A Remark On Positively Curved 4-Manifolds

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Abstract

Let M be an oriented connected compact Riemannian 4-manifold. We show that if the sectional curvature satisfies $K \geq 1$ and the covariant differential of the curvature tensor satisfies $\|\nabla R\|_{L^\infty} \leq \frac{2}{\pi}$, then the intersection form of M is definite.

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1 Introduction

The relation between the topology and the geometry of Riemannian manifolds is currently an object of intense research in differential geometry. To see how hard this kind of questions can be, one should recall that even the following classical conjecture still remains unsolved:

Conjecture (H. Hopf).
$S^2 \times S^2$ does not admit a Riemannian metric of positive sectional curvature.

By Synge's Lemma one knows that on $RP^2 \times RP^2$ there is no Riemannian metric of positive sectional curvature; the only known examples of positively curved compact connected 4-manifolds are S^4, RP^4, and CP^2.

There have been various attempts to prove or disprove Hopf's conjecture; one was to start with the standard product metric (which is nonnegatively curved) and try to deform it to a positively curved metric, see [3] and [2]. Although one can make the curvature of mixed planes positive, there appear new planes of zero or even negative curvature; hence this method seems not to answer the question.

From the classical Sphere Theorem it is clear that $S^2 \times S^2$ cannot carry a metric with sectional curvature K satisfying $1 \leq K < 4$. By adapting a Bochner type argument by Berger, Bourguignon could show (see [1][p.351])

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Theorem (Berger, Bourguignon).

Let M be an oriented connected compact 4-manifold with indefinite intersection form (e.g. $M = S^2 \times S^2$). Then there is no Riemannian metric on M such that

$$1 \leq K \leq \frac{19}{4}.$$

It is not clear, however, how one could get rid of the upper curvature bound. Since the attempts to explicitely construct positively curved metrics on $S^2 \times S^2$ have also failed, Hopf's conjecture remains an open question. In the present paper we are not going to resolve this question either; roughly speaking, we show that if $S^2 \times S^2$ admits a positively curved metric, then this metric cannot be very symmetric. More precisely, if we denote by R the Riemannian curvature tensor and by ∇R its covariant differential, then the result is

Theorem. *Let M be an oriented connected compact 4-manifold with indefinite intersection form (e.g. $M = S^2 \times S^2$). Then there is no Riemannian metric on M such that*

(i) $K \geq 1$,

(ii) $\|\nabla R\|_{L^\infty} \leq \frac{2}{\pi}$.

In fact, those two known examples with $K \geq 1$ and $\nabla R = 0$, namely S^4 and CP^2, have definite intersection form.

2 The proof

In this section we give the proof of the Theorem up to some technical details which are carried out in the last section.

Let M be an oriented connected compact 4-manifold with a Riemannian metric such that

(i) $K \geq 1$,

(ii) $\|\nabla R\|_{L^\infty} \leq \frac{2}{\pi}$.

We want to show that $b_2^+ = 0$ or $b_2^- = 0$.

The Riemannian metric on the tangent bundle induces a Euclidean inner product on each tensor space characterized by the property that if $\{e_i\}_i$ is an orthonormal basis of T_pM, then $\{e_{i_1} \otimes \cdots \otimes e_{i_k}\}_{i_1 \cdots i_k}$ is orthonormal for the tensor space. We always work with the norm induced by these inner products.

The Laplace operator $\triangle$ acting on 2-forms commutes with the Hodge star operator $*$, hence it maps (anti-)self-dual forms into (anti-)self-dual forms. Let $\triangle^\pm$ be the restriction of $\triangle$ on sections of $\Lambda^\pm$. The decomposition $\Lambda^2 T^* M =$

$\Lambda^+ \oplus \Lambda^-$ is parallel, hence the rough Laplacian $\nabla^*\nabla$ also respects this splitting. We have the Weitzenböck formulas (compare Prop. 1)

$$\triangle^{\pm} = \nabla^*\nabla + \mathcal{K}^{\pm}.$$

At every point $p \in M$ let $\mu^{\pm}(p)$ be the smallest eigenvalue of $\mathcal{K}^{\pm}$; μ^+ and μ^- are continuous functions on M.

