HARMONIC SPINORS AND TOPOLOGY

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Abstract. We discuss relations between the dimension of the solution space of the Dirac equation and the topology of the underlying manifold. It is shown that in certain dimensions existence of metrics with harmonic spinors is not topologically obstructed. In this respect the Dirac operator behaves very differently from the Laplace-Beltrami operator.
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1. Introduction

On a closed n-dimensional Riemannian manifold we have the *exterior differential* d mapping k-forms into $(k+1)$-forms, its L^2-adjoint δ mapping k-forms into $(k-1)$-forms, and the *Laplace-Beltrami operator* $\Delta = d\delta + \delta d$ respecting the degree of forms. The Laplace operator is an elliptic second order differential operator. By general elliptic theory it has a discrete eigenvalue spectrum. Since Δ is nonnegative so are the eigenvalues. If one changes the Riemannian metric the eigenvalues of Δ will change except for the eigenvalue zero. Classical Hodge and deRham theory tells us that the dimension $b^k(M)$ of the kernel of the Laplace operator acting on k-forms is a topological invariant, the k^{th} *Betti number*. Harmonic k-forms can be counted topologically! This constitutes a strong link between the analysis of the Laplace operator and the topology of the underlying manifold.

The question arises whether something similar is true for other natural differential operators like the Dirac operator acting on spinors on a

Riemannian spin manifold. Is there topological information stored in the dimension of the space of harmonic spinors?

Let us start with a few remarks on spinors and the Dirac operator. For details the reader may consult the excellent introductions [9] or [23]. In contrast to exterior form bundles the spinor bundle does not exist on every Riemannian manifold. The manifold has to satisfy a certain global topological condition, the *spin condition.* This can be thought of as a sharpening of orientability. Many well-known manifolds are spin, such as spheres, $\mathbb{CP}^m$ if m is odd, quaternionic projective spaces, orientable surfaces, and many more. But there are also familiar nonspin spaces, like nonorientable manifolds, $\mathbb{CP}^m$ if m is even, and others.

From now on let us assume that the manifold M under consideration is spin. If M is not simply connected there might be different spin structures. There are $\#H_1(M;\mathbb{Z}_2)$ many different ones just like there are $\#H_0(M;\mathbb{Z}_2)$ many different orientations. Pick one spin structure. Then we obtain the spinor bundle ΣM over M, a complex vector bundle of rank $2^{[n/2]}$. Sections in this bundle are called *spinor fields* or simply *spinors.* There is a pointwise algebraic action of tangent vectors on spinors, $T_pM \otimes \Sigma_p M \to \Sigma_p M, X \otimes \phi \to X \cdot \phi$, called *Clifford multiplication* satisfying certain relations. This can be thought of as similar to the exterior or interior product of forms.

The *Dirac operator* then acts on spinor fields by the formula

$$D\phi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\phi$$

where $e_1, \dots, e_n$ denotes a local orthonormal basis of TM and ∇ is a covariant derivative naturally induced by the Levi-Civita connection.

The Dirac operator is an elliptic first order differential operator. It hence has a discrete eigenvalue spectrum with eigenvalues tending to $+\infty$ and to $-\infty$. Spinor fields in the kernel of D are called *harmonic spinors.* Let us denote the dimension of the space of harmonic spinors by $h(M,g,S)$ where M denotes the differential manifold, g the Riemannian metric, and S the spin structure. Since M is closed the kernel of D is the same as that of D^2 which is a nonnegative elliptic second order differential operator just like the Laplace-Beltrami operator acting on forms. Our main question may now be rephrased as follows:

Does $h(M,g,S)$ really depend on g and/or S, does $h(M,g,S)$ tell us anything about the topology of M?

Hitchin showed [16, Prop. 1.3] that $h(M,g,S)$ is a conformal invariant, i.e. $h(M,g_1,S) = h(M,g_2,S)$ if g_1 and g_2 are conformally equivalent.

2. Atiyah-Singer index theorem

Since all Betti numbers $b^k(M)$ are topological invariants so is their alternating sum $\chi(M) = \sum_{k=0}^{n}(-1)^k b^k(M)$, the *Euler number.* This has the following interpretation. The Laplace-Beltrami operator Δ can be written as a square, $\Delta = (d+\delta)^2$. The elliptic first order operator $d+\delta$ does not respect the grading of the form bundle any more but it still respects the splitting into even and odd forms. Thus $\chi(M)$ is the dimension of the kernel of $d+\delta$ restricted to even forms minus the dimension of $d+\delta$ restricted to odd forms. Hence $\chi(M)$ is the Fredholm index of $d+\delta$ restricted to even forms.

