CHAPTER 1

Preliminaries

1.1. Basic Relevant Algebra

1.2. Introduction to Differential Geometry

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In this lecture we give a brief introduction to the theory of manifolds and related basic concepts of differential geometry. Since a lot of material will have to be covered in a very short time we can hardly give any proofs. Most proofs are fairly straightforward anyway. The reader interested in details may consult any introduction to differential geometry such as [**?, ?**].

Definition 1.1. A topological space X is called an *n-dimensional topological manifold* if and only if

i) X is Hausdorff and has a countable basis.
ii) X is locally homeomorphic to $\mathbb{R}^n$, i. e., for every point $p \in X$ there exists an open neighborhood U of p in X, an open subset $V \subset \mathbb{R}^n$ and a homeomorphism $x : U \to V$.

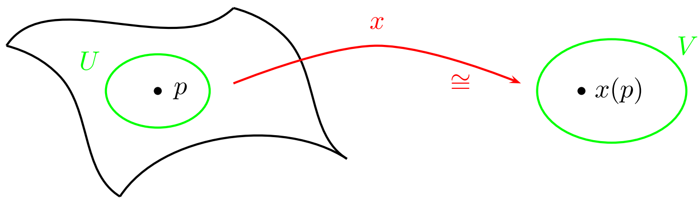

Fig. 1

The first condition is of technical nature and sometimes omitted. For subsets of $\mathbb{R}^N$ it is automatically satisfied. The crucial condition for a topological space to be a manifold is the second one. Topological manifolds are precisely those spaces that look locally like $\mathbb{R}^n$.

The homeomorphisms $x : U \to V$ are called *charts*. Given a chart x the standard coordinates on $\mathbb{R}^n$ can be used to specify points in U. Namely, a point p in U is uniquely determined by the n numbers $(x^1(p), \dots, x^n(p))$. A collection of charts $x_\alpha : U_\alpha \to V_\alpha$ is called an *atlas* of M if the U_α cover M, i. e., $M = \bigcup_\alpha U_\alpha$.

Examples 1.2.

1. *Euclidean space* $\mathbb{R}^n$ *itself is of course an* n*-dimensional topological manifold. Here we can use an atlas consisting of only one chart, namely* $x = \mathrm{Id} : \mathbb{R}^n \to \mathbb{R}^n$.
2. *The sphere* S^n *of unit vectors in* $\mathbb{R}^{n+1}$ *is an* n*-dimensional topological manifold. To find a chart containing a given point* $p \in S^n$ *one can project the hemisphere about* p *onto the* n*-dimensional disk perpendicular to* p *and then rotate this disk into* $\mathbb{R}^n$.

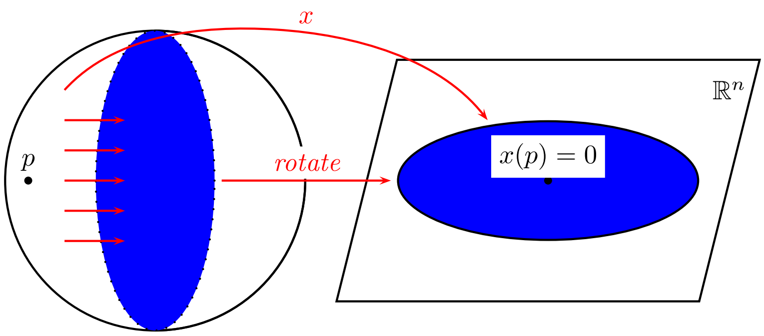

Fig. 2

3. *2-dimensional topological manifolds are surfaces. Compact connected oriented*[1] *surfaces are classified up to homeomorphism by their* genus *(which can be any number in* $\mathbb{N}_0$*).*

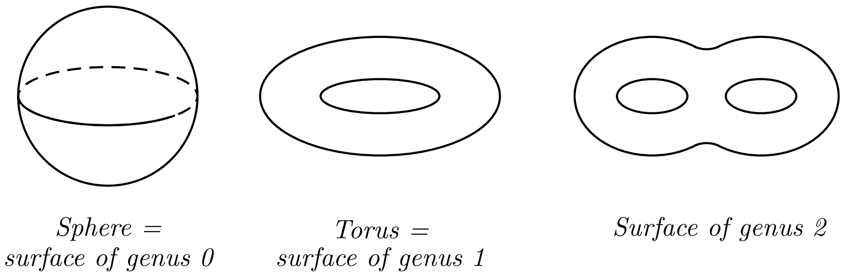

Fig. 3

To get a better feeling for the concept of a manifold let us also look at some topological spaces which are not manifolds.

Non-Examples.

