Elliptic operators and representation theory of compact groups

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1. Introduction

In a famous series of papers [3] - [7] Atiyah and Singer solved the problem to compute the Fredholm index of elliptic operators on closed manifolds in terms of topological data. Applications have been numerous; for example applying the index formula to the Dirac operator on a closed spin manifold shows that on such a spin manifold the $\hat{A}$-genus is an integer. To understand this fact was part of the motivation to work out the index formula.

Now it is natural to ask how one can get interesting elliptic operators. So far operators have always been constructed 'by hand'. For example, on a Riemannian manifold we have the Euler operator $d+\delta$ mapping even forms to odd forms and vice versa, on an oriented Riemannian manifold of appropriate dimension we have the signature operator, on a Riemannian spin manifold there is the Dirac operator, and on an almost complex manifold there is the Cauchy-Riemann operator.

We see that the condition for existence of a certain operator is always a condition on the structure group of the manifold, for example being spin or being almost complex. Thus we suspect that there should be a conceptional way to determine elliptic operators associated to a given structure group. This is in fact possible and will be carried out in the next section. In some sense the natural elliptic operators associated to a structure group G are parametrized by a certain ideal in the representation ring of G which can easily be calculated in concrete situations. In particular, the ideal is finitely generated and the generators correspond to operators which we may well call *fundamental* for G.

2. The construction

Let us first set up the notation. Let M be a Riemannian manifold of dimension n. For the moment we do not make any assumption on compactness or even completeness. Let G be a compact Lie group, let $\mathcal{P} \to M$ be a principal bundle with structure group G. Moreover, we need an orthogonal representation $\tau : G \to O(n)$ and we assume that the associated vector bundle $\mathcal{P} \times_\tau \mathbb{R}^n$ is the tangent bundle of M. In such a situation we say that G is the *structure* *group* of M. For example, $G = Spin(n)$ means that the manifold is spin, $G = U(m), n = 2m$, means that the manifold is almost complex.

We emphasize that we do *not* assume that the holonomy group be reduced to G. In the case $G = U(m)$ this would mean that M is Kähler which is much stronger than being almost complex.

We call the G-structure on M *transitive* if G acts transitively on $S^{n-1} \subset \mathbb{R}^n$ via τ. The examples we have mentioned so far are all transitive. Now pick an arbitrary point $x_0 \in S^{n-1}$ and look at the isotropy subgroup $H = \{g \in G \mid \tau(g)x_0 = x_0\}$. Since G acts transitively H is unique up to conjugation. Now we can formulate the main existence result.

Theorem 1. *Let M, G, $\mathcal{P}$, and H be as above. Let V_1 and V_2 be two G-modules, let $E_i = \mathcal{P} \times_G V_i$ be the associated vector bundles.*

If V_1 and V_2 are equivalent as H-modules, then there exists an elliptic pseudodifferential operator $C^\infty(M, E_1) \to C^\infty(M, E_2)$ (of arbitrary order).

Sketch of proof. It is enough to construct an elliptic symbol. So if $\pi : T^1M \to M$ is the unit tangent bundle we need to construct a vector bundle isomorphism $\sigma : \pi^*E_1 \to \pi^*E_2$. This σ will be the asymptotic principal symbol; it can be extended to the whole tangent bundle minus the zero section by just extending it homogeneously of some arbitrary degree. This degree is then the order of the operator. At this point it should be mentioned that we use the Riemannian metric to identify tangent and cotangent bundle so that we don't need to make any difference here.

The construction of σ can be carried out pointwise over the manifold. Let us think of S^{n-1} as being the fibre of T^1M over some point. We want to have a natural function on S^{n-1} mapping each point of the sphere to an isomorphism $V_1 \to V_2$ since those are the fibres of E_1 and E_2 resp. At x_0 there is a natural choice, namely just take the H-isomorphism A which we have by assumption. Any other point on S^{n-1} may be written in the form $\tau(g)x_0$ and we take $\tau(g) \circ A \circ \tau(g^{-1})$. Since A is H-equivariant the whole mapping is well defined. □

Remark. In Theorem 1 we did not specify over which field we have to take the modules V_1 and V_2. In fact, we can use $\mathbb{R}, \mathbb{C}$ and even $\mathbb{H}$ and we get real, complex, or quaternionic operators. For simplicity, we will from now on restrict our attention to complex G-modules.