This has an analog for the Dirac operator as follows. If the dimension n of M is even then the spinor bundle splits naturally into the so-called positive and negative *half-spinor bundles*, $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$. The Dirac operator interchanges these bundles, hence D^2 respects the splitting. Consequently, there is a splitting of the space of harmonic spinors, $h(M,g,S) = h^+(M,g,S) + h^-(M,g,S)$.

The Fredholm index of D restricted to positive half-spinors is then given by $h^+(M,g,S) - h^-(M,g,S)$ and can be expressed topologically.

Atiyah-Singer index theorem. [2, Thm. 5.3]
Let M be an even-dimensional closed Riemannian spin manifold. Then the Fredholm index of the Dirac operator restriced to positive half-spinors is given by

$$h^+(M,g,S) - h^-(M,g,S) = \hat{A}(M).$$

Here $\hat{A}(M)$ is the $\hat{A}$*-genus* of M, a topological invariant computable in terms of Pontryagin numbers.

Corollary. *The dimension of the kernel of D can be bounded from below by a topological invariant*

$$h(M,g,S) \geq |\hat{A}(M)|.$$

A remarkable application of this index theorem was found by Lichnerowicz. He proved the formula

$$D^2 = \nabla^*\nabla + \frac{s}{4}$$

where $s : M \to \mathbb{R}$ denotes scalar curvature. Hence if the scalar curvature is positive, $s > 0$, then D^2 is strictly positive and $h(M,g,S) = 0$. By the index theorem $\hat{A}(M) = 0$. Scalar curvature is a very weak geometric invariant and it is not possible to prove this topological obstruction against positive scalar curvature without use of harmonic spinors. Nonspin manifolds can carry positive scalar curvature metrics and still have $\hat{A}(M) \neq 0$, e.g. $M = \mathbb{CP}^2$! See the end of this paper for further discussion of scalar curvature.

The above corollary gives a nontrivial estimate only if n is divisible by 4 because otherwise always $\hat{A}(M) = 0$. In certain dimensions there is a refinement of the index theorem [4] using Milnor's *α-genus* [26]:

$$\begin{aligned} &\text{If } n \equiv 1 \bmod 8: \quad h(M,g,S) = \alpha(M,S) \mod 2, \\ &\text{if } n \equiv 2 \bmod 8: \quad h^+(M,g,S) = \alpha(M,S) \mod 2. \end{aligned}$$

The α-genus is a subtle invariant. For $n \equiv 1$ or $2 \bmod 8$ it takes values in $\mathbb{Z}_2$ rather than in $\mathbb{Z}$ and it depends on the *differential* topology of M. There are exotic spheres with nonvanishing α-genus which proves that these spheres do not carry metrics of positive scalar curvature.

3. Surfaces

What can we say about $h(M,g,S)$ in case $n = \dim(M) = 2$? Let us start with the case $\text{genus}(M) = 0$, i.e. M is topologically a 2-sphere. One has the following eigenvalue estimate for the Dirac operator.

Theorem (Bär [5, Thm. 2]). *Let M be a closed surface of genus 0. Then all eigenvalues λ of the Dirac operator on M satisfy*

$$\lambda^2 \geq \frac{4\pi}{\text{area}(M)}.$$

Equality holds for the eigenvalue of smallest absolute value if and only if M carries a metric of constant curvature.

In particular, λ is never zero, i.e. $h(S^2,g,S) = 0$ for all Riemannian metrics g. The spin structure is unique in this case because S^2 is simply connected.

The conclusion $h(S^2,g,S) = 0$ can also be deduced from conformal invariance of $h(S^2,g,S)$ and the fact that all metrics on S^2 are conformally equivalent. Hence $h(S^2,g,S)$ cannot depend on g. Since metrics of positive scalar curvature don't admit harmonic spinors we conclude for the canonical metric g_0 on S^2 of constant (positive) curvature that $h(S^2,g_0,S) = 0$.