Case 1: $\mu^- > 0$ everywhere.

Let $\epsilon > 0$ be such that $\mu^- \geq \epsilon$ on M. By $\|\cdot\|$ we denote the L^2-norm and by $(\cdot,\cdot)$ the L^2-scalar product. If ω is a harmonic anti-self-dual 2-form, then

$$\begin{aligned} 0 &= (\triangle^-\omega,\omega) \\ &= (\nabla^*\nabla\omega,\omega) + (\mathcal{K}^-\omega,\omega) \\ &\geq \|\nabla\omega\|^2 + \int_M \mu^-|\omega|^2 \\ &\geq \epsilon\cdot\|\omega\|^2. \end{aligned}$$

Hence $\omega = 0$ and we have shown $b_2^- = 0$.

Case 2: There exists a point $p \in M$ such that $\mu^-(p) \leq 0$.

By Proposition 2 we know that $\mu^+ + \mu^- \geq 8$, hence $\mu^+(p) \geq 8$. Let $q \in M$ be arbitrary. From Grove's and Shiohama's Sphere Theorem, see [4], we know $l = d(p,q) \leq \frac{\pi}{2}$. Let $c : [0,l] \to M$, $c(0) = p$, $c(l) = q$, be a shortest geodesic from p to q. Let ω be an eigenvector of $\mathcal{K}^+(q)$ for the eigenvalue $\mu^+(q)$, $|\omega| = 1$. By parallel translation we get $\omega(t)$ along $c(t)$.

Using Proposition 3 we get

$$\begin{aligned} \mu^+(q) &= \langle\mathcal{K}^+(q)\cdot\omega,\omega\rangle \\ &= \langle\mathcal{K}^+(p)\cdot\omega,\omega\rangle - \int_0^l \frac{d}{dt}\langle\mathcal{K}^+(c(t))\cdot\omega(t),\omega(t)\rangle \\ &\geq \mu^+(p) - \int_0^l \langle\nabla_{\dot c(t)}\mathcal{K}^+\cdot\omega,\omega\rangle \\ &\geq 8 - l\cdot\frac{16}{\pi}. \end{aligned}$$

Hence $\mu^+ \geq 0$ and $\mu^+(q) = 0$ only if $d(p,q) = \frac{\pi}{2}$. Now an argument similar to that of Case 1 yields $b_2^+ = 0$.

3 The calculations

We keep the notations of the previous section. The following Weitzenböck formula is well known, see [1][p. 319 and p. 328].

Proposition 1. *In dimension 4 the Laplace operator acting on 2-forms decomposes into*

$$\triangle = \nabla^*\nabla + \mathcal{K}$$

where

$$\begin{aligned}\mathcal{K}(\omega)(X,Y) &= \sum_{i,j=1}^{4}(\frac{1}{2}Ric \bullet g - R)(e_i, e_j, X, Y)\omega(e_i, e_j) \\ &= \sum_{i,j=1}^{4}(\frac{1}{12}Sg \bullet g - W)(e_i, e_j, X, Y)\omega(e_i, e_j).\end{aligned}$$

Here W is the Weyl tensor, S the scalar curvature, g the Riemannian metric, $\bullet$ the Kulkarni-Nomizu product, and $e_1, \ldots, e_4$ an orthonormal basis.

Corollary. *If $K \geq 1$ and if $\omega \in \Lambda^2 T_p^* M$ is decomposable, then*

$$\langle \mathcal{K}\omega, \omega \rangle \geq 4 \cdot |\omega|^2.$$

Proof. We write $\omega = e_1 \wedge e_2$ where $e_1, \ldots, e_4$ is an orthonormal basis. By K_{ij} we denote the sectional curvature of the plane spanned by e_i and e_j.

$$\begin{aligned}\langle \mathcal{K}\omega, \omega \rangle &= Ric(e_1, e_1) + Ric(e_2, e_2) - 2K_{12} \\ &= K_{12} + K_{13} + K_{14} + K_{12} + K_{23} + K_{24} - 2K_{12} \\ &= K_{13} + K_{14} + K_{23} + K_{24} \\ &\geq 4.\end{aligned}$$