1. The union M of the x-axis and the y-axis in the plane is not a manifold. The problems arise at the origin where the two lines intersect. In fact,

[1] see Definition 1.31

if we remove the origin, then the remaining space $M - \{(0,0)\}$ is a 1-dimensional topological manifold. That M itself is not a manifold can be seen as follows. If M were a 1-dimensional topological manifold, then there would be a chart about $(0,0)$, i. e., a homeomorphism $x : U \to V$ where U is an open neighborhood of $(0,0)$ and V is some open subset of $\mathbb{R}$. By restricting x to a smaller open neighborhood if necessary we may assume that V is an open interval. If we now remove $x(0,0)$ from V, then we are left with two components whereas $U - \{(0,0)\}$ has at least four components, a contradiction to x being a homeomorphism.

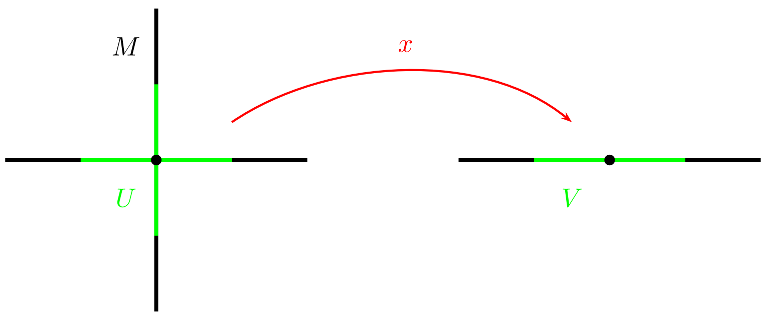

FIG. 4

2. Now let us look at the upper half-plane $\mathcal{H} = \{(x^1, x^2) \in \mathbb{R}^2 \mid x^2 \geq 0\}$. This time the boundary points $(x^1, 0)$ are the ones which are not contained in a chart. $\mathcal{H}$ is what is called a *manifold-with-boundary*, a concept which we will meet again when we get to the Stokes theorem.

There are some standard methods to construct new manifolds out of given ones.

EXAMPLES FOR CONSTRUCTION METHODS.

1. If M is an m-dimensional topological manifold and N an n-dimensional one, then $M \times N$ is an $(n+m)$-dimensional topological manifold. To obtain charts for $M \times N$ one can simply take the products of charts $x : U \to V$ for M and $\tilde{x} : \tilde{U} \to \tilde{V}$ for N,
$$x \times \tilde{x} : U \times \tilde{U} \to V \times \tilde{V} \subset \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}.$$
2. If M is an n-dimensional manifold and $U \subset M$ an open subset, then U is also an n-dimensional manifold. Charts for U are obtained from charts for M by restricting their domain to the intersection with U.

For a topological space X it makes sense to talk about *continuous* functions
$$f : X \longrightarrow \mathbb{R}\,.$$
Since we plan to do *differential* geometry we would like to know:

QUESTION 1.3. *What are* differentiable *functions?*

We do know what differentiability of a function means if it is defined on an open subset of $\mathbb{R}^n$. Since by definition manifolds look locally like $\mathbb{R}^n$ the natural attempt would be to call a function $f : M \to \mathbb{R}$ defined on a manifold M differentiable at $p \in M$ if and only if $f \circ x^{-1} : V \to \mathbb{R}$ is differentiable at $x(p)$ for some chart $x : U \to V$ whose domain U contains p. But then we have to check if this definition depends on the choice of chart x. For a second chart y we have near $y(p)$

$$f \circ y^{-1} = (f \circ x^{-1}) \circ (x \circ y^{-1}).$$

If we knew that $x \circ y^{-1}$ is a *diffeomorphism* and not just a homeomorphism we could conclude differentiability of $f \circ y^{-1}$ from differentiability of $f \circ x^{-1}$. For this reason we make the following definitions.

Let M be an n-dimensional topological manifold.

DEFINITION 1.4. Two charts $x : U_1 \to V_1$ and $y : U_2 \to V_2$ of a topological manifold are called C^∞*-compatible* if and only if $y \circ x^{-1} : x\,(U_1 \cap U_2) \to y\,(U_1 \cap U_2)$ is a C^∞-diffeomorphism.