The 2-torus has four different spin structures one of which is trivial (biinvariant). Friedrich [11] computed the Dirac spectrum for flat metrics on T^2 for all four spin structures. Since every metric on T^2 is conformally equivalent to a flat metric one concludes from this computation

$$h(T^2, g, S) = \begin{cases} 2, & \text{if } S \text{ is trivial} \\ 0, & \text{otherwise.} \end{cases}$$

Hence $h(T^2, g, S)$ depends on S but not on g.

The case genus = 2 turns out to be similar to the torus case, $h(M, g, S)$ depends on the spin structure S but not on the metric g, [16, Prop. 2.3]. But if the genus of M is larger than 2, then $h(M, g, S)$ depends on both the spin structure and the metric [16, Thm. 2.6]. The number $h(M, g, S)$ can be bounded from above in terms of the genus [16, Rem. 4]:

$$h(M, g, S) \leq 2 \cdot \left\lceil \frac{\text{genus}(M) + 1}{2} \right\rceil .$$

This estimate is sharp. For hyperelliptic metrics g on M one can compute $h(M, g, S)$ for all spin structures [8, Thm. 3 and 4].

The discussion of surfaces shows that unlike Betti numbers $h(M, g, S)$ does in general depend on the metric. There is topological information contained in $h(M, g, S)$ however. The 2-sphere is characterised among surfaces by the fact that $h(M, g, S) = 0$ for all metrics g and all spin structures S. But this may simply reflect the fact that on S^2 all metrics are conformally equivalent. Since S^2 is the only closed manifold with this property we are led to the

Conjecture. *Let M be a closed Riemannian spin manifold of dimension $n \geq 3$. Let S be a spin structure on M. Then there exists a metric g on M such that there are nontrivial harmonic spinors,*

$$h(M, g, S) > 0.$$

In other words, we believe that harmonic spinors are not topologically obstructed in dimension $n \geq 3$. We will see in the next two sections that the conjecture has been proven for $n \equiv 0, 1, 3, 7 \bmod 8$ while it is still open in the remaining cases.

4. The topological approach

Even though Hitchin did not explicitly state the conjecture in this generality he proved it in certain dimensions [16] using topological methods which we now describe. Let M be a closed spin manifold. To obtain a Riemannian metric with nontrivial harmonic spinors we proceed in two steps. First show

Step 1. *If M can be put as a fiber into a fiber bundle of spin manifolds, $M \to Z \to B$, such that the total space has nontrivial α-genus, $\alpha(Z) \neq 0$, then there is a Riemannian metric g on M such that $h(M, g, S) > 0$.*

Namely, assume we have such a fiber bundle of spin manifolds, $M \to Z \to B$. Put any metric on the total space Z. Restriction to the fibers gives us for every $b \in B$ a metric on M.

Assume that for all these metrics there are no nontrivial harmonic spinors. This means that the Dirac operator along the fibers is invertible for every $b \in B$. Hence the family index of this family of operators is trivial [3]. One can compute $\alpha(Z)$ in terms of the family index. In particular, if the family index vanishes so does $\alpha(Z)$, a contradiction.

Hence for some fiber the Dirac operator is not invertible. The metric of Z restricted to this fiber does the job.

The question now is how to find such fiber bundles $M \to Z \to B$. This is

Step 2. *For a closed n-dimensional spin manifold M there exists a fiber bundle of spin manifolds*

$$\begin{aligned} M \to Z \to S^1 \quad &\text{if} \quad n \equiv 0, 1 \bmod 8, \\ M \to Z \to S^2 \quad &\text{if} \quad n \equiv 0, 7 \bmod 8, \end{aligned}$$

for which $\alpha(Z) \neq 0$.

To construct Z start with the trivial fiber bundle $M \to M \times S^i \to S^i$. Of course, this is not good enough because $\alpha(M \times S^i) = \alpha(M) \cdot \alpha(S^i) = \alpha(M) \cdot 0 = 0$. The basic idea is now to pick an exotic sphere Σ^{n+i} with $\alpha(\Sigma^{n+i}) \neq 0$ and to put $Z = (M \times S^i) \# \Sigma^{n+i}$. Then $\alpha(Z) = \alpha(M \times S^i) + \alpha(\Sigma^{n+i}) = \alpha(\Sigma^{n+i}) \neq 0$.

