{r\to\infty}\coth(r) = \infty$ there exists some $K_0 > 0$ such that for any $K \in (0, K_0]$ the following holds:

$$\coth(2K) \geq \coth(2K_0) \geq 1 \quad \text{and} \quad f_\rho(K) \leq f_0(K) \leq \tfrac{1}{2} \text{ for any } \rho > 0. \tag{37}$$

As $\frac{1}{2}\big|\partial_\nu|\varphi|^2\big| \leq |\nabla^M\varphi|\cdot|\varphi|$ the Cauchy-Schwarz inequality yields

$$\left(\int_{T[\rho,K]}\tfrac{1}{2}f_\rho(r)\,\partial_\nu|\varphi|^2\right)^2 \leq \left(\int_{T[\rho,K]} f_\rho(r)\,|\nabla^M\varphi|^2\right)\cdot\left(\int_{T[\rho,K]} f_\rho(r)\,|\varphi|^2\right).$$

Taking the square root and using $f_\rho(r) \leq f_\rho(K)$ for $r \leq K$ this shows that

$$\Big|\int_{T[\rho,K]} f_\rho(r)\,\partial_\nu|\varphi|^2\Big| \leq 2\,f_\rho(K)\cdot\|\nabla^M\varphi\|_{L^2(T[\rho,K])}\cdot\|\varphi\|_{L^2(T[\rho,K])}\,.$$

We combine this and (36) and let $\rho \to 0$ to get

$$\|\varphi\|^2_{L^2(T[0,K])} \leq f_0(K)\,\|\varphi\|^2_{L^2(T_K)} + 2\,f_0(K)\,\|\varphi\|_{L^2(T[0,K])}\cdot\|\nabla^M\varphi\|_{L^2(T[0,K])}. \tag{38}$$

To proceed we need an elementary lemma.

Lemma 6.2. *Let $\xi, \eta, \zeta, f \geq 0$ such that*

$$\xi \leq f \cdot \eta + 2f \cdot \sqrt{\xi} \cdot \sqrt{\zeta}. \tag{39}$$

Then it holds: $\xi \leq 2f \cdot (\eta + 2f \cdot \zeta)$.

Proof. By (39) we get $\left(\sqrt{\xi} - f \cdot \sqrt{\zeta}\right)^2 = \xi - 2f \cdot \sqrt{\xi} \cdot \sqrt{\zeta} + f^2 \cdot \zeta \leq f \cdot \eta + f^2 \cdot \zeta$. We take the square root $|\sqrt{\xi} - f \cdot \sqrt{\zeta}| \leq \sqrt{f} \cdot \sqrt{\eta + f \cdot \zeta}$ and obtain $\sqrt{\xi} \leq \sqrt{f} \cdot \left(\sqrt{\eta + f \cdot \zeta} + \sqrt{f \cdot \zeta}\right)$. We square this and use that $(x+y)^2 \leq 2(x^2 + y^2)$ for all $x, y \in \mathbb{R}$ to finish the proof. □

Let $0 < K \leq K_0$ with K_0 as in (37). Then from Lemma 6.2 and from (38) it follows that

$$\begin{aligned} \|\varphi\|^2_{L^2(T[0,K])} &\leq 2f_0(K) \cdot \left(\|\varphi\|^2_{L^2(T_K)} + 2f_0(K)\|\nabla^M \varphi\|^2_{L^2(T[0,K])}\right) \\ &\leq 2f_0(K) \cdot \left(\coth(2K)\|\varphi\|^2_{L^2(T_K)} + \|\nabla^M \varphi\|^2_{L^2(T[0,K])}\right) \end{aligned}$$

and, hence, $\coth(2K)\|\varphi\|^2_{L^2(T_K)} + \|\nabla^M \varphi\|^2_{L^2(T[0,K])} \geq \frac{1}{2f_0(K)} \|\varphi\|^2_{L^2(T[0,K])}$.

This is a lower bound for the last two terms in (35) and we have proved the following:

Proposition 6.3. *Let T be a distance tube of radius R around a simply closed geodesic in a 3-dimensional hyperbolic spin manifold M. The boundary ∂T is a flat torus, and let μ denote the smallest eigenvalue of the square of the induced Dirac operator of ∂T. Let $K \in (0, R)$ with $K \leq K_0$ where K_0 is as in (37), and let $\mu \geq 4\coth(2K)^2$. Then for every smooth spinor on T satisfying Dirichlet boundary conditions it holds:*

$$\|D^M \varphi\|^2_{L^2(T[0,R])} \geq (\mu - 2\sqrt{\mu}\,\coth(2K))\,\|\varphi\|^2_{L^2(T[K,R])} + \tfrac{1}{2}\left(\tfrac{\sinh(2K)}{\cosh(2K)-1} - 3\right)\|\varphi\|^2_{L^2(T[0,K])}.$$

