Elliptic Symbols

Christian Bär *

December, 1995

Abstract

If G is the structure group of a manifold M it is shown how a certain ideal in the character ring of G corresponds to the set of geometric elliptic operators on M. This provides a simple method to construct these operators. For classical structure groups like $G = O(n)$ (Riemannian manifolds), $G = SO(n)$ (oriented Riemannian manifolds), $G = U(m)$ (almost complex manifolds), $G = Spin(n)$ (spin manifolds), or $G = Spin^c(n)$ (spin^c manifolds) this yields well known classical operators like the Euler-deRham operator, signature operator, Cauchy-Riemann operator, or the Dirac operator. For some less well studied structure groups like $Spin^h(n)$ or $Sp(q)Sp(1)$ we can determine the corresponding operators.

As applications, we obtain integrality results for such manifolds by applying the Atiyah-Singer Index Theorem to these operators. Finally, we explain how immersions yield interesting structure groups to which one can apply this method. This yields lower bounds on the codimension of immersions in terms of topological data of the manifolds involved.

AMS Subject Classification: 58G30, 57R15

Introduction

Manifolds with different kinds of geometric structure have been studied for a long time. In particular, one has found certain associated elliptic differential operators on these manifolds. For example, on any Riemannian manifold there is the *Euler-deRham operator* $d + \delta$ acting on differential forms. If in addition, the manifold is oriented and even dimensional, then there also exists the *signature operator*, also acting on forms. If the manifold is almost complex, then there is

*Partially supported by SFB 256 and by the GADGET program of the EU

the *Cauchy-Riemann operator* $\partial + \partial^*$ and if the manifold is spin, we can talk about the *Dirac operator* acting on spinors.

These operators have been found over the time and they are not only interesting in themselves but they also had numerous applications to the study of the topology and the geometry of these manifolds. For example, for a long time it was not clear under what conditions the classical definition of the Dirac operator on $\mathbb{R}^4$ could be generalized to manifolds. The discovery of the Dirac operator and the Atiyah-Singer Index Theorem finally explained the integrality of the $\hat{\mathcal{A}}$-genus of closed spin manifolds.

If we want to study other geometric structures on manifolds, like e.g. almost quaternionic structures, it would obviously be desirable to have a conceptional way to find the corresponding elliptic operators. Looking at the conditions on the manifolds for the classical operators to exist we notice that it is always a condition on the structure group of the manifold.

We say that G is a structure group of the manifold M if the tangent bundle of M is induced by a G-principal bundle. This is a topological concept and is not to be confused with the much stronger notion of holonomy group. Being Riemannian can be expressed by saying that the structure group is $G = O(n)$, being oriented Riemannian by $G = SO(n)$, being almost complex by $G = U(m)$, and being spin by $G = Spin(n)$. We get the suspicion that one should be able to read off the elliptic operators on a manifold with structure group G directly from G.

In fact, we will see that the set of elliptic operators corresponds to an ideal $R(G, H)$ in the character ring $R(G)$ of G. For this to be true we need the condition that G act transitively on the unit sphere via the representation which induces the tangent bundle. For all groups mentioned above this condition is satisfied. The ideal $R(G, H)$ is the kernel of a restriction map $R(G) \to R(H)$ for a suitable subgroup $H \subset G$ and can easily be computed for concrete G. It is not such a big surprise that in the classical cases the classical operators essentially correspond to generators of $R(G, H)$.

To get something new from this construction we look at several less well studied structure groups. We determine the elliptic operators on almost quaternionic manifolds ($G = Sp(q)Sp(1)$) and on spin^h manifolds ($G = Spin^h(n)$).

Of course, there are potentially many applications of these operators. In this paper we focus on integrality results. By applying the Atiyah-Singer Index Theorem to these operators we can express their Fredholm index in topological data. In particular, the resulting characteristic number must be integral.

As an application we show that the projective Cayley plane does not admit an almost quaternionic structure. Non-existence of an almost complex structure was shown by Borel and Hirzebruch in [11].

Spin^h manifolds are in some sense the quaternionic analogue to spin^c manifolds. They constitute a very large class of manifolds. In particular, all spin^c manifolds (hence all spin manifolds and all almost complex manifolds) and all almost quaternionic manifolds are spin^h. We will see that any spin^h manifold possesses a twistor space which turns out to be a spin^c manifold.

In the last part we look at applications to immersion problems. An immersion of a manifold into a spin manifold (like Euclidean space) yields a reduction of the structure group. Applying our results to these reductions yields integrality of certain expressions in characteristic numbers of the manifold and the normal bundle. This can be used to derive lower bounds on the codimension of the immersion. We explain how this method can be adapted to different situations where one imposes additional structure on the manifold and/or on the normal bundle. One can also weaken the spin condition on the target manifold to e.g. the spin^c condition.

For all the applications it is very helpful that one no longer needs a direct construction of the relevant operators in every situation but that one has a general and simple method for their determination.

Contents

I. Elliptic symbols

We show how to construct elliptic symbols and hence elliptic operators on a manifold using the representation theory of the structure group. The proof of Theorem 1 is surprisingly simple and follows directly from an easy reduction lemma.

Then we reformulate this construction recipe in K-theoretical language. This makes explicit computations easier and helps to study the question whether we get all elliptic operators this way (up to deformations and K-theoretically trivial manipulations).

It turns out that the answer is affirmative for those groups for which a certain representation theoretical invariant, the surjectivity exponent, vanishes. This includes for example $G = Spin(n)$ and $G = U(m)$, i.e. spin manifolds and almost complex manifolds. In general, one cannot expect this because the structure group may not be optimally chosen. But we will see that up to a suitable power of 2 the construction method yields all elliptic operators.