The problem is that we cannot take every exotic sphere Σ^{n+i} because we have to make sure that Z still fibers over S^i with fiber M. Taking the connected sum with an exotic sphere is the same as removing a ball D^{n+i} and gluing it back via some diffeomorphism of the boundary sphere S^{n+i-1}. In order not to loose the fiber bundle structure we must take a diffeomorphism which, roughly speaking, only twists in the vertical direction. In differential topological terms this means that the exotic sphere must be in the image of a suitable *Novikov map*, in a certain *Gromoll group*.

Now one has to consult results from differential topology to see for which n and for which choices of i there are exotic spheres Σ^{n+i} in these Gromoll groups satisfying $\alpha(\Sigma^{n+i}) \neq 0$.

Combining steps 1 and 2 yields

Theorem (Hitchin [16, Thm. 4.5(1)]).
Let M^n be a closed spin manifold of dimension n, $n \equiv 0, 1, 7 \bmod 8$. Let a spin structure S on M be fixed. Then there exists a Riemannian metric g on M such that the corresponding Dirac operator has a nontrivial kernel, i.e.

$$h(M, g, S) > 0.$$

5. The analytic approach

Now we describe a different approach to prove the conjecture which will work in dimension $n \equiv 3 \bmod 4$. Details can be found in [6, 7].

Step 1 (Gluing Theorem, [7, Thm. 2.1]).
Let M_1 and M_2 be n-dimensional closed Riemannian spin manifolds of dimension $n \geq 3$. Let $U_i \subset M_i$ be open balls, let D_i be the Dirac operators of M_i. Let $\Lambda > 0$ such that $\pm\Lambda \notin spec\,(D_1) \cup spec\,(D_2)$. Let $\epsilon > 0$.

Then there exists a Riemannian metric on $X = M_1 \# M_2$ such that X is a disjoint union $X = X_1 \dot\cup X_2 \dot\cup X_3$ where

(i) *X_1 is isometric to $M_1 - U_1$,*
(ii) *X_2 is isometric to $M_2 - U_2$,*
(iii) *X_3 is diffeomorphic to $(0, 1) \times S^{n-1}$,*

and such that all eigenvalues of the Dirac operator D of M in the range $[-\Lambda, \Lambda]$ are ϵ-close to eigenvalues of D_1 or D_2 and vice versa.

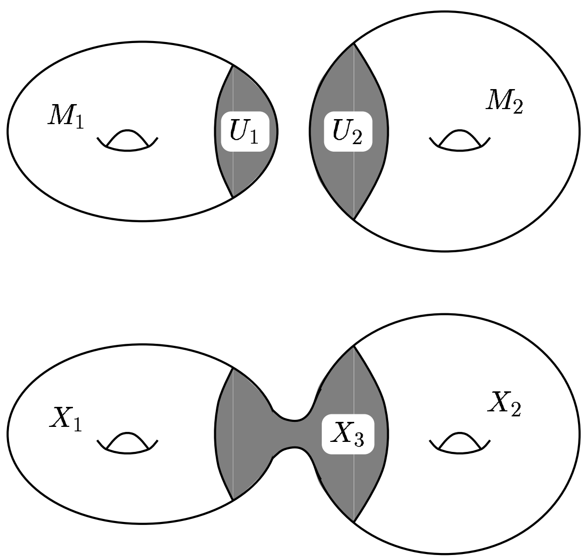

Fig. 1

In other words, up to a prescribed error ϵ, the spectrum of D on X in the range $[-\Lambda, \Lambda]$ is the same as the disjoint union of the spectra of D_1 and D_2 in this range. The proof uses the variational characterisation of eigenvalues. To compare test-spinors on X with those on $M_1 \dot{\cup} M_2$ one has to use cut-off functions. These cut-off functions introduce bad error terms in the Rayleigh quotient. One has to show that these error terms are over-compensated by smallness of the eigenspinors in the support of the gradient of the cut-off function. This requires certain a-priori estimates on the distribution of the L^2-norm of eigenspinors on manifolds of the type $(0,1) \times S^{n-1}$ with a suitable warped-product metric.