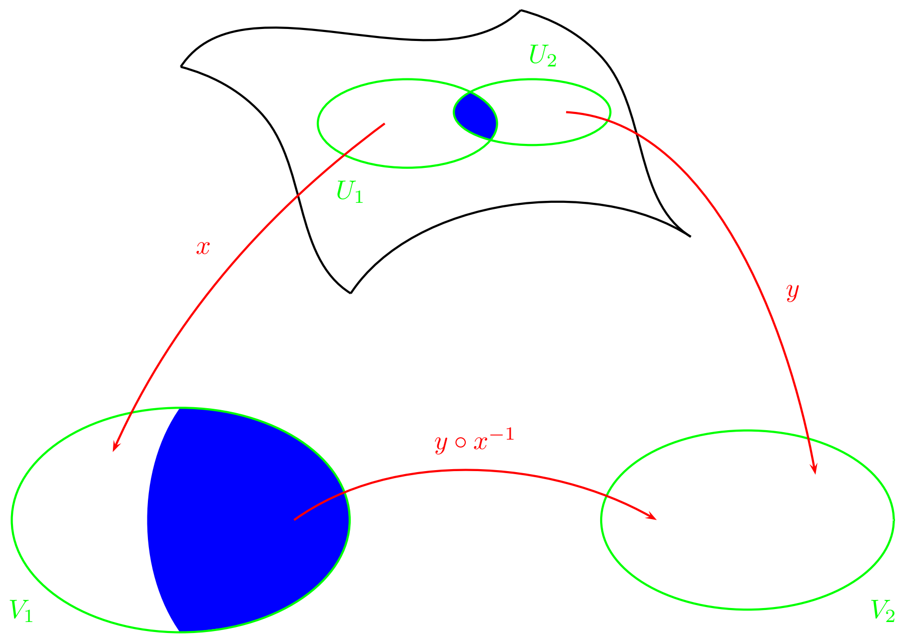

FIG. 5

DEFINITION 1.5. A collection $\mathcal{A} = \{x_\alpha : U_\alpha \to V_\alpha\}$ of charts of M is called a C^∞*-atlas* if and only if

$$\bigcup_\alpha U_\alpha = M$$

and all charts in $\mathcal{A}$ are mutually C^∞-compatible.

DEFINITION 1.6. A pair $(M, \mathcal{A})$ is called an n-dimensional *differentiable manifold* if and only if M is an n-dimensional topological manifold and $\mathcal{A}$ is a maximal C^∞-atlas of M.

Here maximality of the atlas means that if x is a chart of M which is C^∞-compatible with all charts in $\mathcal{A}$, then x itself also belongs to $\mathcal{A}$.

All examples of topological manifolds we have seen so far are in fact differentiable manifolds. In dimensions larger than 3 there exist topological manifolds which do not have a C^∞-atlas but we will not be concerned about them.

DEFINITION 1.7. A function $f : M \to \mathbb{R}$ is called *differentiable* (or C^k, $C^\infty, \dots$) if and only if

$$f \circ x^{-1} : V \longrightarrow \mathbb{R}$$

is differentiable (or C^k, $C^\infty, \dots$) for all charts $(x : U \to V) \in \mathcal{A}$.

More generally,

DEFINITION 1.8. Let $(M, \mathcal{A})$ and $(\widetilde{M}, \widetilde{\mathcal{A}})$ be two differentiable manifolds. A continuous map $\Phi : M \to \widetilde{M}$ is called *differentiable* (or C^k, $C^\infty, \dots$), if and only if

$$\widetilde{x} \circ \Phi \circ x^{-1} : x(U \cap \Phi^{-1}(\widetilde{U})) \longrightarrow \widetilde{V}$$

is differentiable (or C^k, $C^\infty, \dots$) for all charts $(x : U \to V) \in \mathcal{A}$ and all charts $(\widetilde{x} : \widetilde{U} \to \widetilde{V}) \in \widetilde{\mathcal{A}}$.

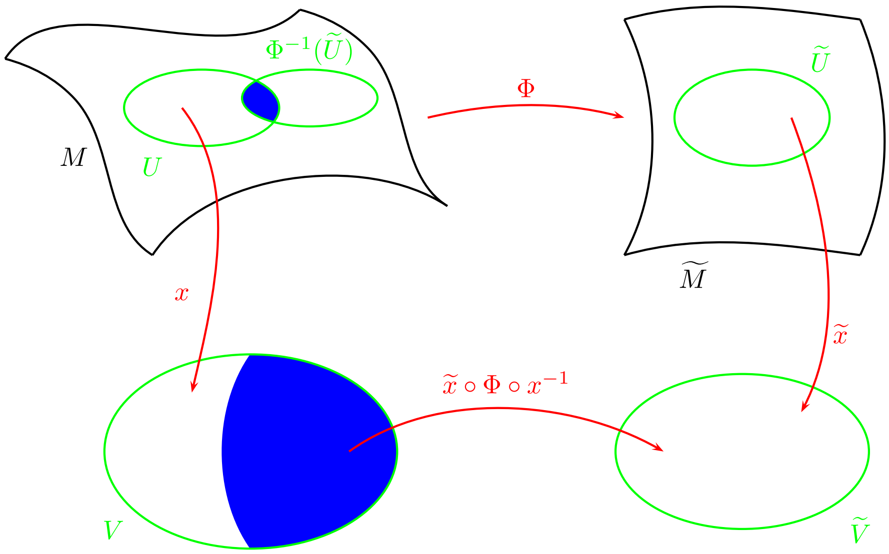

FIG. 6

Now that we know what differentiable maps are the next question arises:

QUESTION 1.9. *What is the derivative of a differentiable map?*

When dealing with differentiable maps between open subsets of Euclidean spaces many people think of the derivative as the Jacobi matrix containing the partial derivatives of the components of the function. This is certainly not a good point of view for generalizing the concept of derivative to maps defined between manifolds because it heavily uses the standard coordinates of $\mathbb{R}^n$. When working with manifolds we should think in a coordinate independent way. The right interpretation of the derivative is to consider it as the linear approximation of the map at a given point. But linear maps are defined between linear spaces (vector spaces) and manifolds are not vector spaces. So the question that has to be answered first is

QUESTION 1.10. *What is the linear approximation of a manifold?*

For the sake of simplicity we will from now on use the following

CONVENTION. From now on *manifold* stands for *differentiable manifold*. A *chart* will mean a chart in the corresponding C^∞-atlas $\mathcal{A}$.

DEFINITION 1.11. Let M be a manifold. Fix a point $p \in M$. Let $\gamma : I \to M$ and $\widetilde{\gamma} : \widetilde{I} \to M$ be two differentiable curves with

$$\gamma(0) = \widetilde{\gamma}(0) = p ,$$

where I, $\widetilde{I}$ are open intervals containing 0. We call the two curves *equivalent* and write $\gamma \sim \widetilde{\gamma}$ if and only if

$$\frac{d}{dt}(x \circ \gamma)\Big|_{t=0} = \frac{d}{dt}(x \circ \widetilde{\gamma})\Big|_{t=0}$$

for some chart $x : U \to V$ with $p \in U$.

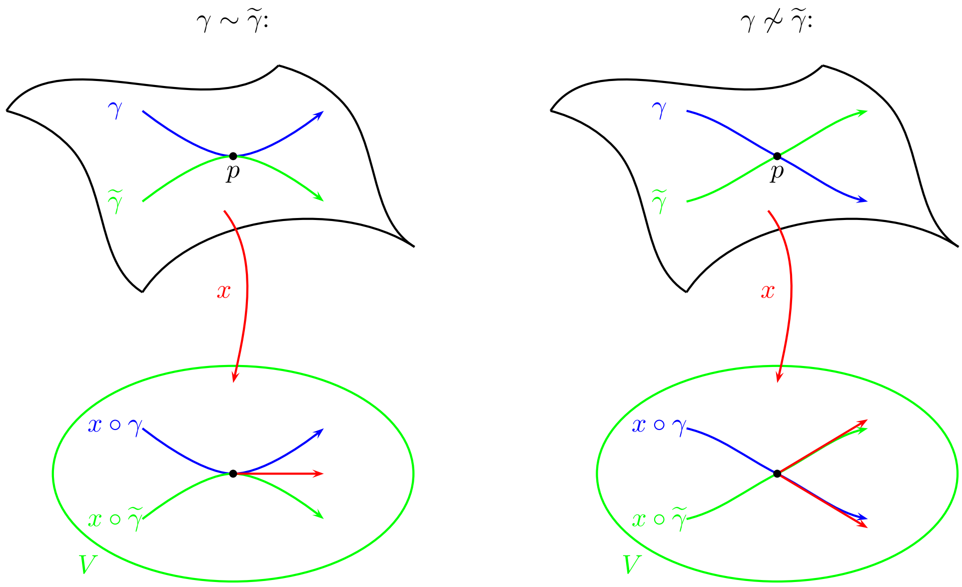

FIG. 7

Obviously this defines an equivalence relation on the set $\mathcal{C}_p$ of differentaible curves γ with $\gamma(0) = p$. We have to check that the definition of $\sim$ does not depend on the choice of chart x. If y is another chart containing p, then we get by the chain rule

$$\begin{aligned}\frac{d}{dt}(y \circ \gamma)\Big|_{t=0} &= \frac{d}{dt}((y \circ x^{-1}) \circ (x \circ \gamma))\Big|_{t=0} \\ &= D(y \circ x^{-1})(x(p))\,\frac{d}{dt}(x \circ \gamma)\Big|_{t=0}.\end{aligned}$$

Hence $\frac{d}{dt}(x \circ \gamma)\big|_{t=0}$ and $\frac{d}{dt}(x \circ \widetilde{\gamma})\big|_{t=0}$ coincide if and only if $\frac{d}{dt}(y \circ \gamma)\big|_{t=0}$ and $\frac{d}{dt}(y \circ \widetilde{\gamma})\big|_{t=0}$ coincide. Here $D(y \circ x^{-1})(x(p))$ denotes the Jacobi matrix of the diffeomorphism $y \circ x^{-1}$ at the point $x(p)$.