Example 1. Let M be a spin manifold of dimension $n = 2m$. Then $G = Spin(2m)$ and $H = Spin(2m-1)$. We choose $V_1 = \Sigma^+$ and $V_2 = \Sigma^-$ the positive and negative half-spinor representations of $Spin(2m)$. Restricted to H, Σ^+ and Σ^- both yield the spinor representation of $Spin(2m-1)$. Hence V_1

and V_2 are equivalent over H and we can apply Theorem 1. Thus there is an elliptic operator $C^\infty(M, \Sigma^+) \to C^\infty(M, \Sigma^-)$. Of course, we are not surprised because we know that there is the Dirac operator.

Example 2. Let M be an oriented 4-manifold. Then $G = SO(4)$ and $H = SO(3)$. We choose $V_1 = \Lambda^1$ and $V_2 = 1 + \Lambda^2_+$ where 1 is the trivial G-module, Λ^1 is the standard representation of $SO(4)$, and Λ^2_+ are self-dual 2-forms. Restricting $V_1 = \Lambda^1$ to $SO(3)$ yields a one-dimensional trivial subspace spanned by x_0 and the three-dimensional orthogonal complement which is just the standard representation Λ^1 for $SO(3)$. On the other hand, Λ^2_+ restricts to a nontrivial three-dimensional $SO(3)$-modul and there is only one, namely Λ^1. We have seen that V_1 and V_2 both restrict to $1 + \Lambda^1$, i.e. they are equivalent over $SO(3)$. By Theorem 1 there is an elliptic operator $C^\infty(M, \Lambda^1) \to C^\infty(M, 1 + \Lambda^2_+)$. In fact, we can choose half the Euler operator $d^+ + \delta$.

In both examples we have been able to choose elliptic *differential* operators. Is it possible to replace the word *pseudodifferential* operator in Theorem 1 by *differential* operator?

In the setting of Theorem 1 this is not possible in general. One can show that for the complex projective plane $M = \mathbb{CP}^2$ there is no elliptic differential operator $C^\infty(\mathbb{CP}^2, \mathbb{R} \oplus \mathbb{R} \oplus \Lambda^2_+ T^*\mathbb{CP}^2) \to C^\infty(\mathbb{CP}^2, \mathbb{R} \oplus T^*\mathbb{CP}^2)$ but applying Theorem 1 as in Example 2 one sees that there exists an elliptic pseudo-differential operator, see [8] for the details.

Algebraically it is much simpler to work with virtual G-modules rather than with actual G-modules. Let $R(G)$ be the representation ring (character ring) of G. The elements of $R(G)$ are virtual finite dimensional complex G-modules. If we look at the difference $V_1 - V_2$, then the condition for existence of elliptic operators in Theorem 1 is that under the restriction mapping $R(G) \to R(H)$ the virtual G-module $V_1 - V_2$ is mapped to 0. Hence every element of the kernel $R(G, H)$ of this restriction mapping gives rise to an elliptic operator.

In the following table we list generators of $R(G, H)$ for some geometrically significant groups G. We call the operators corresponding to generators *fundamental*.