Step 2 (Computation of the Dirac spectrum of Berger spheres).
The *Hopf fibration* $S^{2m+1} \to \mathbb{CP}^m$ is a Riemannian submersion if one equips S^{2m+1} with its standard metric of constant curvature 1 and $\mathbb{CP}^m$ with the Fubini-Study metric. The fibers are circles S^1. Now one can rescale the length of the fibers by some positive constant T and keep fixed the metric on the orthogonal complement to the fibers. This yields a one-parameter family of metrics g_T on S^{2m+1}, called *Berger metrics.*

It is important that all Berger metrics are homogeneous under the unitary group $U(m+1)$. Hence one can apply methods from harmonic analysis to explicitly compute the Dirac spectrum of (S^{2m+1}, g_T). The formulas are given in [6, Thm. 3.1]. For our purposes only the following conclusion is of importance.

For $n \equiv 3$ mod 4 there is a smooth family g_T of Riemannian metrics on S^n, $T \in [a,b]$, such that

(i) There is $\lambda(T) \in spec\,(D_T)$ where D_T is the corresponding Dirac operator on (S^n, g_T) with $\lambda(a) = -1$, $\lambda(b) = +1$ (or vice versa).
(ii) $\lambda(T)$ depends smoothly (actually linearly) on T.
(iii) The multiplicity k of $\lambda(T)$ is constant in T.
(iv) $\lambda(T)$ is the only eigenvalue of D_T in the range $[-2,2]$.

Steps 1 and 2 yield

Theorem (Bär [6, Thm. A]).
Let M^n be a closed spin manifold of dimension n, $n \equiv 3$ mod 4. Let a spin structure S on M be fixed. Then there exists a Riemannian metric g on M such that the corresponding Dirac operator has a nontrivial kernel, i.e.

$$h(M, g, S) > 0.$$

Proof. Pick any metric on M. If there are no nontrivial harmonic spinors for this metric rescale it such that all Dirac eigenvalues become very large, greater than 10 say.

By Steps 1 and 2 there exist Riemannian metrics $\tilde{g}_T$ on $M^n \# S^n$ such that

(i) $\operatorname{spec}(\tilde{D}_T) \cap \left[-\frac{3}{2}, \frac{3}{2}\right] = \{\mu_1(T) \leq \ldots \leq \mu_k(T)\}$,
(ii) $|\mu_i(T) - \lambda(T)| < \epsilon = \frac{1}{2}$, $i = 1, \ldots, k$.

Here $\tilde{D}_T$ is the Dirac operator of $\tilde{g}_T$. In particular, $\mu_i(a) < 0$ and $\mu_i(b) > 0$, $i = 1, \ldots, k$.

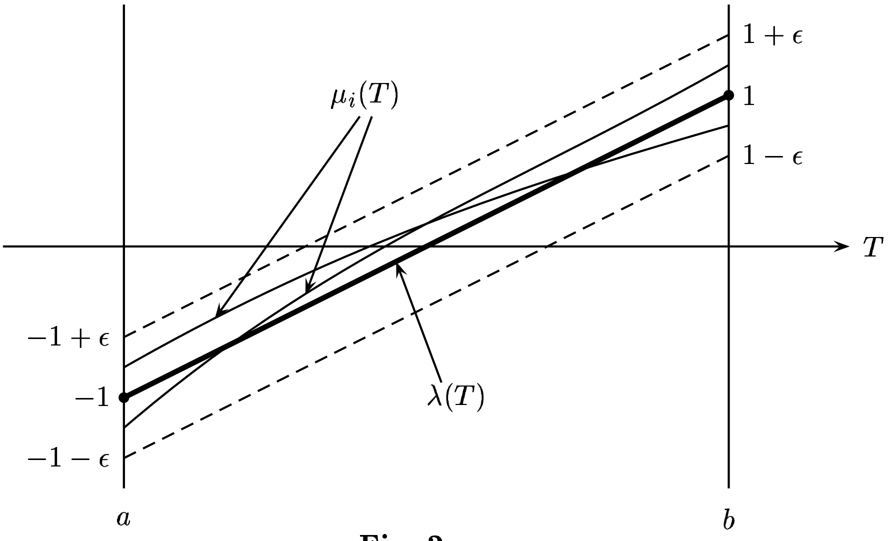

Fig. 2

For some value $T = T_0$ one of the $\mu_i(T)$'s must be zero. Hence $\tilde{g}_{T_0}$ is a metric on $M^n \# S^n$ with harmonic spinors. But of course, $M^n \# S^n = M^n$ and we are done.