Denote the equivalence class of γ by $\dot{\gamma}(0)$. We think of $\dot{\gamma}(0)$ as being the velocity vector of γ at p. Now we are ready to define the linear approximation of a manifold at a given point p.

Definition 1.12. The set $T_pM := \{\dot{\gamma}(0) \mid \gamma \in \mathcal{C}_p\}$ is called the *tangent space* of M at p.

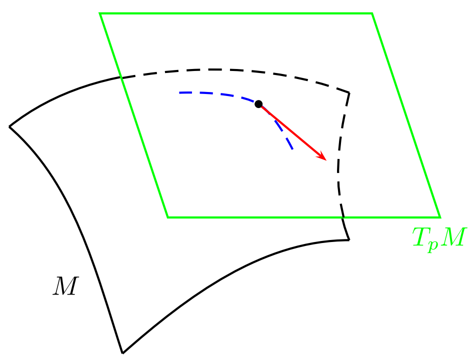

Fig. 8

In case the manifold happens to be a (finite dimensional real) vector space, then the tangent space of V at any point $p \in V$ can be *canonically* identified with V itself by the isomorphism

$$V \to T_pV, \quad v \mapsto \dot{\gamma}_{v,p}(0),$$

where $\gamma_{v,p}(t) = p + t \cdot v$.

Definition 1.13. For $X = \dot{\gamma}(0) \in T_pM$ and a differentiable function $f : M \to \mathbb{R}$ define the *directional derivative* of f in direction X by

$$\partial_X f := \frac{d}{dt}(f \circ \gamma)\Big|_{t=0}.$$

Properties.

1. $\partial_X f$ is well-defined, that is, it gives the same result for all $\gamma \in \mathcal{C}_p$ with $\dot{\gamma}(0) = X$.

2. ∂_X is local, i.e., if f and g coincide in a neighborhood of p, then $\partial_X f = \partial_X g$.
3. ∂_X is linear, i.e.,

$$\partial_X(\alpha \cdot f + \beta \cdot g) = \alpha \cdot \partial_X f + \beta \cdot \partial_X g$$

for all differentiable functions f and g and constants α and β.
4. ∂_X satisfies the Leibnitz rule,

$$\partial_X(f \cdot g) = \partial_X f \cdot g(p) + f(p) \cdot \partial_X g \,.$$

for all differentiable functions f and g.

An operator mapping differentiable functions defined near p to real numbers which satisfies properties 2., 3. and 4. is called a *derivation*. Hence ∂_X is a well-defined derivation. Write Der_p for the set of all derivations (at p).

FACTS.

1. The map $T_pM \to \mathrm{Der}_p$, $X \mapsto \partial_X$, is bijective.
2. For a chart $x : U \to V$ with $p \in U$ the derivations

$$\frac{\partial}{\partial x^i}(p) : f \longmapsto \frac{\partial f}{\partial x^i}(p) := \frac{\partial(f \circ x^{-1})}{\partial x^i}(x(p))$$

form a basis of the vector space Der_p, $i = 1, \ldots, n$, where n is the dimension of M.

COROLLARY 1.14. Der_p *is an n-dimensional vector space.*

Because of Fact 1 we can identify T_pM and Der_p which we will from now on do. This means that we will not distinguish between a tangent vector X and the corresponding derivation ∂_X. In particular, the tangent space T_pM inherits the structure of an n-dimensional vector space from Der_p. Very often in the literature T_pM is simply defined to be Der_p.

DEFINITION 1.15. Let $f : M \to N$ be a differentiable map. Fix $p \in M$. Then the *differential* of f at p is the linear map

$$df(p) : T_pM \to T_{f(p)}N, \quad \dot\gamma(0) \longmapsto \dot{\widehat{(f \circ \gamma)}}(0).$$

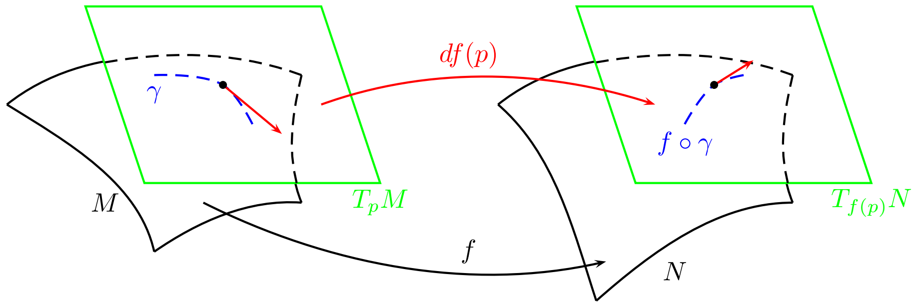

Fig. 9

Many other notations for the differential of a map are used in the literature, e. g. $df(p) = Df(p) = T_pf = f_{*,p}$.

