Heat Operator and ζ-Function Estimates for Surfaces

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Abstract

Using Kato's comparison principle for heat semi-groups we derive estimates for the trace of the heat operator on surfaces with variable curvature. This estimate is from above for positively curved surfaces of genus 0 and from below for genus $g \geq 2$. It is shown that the estimates are asymptotically sharp for small time and in the case of positive curvature also for large time. As a consequence we can estimate the corresponding ζ-function by the Riemann ζ-function.

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0 Introduction

In this paper we want to derive bounds on the Laplace spectrum of closed oriented surfaces. There is huge literature on estimates for particular eigenvalues λ_j, especially in the case of hyperbolic surfaces, see [4] and the references therein. We allow the curvature to vary and look for bounds on the spectrum as a whole. This can be done in terms of estimates on the trace of the heat operator, i.e. we try to bound

$$\mathbf{Tr}\ e^{-t\Delta} = \sum_j e^{-t\lambda_j}$$

by geometric data. Of course, if one has bounds on all eigenvalues, then one also has an estimate for $\mathbf{Tr}\ e^{-t\Delta}$. For example, Korevaar [8] shows for the Laplace eigenvalues on a closed oriented surface M of genus g

$$\lambda_j \leq C \frac{(g+1)j}{\text{area}(M)}$$

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where C denotes some (nonexplicit) universal constant. This done by covering arguments and constructing sufficiently many test functions for the Rayleigh quotient. As a consequence, we obtain

$$\begin{aligned}\sum_j e^{-t\lambda_j} &\geq \sum_j \left(e^{-\frac{tC(g+1)}{\mathrm{area}(M)}}\right)^j \\ &= \frac{1}{1-e^{-\frac{tC(g+1)}{\mathrm{area}(M)}}}\end{aligned}$$

In the case of a hyperbolic surface this yields

$$\sum_j e^{-t\lambda_j} \geq \frac{1}{1-e^{-\frac{tC(g+1)}{4\pi(g-1)}}}$$

A similar estimate has been obtained by Gromov [6] for Kähler manifolds using Kato's comparison principle to be explained later.

This is a beautiful estimate because it involves besides genus only the "soft" geometric invariant area(M) and no curvature assumption is made. On the other hand, no sharpness discussion can be made because of the nonexplicit constant C. Moreover, for large g the lower bound does not increase while $\mathbf{Tr}\ e^{-t\Delta}$ does as we shall see. We will show (Theorem 2.1)

$$\sum_j e^{-t\lambda_j} \geq \frac{g-1}{e^{-t\kappa}-1}$$

where κ is the infimum of Gauss curvature. This estimate has the disadvantage of involving curvature but it is asymptotically sharp for $t \searrow 0$ if $K \equiv \kappa$ (Theorem 2.2). We see that the lower bound increases with g.

Moreover, in the positively curved case $\kappa > 0$ we obtain a bound in the opposite direction

$$\sum_j e^{-t\lambda_j} \leq \frac{1}{1-e^{-t\kappa}}$$

which turns out to be asymptotically sharp for $t \searrow 0$ (and also for $t \nearrow \infty$) if the curvature is constant.

Our proof uses Kato's comparison principle but in a nonstandard way. Usually one looks for a bundle of rank r equipped with an operator L on the manifold such that one has a "Weitzenböck formula"

$$L = \nabla^*\nabla + \mathcal{K}$$

with $\mathcal{K} \geq \kappa$ and one knows (e.g. by index theory) that L has an N-dimensional kernel. Then Kato's principle immediately yields

$$N \leq \mathbf{Tr}\ e^{-tL} \leq r \cdot \mathbf{Tr}\ e^{-t(\Delta+\kappa)},$$

hence

$$\mathbf{Tr}\ e^{-t\Delta} \geq \frac{N}{r} \cdot e^{t\kappa}.$$

Gromov's version of Korevaar's estimate is also based on this approach using some suitable other eigenvalue of L instead of zero.

We will use Kato's inequality to compare $\mathbf{Tr}\ e^{-t\Delta}$ with $\mathbf{Tr}\ e^{-t\Delta_1}$ where Δ_1 denotes the Laplace-Beltrami operator acting on 1-forms. Then we observe that Δ and Δ_1 have essentially the same eigenvalues. This is already enough to derive our estimate.