Remarks. The topological approach together with the analytic appraoch prove the conjecture in dimension $n \equiv 0, 1, 3, 7 \bmod 8$. Can one extend the analytic approach to the remaining dimensions?

The gluing theorem (Step 1) makes no problems, it holds for $n \geq 3$. The problem is to find a one-parameter family of metrics on S^n such that one Dirac eigenvalue crosses 0. The Berger metrics do the job only for $n \equiv 3 \bmod 4$. This is a serious problem because if $n \not\equiv 3 \bmod 4$, then the Dirac spectrum is automatically symmetric about 0 [1]. Even if we can find a family of metrics on S^n for which one Dirac eigenvalue crosses the zero line another eigenvalue must cross in the opposite direction. Then the eigenvalues of $M \# S^n = M$ which are close to them need not cross the zero line.

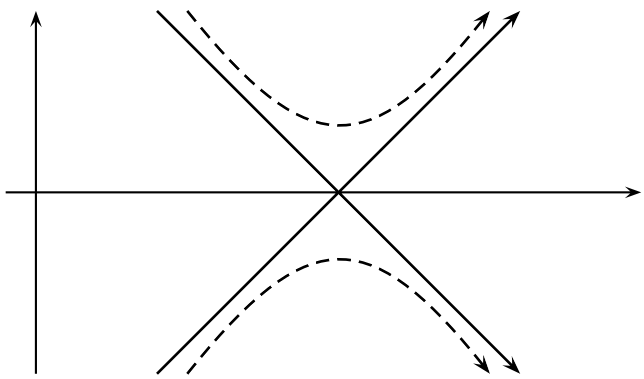

Fig. 3

The analytic approach has the advantage that it is essentially local. It can therefore easily be adapted to twisted versions of the Dirac operator. For example, one can show

Theorem (Bär [7]).
Let M^n be a closed spin^c manifold of dimension n, $n \equiv 3 \bmod 4$. Let a spin^c structure S on M be fixed. Let a $U(1)$-connection A on the canonical line bundle be fixed. Then there exists a Riemannian metric g on M such that the corresponding Dirac operator D_A has a nontrivial kernel, i.e.

$$h(M, g, A, S) > 0.$$

Compare the subsection on Seiberg-Witten theory below for the notion of spin^c manifolds and their Dirac operators.

6. Further aspects of harmonic spinors

Generic Metrics. Our conjecture, which we have seen to be true in many dimensions, tells us that for specific choices of the Riemannian metric there are nontrivial harmonic spinors. On the other hand, all examples which one can explicitly compute, like the Berger metrics on odd-dimensional spheres, indicate that for *generic metrics* the number of linearly independent harmonic spinors is minimal in the sense that there are not more than there must be by the index theorems. This has recently proven to be true in low dimensions at least.

Theorem (Maier [25]).
Let M be a closed spin manifold of dimension n with fixed spin structure S. For generic metrics g on M we have

$$h(M,g,S) = \begin{cases} 0 \text{ or } 2, & \text{depending on } \alpha(M,S), \text{ if } n=2, \\ 0, & \text{if } n=3, \\ |\hat{A}(M)|, & \text{if } n=4. \end{cases}$$

Kähler Metrics. As we have seen our conjecture on existence of metrics with nontrivial harmonic spinors is not true in dimension 2. One possible explanation for this special behavior of surfaces could be the fact that oriented surfaces are automatically Kähler. Does restriction to the class of Kähler metrics really change things?

Hitchin [16] studied the case of complex dimension 2. He showed that $h(M,g,S)$ is minimal (in the sense above) for simply connected algebraic spin surfaces not of general type, for complete intersections, for rational surfaces, and for cyclic ramified coverings over $\mathbb{CP}^2$ branched over a non-singular curve. Here g always denotes a Kähler metric compatible with the given complex structure.

Tempted by these examples Hitchin conjectured that $h(M,g,S)$ might be minimal for generic complex structures on simply connected algebraic spin surfaces.

But Kotschick [20, 21] gave counterexamples to this conjecture. He showed that there exist simply connected algebraic surfaces such that for generic complex structures $h(M,g,S)$ still exceeds the minimal number of linearly independent harmonic spinors (enforced by the index theorems) arbitrarily much.