Expression with respect to charts. Given a chart x on M and a chart y on N one easily computes

$$df(p)\left(\frac{\partial}{\partial x^i}(p)\right) = \sum_j \frac{\partial(y^j \circ f \circ x^{-1})}{\partial x^i}(x(p)) \cdot \frac{\partial}{\partial y^j}(f(p)) \, .$$

Hence with respect to the basis $\frac{\partial}{\partial x^i}(p)$ of T_pM and the basis $\frac{\partial}{\partial y^j}(f(p))$ of $T_{f(p)}N$ the linear map $df(p)$ is given by the Jacobi matrix of $y \circ f \circ x^{-1}$ at $x(p)$.

Definition 1.16. A *vector field* on M is a map X mapping each point $p \in M$ to a tangent vector $X(p) \in T_pM$.

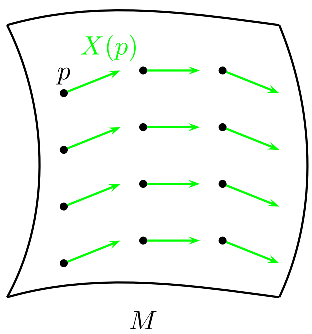

Fig. 10

Expression in charts. If $x : U \to V$ is a chart, then X can be written on U as

$$X(p) = \sum_{i=1}^{n} f_i(p) \cdot \frac{\partial}{\partial x^i}(p)$$

with uniquely determined functions

$$f_i : U \longrightarrow \mathbb{R} \,.$$

Definition 1.17. A vector field is called *continuous* (or differentiable, $C^k, C^\infty, \dots$) if and only if for each chart the corresponding functions $f_1, \dots, f_n$ are continuous (or differentiable, $C^k, C^\infty, \dots$).

After so many definitions it is time for a result.

Theorem 1.18 (Hedgehog Combing Theorem). *Every continuously combed hedgehog has at least one bald point.*

Perhaps we should give a more mathematical formulation of this theorem. The surface of the hedgehog is modelled by a (fairly deformed) 2-dimensional sphere. When combed the pricks of the hedgehog form a continuous vector field. A bald point is a point at which the vector field has a zero. Hence the mathematical way of expressing Theorem 1.18 is

Theorem 1.19. *Every continuous vector field on S^2 has at least one zero.*

In fact, this theorem is true for all even-dimensional spheres.

Exercise 1.20. *Find continuous vector fields on S^n, n odd, which vanish nowhere.*

Definition 1.21. A *k-covector* at p is a multilinear alternating map

$$\underbrace{T_pM \times \cdots \times T_pM}_{k \text{ factors}} \longrightarrow \mathbb{R}.$$

We write $\Lambda^k T_p^* M := \{k\text{-covectors at } p\}$.

Examples 1.22.
1. *By convention we put $\Lambda^0 T_p^* M := \mathbb{R}$.*
2. $\Lambda^k T_p^* M = 0$ *for* $k > n$.
3. $\Lambda^1 T_p^* M = (T_pM)^* =$ *dual space of* T_pM.

3'. *Let $f : M \to \mathbb{R}$ be differentiable. Then the differential of f at p, $T_pM \xrightarrow{df(p)} T_{f(p)}\mathbb{R} \cong \mathbb{R}$, is a linear map from tangent space to $\mathbb{R}$, hence we may consider $df(p)$ as a 1-covector, $df(p) \in \Lambda^1 T_p^* M$.*

For a chart x we know that $\frac{\partial}{\partial x^1}(p), \dots, \frac{\partial}{\partial x^n}(p)$ is basis of T_pM. Since the coordinate functions $x^1, \dots, x^n$ are smooth functions defined near p we can form the differentials $dx^1(p), \dots, dx^n(p) \in \Lambda^1 T_p^* M$. It turns out that they form the dual basis of $\Lambda^1 T_p^* M = (T_pM)^*$. Namely,

$$dx^i(p)\left(\frac{\partial}{\partial x^j}(p)\right) = \frac{\partial x^i}{\partial x^j}(p) = \delta^i_j \,.$$

Hence every 1-covector at p can be expressed in a unique way as a linear combination of the $dx^1(p), \ldots, dx^n(p)$. For the differential of a function one checks

$$df(p) = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}(p) \cdot dx^j(p) .$$

WEDGE PRODUCT. For 1-covectors $\varphi_1, \ldots, \varphi_k \in \Lambda^1 T_p^* M$ we define the k-covector $\varphi_1 \wedge \cdots \wedge \varphi_k \in \Lambda^k T_p^* M$ by

$$(\varphi_1 \wedge \cdots \wedge \varphi_k)(X_1, \ldots, X_k) := \det\left(\varphi_i(X_j)\right)_{i,j=1,\ldots,k} .$$