Elementary linear algebra yields

Lemma. *$\omega \in \Lambda^2 R^4$ is decomposable if and only if $\langle *\omega, \omega \rangle = 0$.*

Proposition 2. *If $K \geq 1$, then $\mu^+ + \mu^- \geq 8$.*

Proof. Let ϕ be an eigenvector of $\mathcal{K}^+$ for μ^+ and let ψ be an eigenvector of $\mathcal{K}^-$ for μ^-, $|\phi| = |\psi| = 1$. We set $\omega = \phi + \psi$. By the Lemma ω is decomposable because $\langle *\omega, \omega \rangle = |\phi|^2 - |\psi|^2 = 0$. Using the Corollary to Proposition 1 we obtain

$$\begin{aligned}8 &= 4 \cdot |\omega|^2 \\ &\leq \langle \mathcal{K}\omega, \omega \rangle \\ &= \langle \mathcal{K}^+\phi, \phi \rangle + \langle \mathcal{K}^-\psi, \psi \rangle \\ &= \mu^+ + \mu^-.\end{aligned}$$

Proposition 3. *Let $p \in M$ be fixed, $\omega \in \Lambda_p^+$, $X \in T_pM$, $|X| = 1$. Then*

$$|\langle(\nabla_X \mathcal{K})\omega, \omega\rangle| \leq \frac{16}{\pi}|\omega|^2.$$

Proof. W.l.o.g. we assume $|\omega|^2 = 2$. Choose an orthonormal basis $e_1, \ldots, e_4$ such that $\omega = e_1 \wedge e_2 + e_3 \wedge e_4$. We extend $e_1, \ldots, e_4$ to a local orthonormal frame such that $\nabla e_j(p) = 0$. Set $T := W - \frac{S}{12} g \bullet g$.

$$\begin{aligned}
|\langle(\nabla_X \mathcal{K})\omega, \omega\rangle| &= |\partial_X(\langle\mathcal{K}(e_1 \wedge e_2), e_1 \wedge e_2\rangle + 2\langle\mathcal{K}(e_1 \wedge e_2), e_3 \wedge e_4\rangle \\
&\quad +\langle\mathcal{K}(e_3 \wedge e_4), e_3 \wedge e_4\rangle)| \\
&= |\partial_X(2T(e_1, e_2, e_1, e_2) + 4T(e_1, e_2, e_3, e_4) \\
&\quad +2T(e_3, e_4, e_3, e_4))| \\
&= 2|(\nabla_X T)(e_1, e_2, e_1, e_2) + 2(\nabla_X T)(e_1, e_2, e_3, e_4) \\
&\quad +(\nabla_X T)(e_3, e_4, e_3, e_4)| \\
&\leq 8|\nabla_X T| \\
&= 4|\nabla_X T||\omega|^2.
\end{aligned}$$

The Riemannian curvature tensor R decomposes

$$R = W + \frac{1}{2} Ric_0 \bullet g + \frac{S}{24} g \bullet g$$

where Ric_0 is the traceless Ricci tensor. Since the corresponding decomposition of the curvature tensor bundle is parallel we get

$$|\nabla_X R|^2 = |\nabla_X W|^2 + |\frac{1}{2}\nabla_X(Ric_0 \bullet g)|^2 + |\frac{\partial_X S}{24} g \bullet g|^2$$

and

$$|\nabla_X T|^2 = |\nabla_X W|^2 + |\frac{\partial_X S}{12} g \bullet g|^2.$$

From $|\nabla R| \leq \frac{2}{\pi}$ we conclude $|\nabla_X T| \leq \frac{4}{\pi}$. Hence $|\langle(\nabla_X \mathcal{K})\omega, \omega\rangle| \leq \frac{16}{\pi}|\omega|^2$.

Remark. The proof shows that we don't really need a bound on the covariant differential of the whole curvature tensor in our Theorem. An L^∞-bound on ∇S and on ∇W^+ or ∇W^- is sufficient.

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