Once one has a bound on the trace of the heat operator on can immediately obtain a bound on the associated ζ-function. Let ζ_R be the Riemann ζ-function. We will show for $\zeta_M(s) = \mathbf{Tr}\ (P_{(0,\infty)}\Delta)^{-s}$, $s > 1$ (Theorems 3.1 and 3.2)

a) If $K \geq 1$:

$$\zeta_M(s) \leq \zeta_R(s).$$

b) If $K \geq -1$ and $g = 2$:

$$\zeta_M(s) \geq \left(\frac{2}{\sqrt{e}} - 1\right) \cdot \left(\frac{1}{2^s\Gamma(s+1)} + \frac{\Gamma_{1/2}(s)}{\Gamma(s)}\zeta_R(s)\right).$$

c) If $K \geq -1$ and $g \geq 3$:

$$\zeta_M(s) \geq \left(\frac{g}{e} - 1\right) \cdot \left(\frac{1}{\Gamma(s+1)} + \frac{\Gamma_1(s)}{\Gamma(s)}\zeta_R(s)\right).$$

Here Γ_T denotes the truncated Γ-function, see Section 3. We conclude the article with a remark on higher dimensions, especially dimension 4.

1 Kato's Comparison Principle

Let M be a closed Riemannian manifold, "closed" meaning compact, connected and without boundary. Let $V \to M$ be a Riemannian vector bundle over M with metric connection ∇. Denote the rank of V by r. For a symmetric endomorphism field $\mathcal{K} \in C^\infty(M, \mathrm{End}(V))$ consider the self-adjoint operator $L = \nabla^*\nabla + \mathcal{K}$ acting as an unbounded operator on $L^2(M, V)$. Let $\kappa : M \to \mathbb{R}$ be a smooth function bounding $\mathcal{K}$ pointwise from below,

$$\mathcal{K} \geq \kappa.$$

Let Δ be the Laplace operator acting on functions. Kato's comparison principle tells us that the eigenvalues of L tend to be larger than those of $\Delta + \kappa$ on r copies of the trivial line bundle. Roughly speaking, the twisting of the bundle V increases the energy states of L as compared to the standard operator $\Delta + \kappa$ on the untwisted bundle of equal rank. More precisely, we have

Theorem 1.1. [7] *The traces of the heat operators satisfy the following estimate for all $t > 0$:*

$$\mathbf{Tr}\ e^{-tL} \leq r \cdot \mathbf{Tr}\ e^{-t(\Delta+\kappa)}.$$

Recall that the trace of the heat operator for a self-adjoint operator A bounded from below can be defined by

$$\mathbf{Tr}\ e^{-tA} = \sum_j e^{-t\lambda_j}$$

where the λ_j are the eigenvalues of A. In fact, one can estimate the heat kernel of L itself pointwise on $M \times M \times (0, \infty)$ by the one of $\Delta + \kappa$ but we will not need this stronger version of Theorem 1.1.

2 The Estimate for Surfaces

Now we restrict our attention to closed 2-dimensional Riemannian manifolds M. The estimate for $\mathbf{Tr}\ e^{-t\Delta}$ will follow from an application of Kato's principle to the Laplace operator on 1-forms together with the observation that the Laplace operator on 1-forms has essentially the same eigenvalues as the Laplace operator on functions.

Theorem 2.1. *Let M be a closed oriented 2-dimensional Riemannian manifold with Gauss curvature $K \geq \kappa, \kappa \in \mathbb{R}$. Let Δ be the Laplace operator acting on functions. Then the following estimates hold for all $t > 0$:*

If $\kappa > 0$:

$$\mathbf{Tr}\ e^{-t\Delta} \leq \frac{1}{1 - e^{-t\kappa}}$$

If $\kappa < 0$:

$$\mathbf{Tr}\ e^{-t\Delta} \geq \frac{g-1}{e^{-t\kappa} - 1}$$

where g denotes the genus of M.