7 Proof of Theorem 1.2

We will apply Theorem 3.9 with $\sigma = \inf\, \mathit{ess\,spec}(D^2, M) = \infty$ as the spectrum is purely discrete. The limit manifold M is hyperbolic, its scalar curvature has a lower bound.
We consider the sequences of escaping sets $(\Omega_i)_i$ and $(\widetilde{\Omega}_i)_i$ of Lemma 4.1. We have to show that for any arbitrarily large $\tau > 0$ it holds:

$$\lambda_1^{\mathit{Dirichlet}}(\widetilde{\Omega}_i, D_i^2) > \tau \qquad \text{for all sufficiently large } i\,. \tag{40}$$

For all $\delta < \delta_0$ lemma 4.1 gives $\widetilde{\Omega}_i \subset \dot{\bigcup_j} T^\delta_{i,j}$ for sufficiently large i, and by domain monotonicity:

$$\lambda_1^{\mathit{Dirichlet}}\left(\widetilde{\Omega}_i, D_i^2\right) \geq \lambda_1^{\mathit{Dirichlet}}\left(\dot{\bigcup_j} T^\delta_{i,j}, D_i^2\right) = \min_j \lambda_1^{\mathit{Dirichlet}}\left(T^\delta_{i,j}, D_i^2\right).$$

To prove (40) we have to find a small $\delta > 0$ such that for all $j = 1, ..., k$ and for all sufficiently large i we have

$$\lambda_1^{\mathit{Dirichlet}}(T^\delta_{i,j}, D_i^2) > \tau. \tag{41}$$

First, we will prove (41) for the 2-dimensional case: The function $x \mapsto x - \sqrt{x}$ is monotonic increasing on $[1, \infty)$ and tends to ∞ for $x \to \infty$. Hence, there is $x_0 > 1$ such that $x - \sqrt{x} > \tau$ for any $x > x_0$. If we have found a small $\delta > 0$ such that the first eigenvalue of the square of the induced Dirac operator on $\partial T^\delta_{i,j}$ is larger than x_0 for all j and all sufficiently large i we get (41) by applying Proposition 6.1 and a Rayleigh quotient argument.

We can find such a δ by the following observation: Let the cusp $\mathcal{E} = N \times [0, \infty)$ carry the warped product metric $g_{\mathcal{E}} = e^{-2t} g_N + dt^2$ and let the spin structure along $\mathcal{E}$ be non-trivial. Then we get for the smallest eigenvalue μ_t of the square of the induced Dirac operator on the leaf $N \times \{t\}$:

$$\lim_{t \to \infty} \mu_t = \infty.$$

For any cusp there exists $t^{\mathcal{E}} > 0$ such that for all leaves $N \times \{t\}$ with $t > t^{\mathcal{E}}$ the smallest eigenvalue μ_t is bigger than x_0. For small δ the boundary of the δ-thick part M^δ_0 of M is the disjoing union of such leaves. If δ is sufficiently small the corresponding parameter t is larger than $t^{\mathcal{E}}$. As the pull back metrics of the δ-thick parts $M^\delta_{i,0}$ of the compact manifolds converge in the C^1-topology one gets also the convergence of the pull back metrics of the boundaries $\partial M^\delta_{i,0}$. By Corollary 2.5 the smallest eigenvalues of the square of the Dirac operator of $\partial M^\delta_{i,0}$ converge to the smallest eigenvalue of the square of the Dirac operator of ∂M^δ_0, and therefore these eigenvalues are larger than x_0 for i sufficiently large.

In the 3-dimensional case we proceed in a similar way: The function $K \mapsto \frac{\sinh(2K)}{\cosh(2K)-1}$ is monotonic decreasing on $(0, \infty)$ and tends to ∞ for $K \to 0$. Therefore, there exists a $K_1 \in (0, K_0)$ such that

$$\tfrac{1}{2}\left(\tfrac{\sinh(2K_1)}{\cosh(2K_1)-1} - 3\right) > \tau.$$

As $\lim_{x\to\infty} (x - 2\sqrt{x}\,\coth(2K_1)) = \infty$ we can find an x_0 with $(x - 2\sqrt{x}\,\coth(2K_1)) > \tau$ for all $x > x_0$. In analogy to the 2-dimensional part there is $\delta > 0$ such that for all j and for all sufficiently large i the first eigenvalue of the square of the Dirac operator of $\partial T^\delta_{i,j}$ is bigger than x_0 and also bigger than $4\coth(2K_1)^2$. For $i \to \infty$ the radii of the tubes $T^\delta_{i,j}$ tend to ∞, hence, for sufficiently large i they are bigger than K_1. By Propositon 6.3 we get for all j and all sufficiently large i:

$$\left\|D^{M_i}\varphi\right\|^2_{L^2(T^\delta_{i,j})} > \tau \cdot \|\varphi\|^2_{L^2(T^\delta_{i,j})},$$

for any smooth spinor φ on $T^\delta_{i,j}$, which implies (41). □

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