G	n	gener. of $R(G,H)$	fundam. op.s
$SO(2m)$	$2m$	$1-\Lambda^1\pm\cdots-\Lambda^{2m-1}+1$, $\Lambda^m_+-\Lambda^m_-$	Euler op. signature op.
$O(2m)$	$2m$	$1-\Lambda^1\pm\cdots+\Lambda^{2m}$	Euler op.
$Spin(2m)$	$2m$	$\Sigma^+-\Sigma^-$	Dirac op.
$Spin^c(2m)$	$2m$	$(\Sigma^+-\Sigma^-)z$	twisted Dirac op.
$Spin^h(2m)$	$2m$	$1-\Lambda^1\pm\cdots-\Lambda^{2m-1}+1$, $\Lambda^m_+-\Lambda^m_-$, $(\Sigma^+-\Sigma^-)\rho$	Euler op. signature op. twisted Dirac op.
$U(m)$	$2m$	$1-\Lambda^{1,0}\pm\cdots+(-1)^m\Lambda^{m,0}$	Cauchy-Riemann op.
$SU(m)$	$2m$	$1-\Lambda^{1,0}\pm\cdots$ $+(-1)^{m-1}\Lambda^{m-1,0}+(-1)^m$ $=\Sigma^+-\Sigma^-$	Cauchy-Riemann op. $\cong$ Dirac op.
$Sp(q)Sp(1)$, q even	$4q$	$\Sigma^+-\Sigma^-$	Dirac op.
$Sp(q)Sp(1)$, q odd	$4q$	$(\Sigma^+-\Sigma^-)\cdot\rho$, $(\Sigma^+-\Sigma^-)\cdot\Lambda^{1,0}$, $(\Sigma^+-\Sigma^-)\cdot\Lambda^{3,0}$, $\vdots$ $(\Sigma^+-\Sigma^-)\cdot\Lambda^{q,0}$	$\frac{q+3}{2}$ twisted Dirac op.s
$Sp(q)U(1)$, q even	$4q$	$\Sigma^+-\Sigma^-$	Dirac op.
$Sp(q)U(1)$, q odd	$4q$	$(\Sigma^+-\Sigma^-)\cdot z$	twisted Dirac op.
$Sp(q)$	$4q$	$\Sigma^+-\Sigma^-$	Dirac op.

Notation should be self-explaining, Λ^k denotes k-forms, $\Lambda^m_\pm$ are (anti-) self-dual m-forms, $\Lambda^{p,q}$ denotes (p,q)-forms, $\Sigma^\pm$ are half-spinor representations, ρ is the canonical representation of $Sp(1)=SU(2)$ on $\mathbb{C}^2$, and z denotes the standard representation of $U(1)$ on $\mathbb{C}$. For background on representation theory see [9]. $Spin^h$ will be explained in the next section.

3. Spin^h manifolds

The class of Spin^c manifolds can be considered the natural class containing spin manifolds and almost complex manifolds. Now we enlarge the class once more and add almost quaternionic manifolds. What we get is the class of Spin^h manifolds.

A *Spin^h manifold* is an n-dimensional manifold with structure group $Spin^h(n)=(Spin(n)\times Sp(1))/\mathbb{Z}_2$ where $\mathbb{Z}_2=\{(1,1),(-1,-1)\}$. We just replace $U(1)$ in the definition of $Spin^c(n)$ by $Sp(1)$. The inclusion $U(1)\subset Sp(1)$

induces an inclusion $Spin^c(n) \subset Spin^h(n)$, hence all Spin^c manifolds are Spin^h. In dimension $n = 4q$ there is an inclusion $Sp(q) \cdot Sp(1) \subset Spin^h(n)$ which implies that every almost quaternionic manifold is Spin^h.

From the exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin^h(n) \longrightarrow SO(n) \times SO(3) \longrightarrow 0 \tag{1}$$

we see that a Spin^h manifold has a *canonical* $SO(3)$*-bundle* E. This canonical $SO(3)$-bundle is analogous to the canonical line bundle of Spin^c manifolds.

From the exact cohomology sequence

$$H^1(M; Spin^h(n)) \longrightarrow H^1(M; SO(n)) \oplus H^1(M; SO(3)) \xrightarrow{w_2} H^2(M; \mathbb{Z}_2) \tag{2}$$

we deduce that the condition on an $SO(3)$-bundle E to be canonical for a Spin^h-structure is

$$w_2(M) = w_2(E) \tag{3}$$

where w_2 denotes the second Stiefel-Whitney class. Applying the Atiyah-Singer index theorem to the third generator of $R(G, H)$, $G = Spin^h(n)$, in the table of the previous section, we obtain the following integrality theorem.