Proof. Let $\lambda_0 = 0, \lambda_1, \lambda_2, \ldots$ denote the eigenvalues of Δ. The Laplace operator Δ_2 acting on 2-forms has the same eigenvalues as Δ because the Hodge star operator $*$ maps eigenfunctions into eigen-2-forms. The operator $d + \delta$: $\Omega^0(M) \oplus \Omega^2(M) \to \Omega^1(M)$ yields an isomorphism on the Laplace-eigenspaces for nonzero eigenvalues λ with inverse $\frac{1}{\lambda}(d + \delta)$. Hence each nonzero eigenvalue λ_j of Δ is also an eigenvalue for the Laplace operator Δ_1 acting on 1-forms with double multiplicity. Hodge theory tells us that the multiplicity of the eigenvalue zero for Δ_1 is given by the first Betti number $b_1(M) = 2g$. Thus Δ_1 has the spectrum

$$\underbrace{0, \ldots, 0}_{2g}, \lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots$$

Using the Bochner formula $\Delta_1 = \nabla^*\nabla + K$ we get from Kato's inequality

$$\begin{aligned}
\mathbf{Tr}\ e^{-t\Delta_1} &\leq 2 \cdot \mathbf{Tr}\ e^{-t(\Delta+\kappa)}, \\
2g + 2 \cdot \sum_{j=1}^{\infty} e^{-t\lambda_j} &\leq 2 \cdot \sum_{j=0}^{\infty} e^{-t(\lambda_j+\kappa)} \\
&= 2 \cdot e^{-t\kappa} \cdot \sum_{j=0}^{\infty} e^{-t\lambda_j}, \\
g - 1 &\leq (e^{-t\kappa} - 1) \cdot \sum_{j=0}^{\infty} e^{-t\lambda_j}.
\end{aligned}$$

If $\kappa > 0$, then necessarily $g = 0$ and since $e^{-t\kappa} - 1 < 0$ we obtain

$$\frac{-1}{e^{-t\kappa} - 1} \geq \sum_{j=0}^{\infty} e^{-t\lambda_j}.$$

If $\kappa < 0$ we have $e^{-t\kappa} - 1 > 0$ and we get

$$\frac{g-1}{e^{-t\kappa} - 1} \leq \sum_{j=0}^{\infty} e^{-t\lambda_j}.$$

This is a nontrivial estimate only if $g \geq 2$. □

Next we examine how sharp these bounds on $\mathbf{Tr}\ e^{-t\Delta}$ are. Let $A, B : (a, b) \to \mathbb{R}$ be two positive functions. We say that the estimate $A(t) \leq B(t)$ is *asymptotically sharp for* $t \searrow a$ if

$$\lim_{t \searrow a} \frac{B(t)}{A(t)} = 1,$$

similarly for $t \nearrow b$. Asymptotic sharpness of the estimates in Theorem 2.1 can of course only be expected if there is sharpness in the assumptions of the theorem, i.e. if the Gauss curvature is equal to κ, $K \equiv \kappa$.

Theorem 2.2. *Let M be a closed oriented 2-dimensional Riemannian manifold with constant Gauss curvature $K \equiv \kappa, \kappa \in \mathbb{R}$.*

If $\kappa > 0$, then the estimate of Theorem 2.1 is asymptotically sharp for $t \searrow 0$ and also for $t \nearrow \infty$.

If $\kappa < 0$, then the estimate of Theorem 2.1 is asymptotically sharp for $t \searrow 0$ but not for $t \nearrow \infty$.

Proof. We first examine the behavior for $t \nearrow \infty$. Since $e^{-t\Delta}$ converges to the orthogonal projection onto the kernel of Δ, i.e. the space of constant functions, we see that

$$\lim_{t \nearrow \infty} \mathbf{Tr}\ e^{-t\Delta} = 1.$$

The RHS of the estimate in Theorem 2.1 converges to 1 if $\kappa > 0$ and to 0 if $\kappa < 0$. This proves the assertion for $t \nearrow \infty$.