By the properties of the determinant we see that with this definition $\varphi_1 \wedge \cdots \wedge \varphi_k$ is indeed a k-covector, namely it is multilinear and alternating in the X_j. Moreover, it is also multilinear and alternating in the φ_j. It is a standard result of multilinear algebra [?] that suitable wedge products of the basis vectors for the space of 1-covectors yields a basis for the space of k-covectors. More precisely,

$$dx^{i_1}(p) \wedge \cdots \wedge dx^{i_k}(p),\ 1 \leq i_1 < \cdots < i_k \leq n,$$

form a basis of $\Lambda^k T_p^* M$.

Hence every k-covector ω at p can be written in a unique manner in the form

$$\omega = \sum_{i_1 < \cdots < i_k} \underbrace{f_{i_1 \cdots i_k}}_{\in \mathbb{R}} dx^{i_1}(p) \wedge \cdots \wedge dx^{i_k}(p) .$$

DEFINITION 1.23. A *differential form of order* k (briefly k*-form*) is a map ω sending each point $p \in M$ to a k-covector $\omega(p)$ at p, $\omega(p) \in \Lambda^k T_p^* M$.

EXAMPLES 1.24.
1. *0-forms = functions.*
2. *For a differentiable function f, its differential df is a 1-form.*

EXERCISE 1.25. *Show that for all $X \in T_p M$*

$$df(p)(X) = \partial_X f.$$

DEFINITION 1.26. A k-form is called *continuous* (or differentiable, $C^k, C^\infty, \ldots$) if and only if for all charts the corresponding functions $f_{i_1 \cdots i_k} : U \to \mathbb{R}$ are continuous (or differentiable, $C^k, C^\infty, \ldots$).

Write $\Omega^k(M)$ for the space of all smooth k-forms defined on M. Differentiating functions is a map

$$d : \Omega^0(M) \longrightarrow \Omega^1(M),\ f \mapsto df .$$

This has a generalization to forms of higher order

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M), \quad \omega \mapsto d\omega ,$$

defined with respect to a local chart x by

$$d\left(\sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) := \sum_{i_1 < \cdots < i_k} df_{i_1 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

One checks that this definition of d is independent of the choice of the chart x. The operator $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ is called *exterior differentiation.*

EXAMPLE 1.27. Let us look at the case $M = \mathbb{R}^2$. We work with the chart $x = \mathrm{Id} : \mathbb{R}^2 \to \mathbb{R}^2$. In other words, we work with the usual Cartesian coordinates. We compute the differential of the 1-form $\omega = x^1 x^2 dx^1$.

$$\begin{aligned} d\omega &= d\left(x^1 x^2 dx^1\right) = d\left(x^1 x^2\right) \wedge dx^1 \\ &= \left(x^1 dx^2 + x^2 dx^1\right) \wedge dx^1 \\ &= x^1 dx^2 \wedge dx^1 + x^2 dx^1 \wedge dx^1 \\ &= -x^1 dx^1 \wedge dx^2 + 0 \,. \end{aligned}$$

Exterior differentiation has an important property, its square is zero, $d^2 = 0$. Note that d^2 increases the order by two,

$$\Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \Omega^{k+2}(M) \,.$$

Property $d^2 = 0$ can also be expressed as $\mathrm{Im}(\Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M)) \subset \mathrm{Ker}(\Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M))$. This allows us to define *deRham cohomology* by taking the quotient space

$$H^k_{dR}(M) := \frac{\mathrm{Ker}(\Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M))}{\mathrm{Im}(\Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M))} \,.$$

For compact manifolds M these cohomology spaces turn out to be finite dimensional,

$$b_k(M) := \dim H^k_{dR}(M) < \infty \,.$$

The dimension $b_k(M)$ is called the k^{th} *Betti number* of M.

Let $\Phi : M \to N$ be a smooth map. The map Φ can be used to pull back functions defined on N to functions defined on M as follows: For $f : N \to \mathbb{R}$ put

$$\Phi^* f := f \circ \Phi : M \to \mathbb{R} \,.$$

This has a generalization to differential forms: Define the *pull back*

$$\Phi^* : \Omega^k(N) \to \Omega^k(M)$$

by

$$(\Phi^* \omega)\,(p)\,(\underbrace{X_1, \ldots, X_k}_{\in T_p M}) := \omega\,(\Phi\,(p))\,(\underbrace{d\Phi(p)(X_1), \ldots, d\Phi(p)(X_k)}_{\in T_{\Phi(p)} N}).$$

Pull back has the following properties:

1. Φ^* is linear.
2. $(\Phi \circ \Psi)^* = \Psi^* \circ \Phi^*$ and $(\mathrm{Id}_M)^* = \mathrm{Id}_{\Omega^k(M)}$.
3. The diagram

$$\begin{tikzcd} \Omega^k(N) \arrow[r, "\Phi^*"] \arrow[d, "d"] & \Omega^k(M) \arrow[d, "d"] \\ \Omega^{k+1}(N) \arrow[r, "\Phi^*"] & \Omega^{k+1}(M) \end{tikzcd}$$

commutes.