Theorem 2. *Let M be a closed Spin^h manifold of dimension $n = 2m$ with canonical $SO(3)$-bundle E. Let $p_1(E) \in H^4(M; \mathbb{Z})$ be the first Pontrjagin class of E.*

Then $p_1(E) \equiv w_2(M)^2$ mod 2 and the rational number

$$2 \int_M \left\{ cosh \left(\frac{\sqrt{p_1(E)}}{2} \right) \hat{A}(TM) \right\}$$

is an integer.□

Since *cosh* is an even power series $cosh\left(\frac{\sqrt{p_1(E)}}{2}\right)$ is in fact a power series in $p_1(E)$, not just in $\sqrt{p_1(E)}$.

Theorem 2 is analogous to well known integrality theorems for spin and for Spin^c manifolds due to Atiyah and Hirzebruch [1].

Remark. At the conference in Sendai I learnt from M. Nagase that he had studied Spin^h manifolds independently (he calls them Spin^q manifolds) including the construction of the twisted Dirac operator and the corresponding integrality theorem, see [11] and [12].

Corollary 1. *Let M be a closed Spin^h manifold of dimension $n = 2m$ with canonical $SO(3)$-bundle E. If the first Pontrjagin class $p_1(E)$ of E is a torsion* *class, then $2\hat{A}(M)$ is an integer.*$\square$

Corollary 2. *Let M be a closed Spin^h manifold of dimension $n = 2m$ with with vanishing forth Betti number, $b_4(M;\mathbb{Q}) = 0$. Then $2\hat{A}(M)$ is an integer.*$\square$

Example. Let us consider the $4q$-dimensional manifold $M = \underbrace{S^4 \times \cdots \times S^4}_{q \text{ factors}}$. Let E be any $SO(3)$-bundle over M.

Then the characteristic number $\int_M p_1(E)^q$ is divisible by $2^{2q-1} \cdot (2q)!$.

The proof is as follows. Since trivially $w_2(E) = 0 = w_2(M)$ we know that E is canonical for some Spin^h structure on M. Theorem 2 and $\hat{\mathcal{A}}(M) = \hat{\mathcal{A}}(S^4)^q = 1$ tells us that the following expression is an integer

$$\begin{aligned} 2\int_M cosh\left(\frac{\sqrt{p_1(E)}}{2}\right) &= 2\int_M \frac{(\sqrt{p_1(E)}/2)^{2q}}{(2q)!} \\ &= \frac{1}{2^{2q-1}\cdot(2q)!}\int_M p_1(E)^q . \square \end{aligned}$$

4. Immersions

To give another application of Theorem 1 we derive topological restrictions on closed manifolds with transitive G^1-structure immersed into a spin manifold such that the normal bundle carries a G^2-structure.

More precisely, let $G^1 \subset SO(n)$, $n = 2m$, and $G^2 \subset SO(k)$ be connected Lie subgroups such that G^1 acts transitively on $S^{n-1} \subset \mathbb{R}^n$. For $x_0 \in S^{n-1}$ denote the isotropy subgroup by $H^1 \subset G^1$. We look at the pre-images $\hat{G}^1 = \pi_1^{-1}(G^1)$, $\hat{H}^1 = \pi_1^{-1}(H^1)$, and $\hat{G}^2 = \pi_2^{-1}(G^2)$ under the twofold coverings $\pi_1 : Spin(n) \to SO(n)$ and $\pi_2 : Spin(k) \to SO(k)$. The two central elements $\pm 1 \in Spin(n \text{ or } k)$ are also contained in $\hat{G}^1$, $\hat{H}^1$, and $\hat{G}^2$.