To study the behavior for $t \searrow 0$ we look at the asymptotic expansion of $\mathbf{Tr}\ e^{-t\Delta}$, see e.g. [5]

$$\mathbf{Tr}\ e^{-t\Delta} \sim (4\pi t)^{-1} \cdot (\text{area}(M) + \mathrm{O}(t)), \qquad t \searrow 0.$$

Hence

$$\lim_{t \searrow 0} t \cdot \mathbf{Tr}\ e^{-t\Delta} = \frac{\text{area}(M)}{4\pi}.$$

As for the RHS we get

$$\begin{aligned} \lim_{t \searrow 0} t \cdot \frac{g-1}{e^{-t\kappa} - 1} &= \frac{1-g}{\kappa} \\ &= \frac{\chi(M)}{2\kappa} \\ &= \frac{(2\pi)^{-1} \int_M K}{2\kappa} \\ &= \frac{\text{area}(M)}{4\pi}. \end{aligned}$$

Here we used the Gauss-Bonnet formula to express the Euler number $\chi(M)$ as the integral over the constant Gauss curvature $K = \kappa$.

This concludes the proof of Theorem 2.2. □

This proof actually shows that the estimates in Theorem 2.1 are asymptotically sharp for $t \searrow 0$ if and only if $K \equiv \kappa$.

3 Application to the ζ-Function

Let ζ_R denote the Riemann ζ-function,

$$\zeta_R(s) = \sum_{n=1}^{\infty} n^{-s}$$

for $s > 1$. Associated to the Laplace operator Δ on a closed Riemannian manifold M there is another ζ-function,

$$\begin{aligned} \zeta_M(s) &= \mathbf{Tr}\ (P_{(0,\infty)}\Delta)^{-s} \\ &= \sum_{j=1}^{\infty} \lambda_j^{-s} \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} (\mathbf{Tr}\ e^{-t\Delta} - 1) t^{s-1} dt \end{aligned}$$

where $P_{(0,\infty)}$ denotes the spectral projection onto the sum of the eigenspaces for the positive eigenvalues $\lambda_1, \lambda_2, \ldots$ of Δ and $\Gamma(s)$ is the usual Γ-function. See [2] or [9] for more details.

We study the two cases in Theorem 2.1 separately.

Theorem 3.1. *Let M be a closed oriented 2-dimensional Riemannian manifold with Gauss curvature $K \geq 1$. Then the following estimate holds for all $s > 1$:*

$$\zeta_M(s) \leq \zeta_R(s)$$

PROOF. The estimate in Theorem 2.1 tells us

$$\begin{aligned} \mathbf{Tr}\ e^{-t\Delta} - 1 &\leq \frac{1}{1-e^{-t}} - 1 \\ &= \sum_{n=1}^{\infty} e^{-nt}. \end{aligned}$$

Using this we get

$$\begin{aligned} \zeta_M(s) &= \frac{1}{\Gamma(s)} \int_0^\infty (\mathbf{Tr}\ e^{-t\Delta} - 1) t^{s-1} dt \\ &\leq \frac{1}{\Gamma(s)} \int_0^\infty \left(\sum_{n=1}^{\infty} e^{-nt} \right) t^s \frac{dt}{t} \\ &= \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \int_0^\infty e^{-u} \left(\frac{u}{n}\right)^s \frac{du}{u} \\ &= \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} n^{-s} \int_0^\infty e^{-u} u^{s-1} du \\ &= \zeta_R(s). \end{aligned}$$

This proves the theorem. □

In the case $K \equiv 1$ the Δ-eigenvalues are explicitly known [1], namely they are $\lambda_n = n(n+1)$ with multiplicity $2n+1$. Therefore Theorem 3.1 then simply says

$$\sum_{n=1}^{\infty} (2n+1)(n(n+1))^{-s} \leq \sum_{n=1}^{\infty} n^{-s}$$

for $s > 1$.

We define the *truncated* Γ*-function* for $T > 0, s > 1$ by

$$\Gamma_T(s) := \int_0^T e^{-t} t^{s-1} dt.$$

Now we can formulate an estimate for the ζ-function for surfaces of genus $g \geq 2$.