Positive Scalar Curvature. The scalar curvature function of a Riemannian manifold is a very weak geometric invariant. It is known that every function f on an n-dimensional closed manifold, $n \geq 3$, which is negative somewhere, is the scalar curvature function for some Riemannian metric on M [18, 19]. In other words, if the scalar curvature s is negative somewhere, then it contains no topological information at all.

But from Lichnerowicz's formula [24]

$$D^2 = \nabla^*\nabla + \frac{s}{4}$$

it follows that if the scalar curvature is positive, then D^2 is a strictly positive operator. Hence, $h(M,g,S) = 0$. In particular, if n is divisible by 4, then $\hat{A}(M) = 0$. We see that nonvanishing of the $\hat{A}$-genus is a topological obstruction against existence of a metric of positive scalar curvature. A similar remark holds for the α-genus in general dimensions.

The remarkable fact is that for simply connected manifolds this is the only obstruction. Combining surgery results obtained independently by Gromov/Lawson and Schoen/Yau with homotopy theoretic work by Stolz one obtains

Theorem (Gromov-Lawson [13], Schoen-Yau [30], Stolz [32, 33]). *Let M be a simply connected closed manifold of dimension $n \geq 5$. Then the following holds.*

(i) If M is not spin, then there is a metric of positive scalar curvature on M.
(ii) If M is spin, then there is a metric of positive scalar curvature on M if and only if the α-genus vanishes.

The nonsimply connected case is still a topic of active research [12]. The corresponding conjecture is known as *Gromov-Lawson-Rosenberg conjecture.* See [28] or [34] for a survey, see also [14] for the noncompact case. Very recently, the Gromov-Lawson-Rosenberg conjecture in its original (unstable) form has been shown to fail in dimension 5,6, and 7 [29].

In the case of zero scalar curvature, $s \equiv 0$, there can be nontrivial harmonic spinors, $h(M, g, S)$ can be positive. But then, again by Lichnerowicz's formula $D^2 = \nabla^*\nabla$, every harmonic spinor must be parallel. This means that the holonomy group of the manifold must have fixpoints under the spinor representation. A holonomy reduction is a very strong restriction on the manifold. Since all possible holonomy groups of Riemannian manifolds are classified one can do a case by case check to see which holonomy groups can occur, see [16, 35, 36].

Seiberg-Witten Theory. The physicists Seiberg and Witten [31] introduced equations which led recently to spectacular results in differential topology of 4-manifolds. It seems that most theorems proved by Donaldson's instanton theory such as his theorem on smooth 4-manifolds with definite intersection form [17] can also be proved using Seiberg-Witten theory, only in a simpler way. Moreover, there have been new important applications such as a proof of Thom's conjecture on the minimal genus of an embedded surface representing a given homology class in $\mathbb{CP}^2$ [22].

To set up the Seiberg-Witten equations one first has to relax the spin condition and replace it by the spin^c condition. The spin^c condition has the advantage of automatically being fulfilled on oriented closed 4-manifolds [15]. Then one can still form the spinor bundle but the definition of the Dirac operator requires the choice of an additional piece of data, a connection A on a certain $U(1)$-bundle. Let us denote the resulting Dirac operator by D_A.

The Seiberg-Witten equations are equations on a closed 4-manifold M for the pair (ϕ, A) where ϕ is a positive spinor field and A is the $U(1)$-connection mentioned above. The first equation is simply the harmonic spinor equation for ϕ with respect to D_A:

$$D_A\phi = 0. \tag{1}$$

Denote the curvature of A by F_A and its self-dual part by F_A^+. By Clifford multiplication one can identify exterior forms with endomorphisms of the spinor space. Taking a suitable part of this identification map yields a canonical map $\sigma : \Sigma^+ \otimes \Sigma^+ \to \Lambda^2_+$. The second equation is then

$$F_A^+ = i\sigma(\phi \otimes \phi). \tag{2}$$

The solution space of these two equations is naturally acted upon by the gauge group $\mathrm{Map}(M, S^1)$. Dividing out this group action yields the Seiberg-Witten moduli space. Topological invariants of this moduli space are important invariants for the differential structure of M. In the simplest case the moduli space is just a finite set. Counting points with the right sign yields the celebrated Seiberg-Witten invariants. See [10] for a survey or [27] for a detailed introduction.

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