Theorem 3. *Let M be an n-dimensional closed manifold with transitive G^1-structure, $n = 2m$ even. Let M be immersed into an $(n+k)$-dimensional spin manifold, e.g. $\mathbb{R}^{n+k}$, such that the normal bundle carries a G^2-structure. Let $\Phi_{TM} : M \to BG^1$ and $\Phi_N : M \to BG^2$ be classifying maps for tangent and normal bundle.*

If $\sigma \in R(\hat{G}^1, \hat{H}^1)$ and $V \in R(\hat{G}^2)$ are such that $(-1,-1)$ acts trivially on $\sigma \cdot V$, then

$$\int_M \left\{ \Phi_N^*\left((\pi_2^*)^{-1} ch(V)\right) \cdot \Phi_{TM}^*\left(\frac{(\pi_1^*)^{-1} ch(\sigma)}{e|BG^1}\right) \cdot \hat{\mathcal{A}}(TM)^2 \right\}$$

is an integer.

Proof. M carries a G^1-structure and hence a $G^1 \times G^2$-structure. Since M is immersed into a spin manifold the $G^1 \times G^2$-structure can be lifted to a G-structure where G is the preimage of $G^1 \times G^2 \subset SO(n+k)$ in $Spin(n+k)$.

We have $G = (G^1 \times G^2)/\mathbb{Z}_2$ where $\mathbb{Z}_2 = \{(1,1),(-1,-1)\}$. Since $\mathbb{Z}_2$ acts trivially on $\sigma \cdot V$ by assumption we can consider $\sigma \cdot V$ as an element of $R(G)$. Since $\sigma \in R(\hat{G}^1, \hat{H}^1)$ the element $\sigma \cdot V$ is actually contained in $R(G,H)$ where $H = (H^1 \times G^2)/\mathbb{Z}_2$ is the isotropy subgroup of G.

Theorem 1 together with the Atiyah-Singer index formula applied to the operator corresponding to $\sigma \cdot V$ finishes the proof. □

In the following tables we list a few examples. We can consider G^1 and G^2 separately and then combine them arbitrarily. V and σ are always such that $-1 \in \hat{G}^1$ or $\hat{G}^2$ acts by multiplication with -1 so that $\mathbb{Z}_2$ acts trivially on $\sigma \cdot V$.

G^1	σ	$\Phi^*_{TM}\left(\frac{(\pi_1^*)^{-1}ch(\sigma)}{e\lvert BG^1} \cdot \hat{\mathcal{A}}^2\right)$
$SO(n)$	$\Sigma^+ - \Sigma^-$	$\hat{\mathcal{A}}(TM)$
$U(m)$	$(1 - \Lambda^{1,0} \pm \cdots) \cdot (\Lambda^{m,0})^{1/2}$	$(-1)^m e^{c_1(TM)/2} \cdot \mathcal{TD}(TM)$
$Sp(q)Sp(1)$, q even	$(\Sigma^+ - \Sigma^-) \cdot \rho$	$2cosh(\sqrt{p_1(E)}/2) \cdot \hat{\mathcal{A}}(TM)$
	$(\Sigma^+ - \Sigma^-) \cdot \Lambda^{k,0}$, k odd	$\lambda^k\left(\frac{ch(TM\otimes\mathbb{C})}{2cosh(\sqrt{p_1(E)/2})}\right)\hat{\mathcal{A}}(TM)$
$Sp(q)Sp(1)$, q odd	$\Sigma^+ - \Sigma^-$	$\hat{\mathcal{A}}(TM)$
	$(\Sigma^+ - \Sigma^-) \cdot \Lambda^{k,0}$, k even	$\lambda^k\left(\frac{ch(TM\otimes\mathbb{C})}{2cosh(\sqrt{p_1(E)/2})}\right)\hat{\mathcal{A}}(TM)$
$Sp(q)U(1)$, q even	$(\Sigma^+ - \Sigma^-) \cdot z$	$e^{c_1(L)/2} \cdot \hat{\mathcal{A}}(TM)$
$Sp(q)U(1)$, q odd	$\Sigma^+ - \Sigma^-$	$\hat{\mathcal{A}}(TM)$

Some explanations: L denotes the canonical $U(1)$-bundle, E denotes the canonical $SO(3)$-bundle, c_1 is the first Chern class, p_1 the first Pontrjagin class, and $\mathcal{TD}$ the total Todd class. $\lambda^k : H^{\text{even}}(M;\mathbb{Q}) \to H^{\text{even}}(M;\mathbb{Q})$ is a homomorphism with the property

$$ch(\Lambda^k E) = \lambda^k ch(E),$$

see [8] for more details.