Theorem 3.2. *Let M be a closed oriented 2-dimensional Riemannian manifold with Gauss curvature $K \geq -1$. Then the following estimates hold for all $s > 1$:*

If $g = 2$, then

$$\zeta_M(s) \geq \left(\frac{2}{\sqrt{e}} - 1\right) \cdot \left(\frac{1}{2^s\Gamma(s+1)} + \frac{\Gamma_{1/2}(s)}{\Gamma(s)}\zeta_R(s)\right)$$

If $g \geq 3$, then

$$\zeta_M(s) \geq \left(\frac{g}{e} - 1\right) \cdot \left(\frac{1}{\Gamma(s+1)} + \frac{\Gamma_1(s)}{\Gamma(s)}\zeta_R(s)\right)$$

Proof. The estimate in Theorem 2.1 says

$$\begin{aligned} \mathbf{Tr}\ e^{-t\Delta} - 1 &\geq \frac{g-1}{e^t - 1} - 1 \\ &= (ge^{-t} - 1) \cdot \sum_{n=0}^{\infty} e^{-nt}. \end{aligned}$$

For $T \in (0, \ln g)$ we obtain

$$\begin{aligned} \zeta_M(s) &= \frac{1}{\Gamma(s)} \int_0^{\infty} (\mathbf{Tr}\ e^{-t\Delta} - 1) t^{s-1} dt \\ &\geq \frac{1}{\Gamma(s)} \int_0^{T} (\mathbf{Tr}\ e^{-t\Delta} - 1) t^{s-1} dt \\ &\geq \frac{1}{\Gamma(s)} \int_0^{T} (ge^{-t} - 1) \cdot \sum_{n=0}^{\infty} e^{-nt} \cdot t^{s-1} dt \\ &\geq \frac{ge^{-T} - 1}{\Gamma(s)} \left(\int_0^T t^{s-1} dt + \sum_{n=1}^{\infty} \int_0^T e^{-nt} \cdot t^s \frac{dt}{t}\right) \\ &= \frac{ge^{-T} - 1}{\Gamma(s)} \left(\frac{T^s}{s} + \sum_{n=1}^{\infty} \int_0^{nT} e^{-u} \cdot \left(\frac{u}{n}\right)^s \frac{du}{u}\right) \\ &= \frac{ge^{-T} - 1}{\Gamma(s)} \left(\frac{T^s}{s} + \sum_{n=1}^{\infty} n^{-s} \cdot \Gamma_{nT}(s)\right) \\ &\geq \frac{ge^{-T} - 1}{\Gamma(s)} \left(\frac{T^s}{s} + \zeta_R(s) \cdot \Gamma_T(s)\right) \\ &= (ge^{-T} - 1) \cdot \left(\frac{T^s}{\Gamma(s+1)} + \frac{\Gamma_T(s)}{\Gamma(s)} \cdot \zeta_R(s)\right). \end{aligned}$$

Choosing $T = \frac{1}{2}$ in case $g = 2$ and $T = 1$ in case $g \geq 3$ proves the theorem. □

4 Higher Dimensions

In dimension $n \geq 3$ the Laplace operator Δ_1 acting on 1-forms has also eigenvalues which are not related to those of Δ. This is the reason why Theorem 2.1 is special for surfaces. In higher dimensions Kato's principle can give us only estimates between traces of heat operators for different Laplace operators. For example, in dimension 4 we can show

Theorem 4.1. *Let M be a closed oriented 4-dimensional Riemannian manifold. Let $\rho \in \mathbb{R}$ be a lower bound for the Ricci curvature, $Ric \geq \rho$. Let $\chi(M)$ denote the Euler number of M. Then the following estimates hold for all $t > 0$:*

a)

$$\mathbf{Tr}\ e^{-t\Delta_1} \leq 4 \cdot e^{-t\rho} \cdot \mathbf{Tr}\ e^{-t\Delta}$$

b)

$$\mathbf{Tr}\ e^{-t\Delta_2} \leq 2 \cdot (4e^{-t\rho} - 1) \cdot \mathbf{Tr}\ e^{-t\Delta} + \chi(M).$$

Statement a) is a direct application of Kato's inequality and is true for all dimensions whereas b) has a proof similar to the one for Theorem 2.1 and is special for dimension 4.

Note that the limit $t \to \infty$ in Theorem 4.1 yields $b_1 = 0$ if $\rho > 0$ and $b_1 \leq 4$ if $\rho = 0$, the well-known Bochner theorem [3].

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