Corollary 1.28. Φ^* *induces well-defined linear map*

$$\Phi^\sharp : H^k_{dR}(N) \longrightarrow H^k_{dR}(M)$$

mapping the cohomology class of ω *to the cohomology class of* $\Phi^*(\omega)$*. This map has the properties*

$$(\Phi \circ \Psi)^\sharp = \Psi^\sharp \circ \Phi^\sharp \quad \text{and} \quad (\mathrm{Id}_M)^\sharp = \mathrm{Id}_{H^k_{dR}(M)}.$$

Corollary 1.29. *If* M *and* N *are diffeomorphic, then*

$$b_k\,(M) \,=\, b_k\,(N)\;.$$

Proof. Let $\Phi : M \to N$ be a diffeomorphism. Then $\Phi^\sharp \circ (\Phi^{-1})^\sharp = (\Phi^{-1} \circ \Phi)^\sharp = (\mathrm{Id}_M)^\sharp = \mathrm{Id}_{H^k_{dR}(M)}$ and similarly $(\Phi^{-1})^\sharp \circ \Phi^\sharp = \mathrm{Id}_{H^k_{dR}(N)}$. Therefore $\Phi^\sharp : H^k_{dR}(N) \to H^k_{dR}(M)$ is an isomorphism with inverse $(\Phi^{-1})^\sharp$. QED

Example 1.30. If M is a compact surface of genus g, then it is known that

$$b_0\,(M) \,=\, b_2\,(M) \,=\, 1\;, \quad b_1\,(M) \,=\, 2g\,.$$

Hence surfaces of different genera cannot be diffeomorphic.

Definition 1.31. A C^∞-atlas of M is called *oriented* if and only if for any two charts x and y

$$\det D\,(y \circ x^{-1}) > 0\;.$$

A pair $(M, \mathcal{A})$ where M is a topological manifold and $\mathcal{A}$ is a maximal oriented atlas is called an *oriented differentiable manifold*.

Let M be an n-dimensional oriented manifold, let $\omega \in \Omega^n\,(M)$ (with compact support). Then

$$\int_M \omega \in \mathbb{R}$$

can be defined using oriented charts $x : U \to V$ by

$$\int_U \omega \;:=\; \int_V (f \circ x^{-1})\, dx^1 \cdots dx^n\;,$$

where we have written

$$\omega \;=\; f \cdot dx^1 \wedge \cdots \wedge dx^n$$

and the right hand side $\int_V (f \circ x^{-1})\, dx^1 \cdots dx^n$ is the usual Lebesgue integral of a function defined on an open subset of $\mathbb{R}^n$.

In the zero dimensional case the underlying manifold is a discrete and countable set of points, $M = \{p_1, p_2, p_3, \ldots\}$. The orientation consists of attaching a sign $\varepsilon(p) = \pm 1$ to each point p. The integral of a function f is then given by

$$\int_M f = \sum_{p \in M} \varepsilon(p)\, f(p).$$

Theorem 1.32 (Stokes). *Let* M *be an* n*-dimensional compact oriented manifold with boundary* ∂M*. Let* $\omega \in \Omega^{n-1}\,(M)$*. Then*

$$\int_M d\omega \;=\; \int_{\partial M} \omega$$

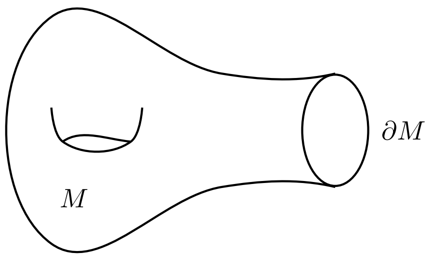

Fig. 11

Example 1.33. Let us look at the Stokes Theorem in the simplest case where M is a 1-dimensional interval, $M = [a, b]$. Then the boundary consists of two points, $\partial M = \{a, b\}$. We pick a smooth 0-form, i. e. a function $\omega = f : [a, b] \to \mathbb{R}$. Its differential then is $d\omega = f' \cdot dx$. The Stokes Theorem now says

$$\int_a^b f' \, dx \; = \; f\,(b) - f\,(a)\ .$$

Hence the Theorem of Stokes is a generalization of the fundamental theorem of calculus to higher dimensions.