Now let us look at the structure group G^2 of the normal bundle. Here we could also take groups which do not act transitively on the unit sphere, but for the sake of simplicity we restrict ourselves to $G^2 = SO(k)$ and $G^2 = U(l)$.

G^2	V	$\Phi_N^*\left((\pi_2^*)^{-1}ch(V)\right)$
$SO(k)$, $k=2l$ even	$\Sigma^+ - \Sigma^-$	$e(N)\cdot\hat{\mathcal{A}}(N)^{-1}$
	$\Sigma^+ + \Sigma^-$	$2^l \cdot \mathcal{M}(N)$
$SO(k)$, $k=2l+1$ odd	Σ	$2^l \cdot \mathcal{M}(N)$
$U(l)$	$(\Lambda^{l,0})^{1/2}$	$e^{c_1(N)/2}$

Here $e(N)$ denotes the Euler class and $\mathcal{M}(N)$ is the multiplicative class for the power series $cosh(x/2)$, i.e. if we write the Pontrjagin class $p(N)$ formally as $p(N)=\prod_{j=1}^{l}(1+x_j^2)$, then

$$\mathcal{M}(N)=\prod_{j=1}^{l} cosh(x_j/2). \tag{4}$$

Combining $G^1=SO(n)$ and $G^2=SO(k)$ we get

Theorem 4. (K.H. Mayer [10, Satz 3.2])
Let M be an n-dimensional closed oriented manifold, $n=2m$ even, which can be immersed into an $(n+k)$-dimensional spin manifold with normal bundle N.
If $k=2l$ is even, then the following expressions are integers:

$$\int_M e(N)\hat{\mathcal{A}}(N)^{-1}\hat{\mathcal{A}}(TM)$$

and

$$2^l\int_M \mathcal{M}(N)\hat{\mathcal{A}}(TM).$$

If $k=2l+1$ is odd, then

$$2^l\int_M \mathcal{M}(N)\hat{\mathcal{A}}(TM)$$

is an integer. □

Of course, one can still twist with coefficient bundles or examine for which n and k the resulting elliptic operators are quaternionic thus improving the integrality result by a factor 2. Applications for immersions of projective spaces into Euclidian spaces may be found in [10].

As a further example let us combine $G^1=Sp(q)Sp(1)$, q even, with $G^2=U(l)$. Then we obtain

Theorem 5. *Let M be a $4q$-dimensional closed almost quaternionic manifold, q even, which can be immersed into a $(4q+2l)$-dimensional spin manifold* *such that the normal bundle N carries an almost complex structure. Let E denote the canonical $SO(3)$-bundle of M.*

Then the following expressions are integers:

$$2\int_M e^{c_1(N)/2} cosh(\sqrt{p_1(E)}/2)\hat{\mathcal{A}}(TM),$$

$$\int_M e^{c_1(N)/2}\lambda^k\left(\frac{ch(TM\otimes\mathbb{C})}{2cosh(\sqrt{p_1(E)}/2)}\right)\hat{\mathcal{A}}(TM), k \text{ odd.}\Box$$

Remark. With this method one can also study immersions into different manifolds such as Spin^c manifolds for example. Then one has to look at modules for the preimages of G^1 and G^2 in $Spin^c(n$ or $k)$ and one obtains in the case $G^1 = SO(n)$ and $G^2 = SO(k)$ the second integrality theorem by K. H. Mayer [10, Satz 3.1].

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