There are many applications of elliptic operators to geometry and topology. We formulate the integrality theorem obtained by applying the Atiyah-Singer Index Theorem.

Finally, we look at the basic example of spin manifolds and, of course, it turns out that the Dirac operator is the fundamental elliptic operator in this case.

1. The construction

We begin by constructing elliptic symbols in terms of the representation theory of the structure group. To make this more precise let us first set up the notation.

Let X be an n-dimensional differentiable manifold, let G be a compact Lie group, let $\mathcal{P}$ be a G-principal bundle over X. Furthermore, let $\tau : G \to O(n)$ be an orthogonal representation of G. We assume that the associated vector bundle equals the cotangent bundle of X, i.e. $\mathcal{P} \times_\tau \mathbb{R}^n = T^*X$. In this situation we say that G is the *structure group* of X or that X has a *G-structure*. Since the representation τ is orthogonal X inherits a Riemannian metric.

Crucial assumption. We assume that G act transitively on $S^{n-1} \subset \mathbb{R}^n$ via τ.

Let $x_0 \in S^{n-1}$ and let $H \subset G$ be its isotropy subgroup, i.e. $H = \{g \in G \mid \tau(g)x_0 = x_0\}$. Since the action of G on S^{n-1} is transitive H is independent of the choice of x_0 up to conjugation. We say that X has a *transitive structure group G with isotropy subgroup H.*

Let $\pi : T^*X \to X$ denote the projection of the cotangent bundle, let π_1 be its restriction to the unit sphere bundle $T_1^*X \subset T^*X$.

Theorem 1. *Let X be an n-dimensional differentiable manifold with transitive structure group G and isotropy subgroup H. Let V_1 and V_2 be G-modules over the field $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Let E_i denote the associated vector bundles, i.e. $E_i = \mathcal{P} \times_G V_i$.*

If V_1 and V_2 are equivalent as H-modules, then there exists an elliptic symbol between the bundles E_1 and E_2. More precisely, there exists a $\mathbb{K}$-linear vector bundle isomorphism $\sigma : \pi_1^*E_1 \to \pi_1^*E_2$.

Proof. Let $\mathcal{P}$ denote the G-principal bundle over X. By Lemma 1 below $\pi_1^*\mathcal{P}$ can be reduced to an H-principal bundle $\hat{\mathcal{P}}$ over T_1^*X. Since V_1 and V_2 are H-equivalent there is an isomorphism $\sigma : \pi_1^*E_1 \to \pi_1^*E_2$.

We can extend this isomorphism homogeneously in radial directions to $T_1^*X -$ {zero section} by $\sigma_{t\xi} := t^k \cdot \sigma_\xi$ where $\xi \in T_1^*X$, $t > 0$, and $k \in \mathbb{R}$ is an arbitrarily chosen degree. This yields the desired elliptic symbol. □

Lemma 1 (Reduction Lemma). *Let X be a topological space, let G be a compact Lie group, let $\mathcal{P}$ be a G-principal bundle over X. Furthermore, let F be a topological space on which G acts transitively. Let $f_0 \in F$ and let $H = \{g \in G \mid gf_0 = f_0\}$. Let $\pi : \mathcal{P} \times_G F \to X$ be the associated fiber bundle.*

Then the structure group of $\pi^*\mathcal{P}$ can be reduced to H.

Proof. The associated fiber bundle $\mathcal{P} \times_G F$ is the set of equivalence classes $[b, f]$ of pairs (b, f), $b \in \mathcal{P}$, $f \in F$ under the equivalence relation $(b, f) \sim (bg, g^{-1}f)$, $g \in G$. The pull-back $\pi^*\mathcal{P}$ is given by

$$\begin{aligned} \pi^*\mathcal{P} &= \{([b, f], b') \mid b, b' \in \mathcal{P} \text{ having the same base point in } X, f \in F\} \\ &= \{([b, f], bg) \mid b \in \mathcal{P}, f \in F, g \in G\} \\ &= \{([b, f_0], bg) \mid b \in \mathcal{P}, g \in G\}. \end{aligned}$$

The reduction $\hat{\mathcal{P}}$ to H is then given by

$$\hat{\mathcal{P}} = \{([b, f_0], bh) \mid b \in \mathcal{P}, h \in H\}. \square$$

This lemma can be applied in many situations. For example, if $\mathcal{P}$ is the $O(n)$-frame bundle of a not necessarily orientable Riemannian manifold X, take $F = O(n)/SO(n) = \mathbb{Z}_2$. Then $\pi : \hat{X} = \mathcal{P} \times_{O(n)} F \to X$ is a twofold covering and the structure group of the frame bundle $\pi^*\mathcal{P}$ of $\hat{X}$ can be reduced to $SO(n)$ by Lemma 1. Hence $\hat{X}$ is orientable. Of course, $\hat{X}$ is nothing but the *orientation covering* of X.

Another example is given by the *splitting principle.* If $H = T$ is a maximal torus of G, $F = G/T$, then the structure group of the pull-back of $\mathcal{P}$ to $Y = \mathcal{P} \times_G F$ can be reduced to T. Therefore any associated vector bundle splits into line bundles.

A third example appears in 4-dimensional geometry. If X is an oriented 4-dimensional Riemannian manifold, i.e. X has structure group $G = SO(4)$, then take $F = SO(4)/U(2) = \mathbb{CP}^1$. By Lemma 1 the pull-back of the frame bundle $\mathcal{P}$ of X to $Z = \mathcal{P} \times_{SO(4)} \mathbb{CP}^1$ can be reduced to $U(2)$. This means that the horizontal tangent bundle of $Z \to X$ carries a complex structure. Since the fiber bundle $Z \to X$ has fibers $\mathbb{CP}^1$ so does the vertical bundle. Thus the *twistor space* Z is an almost complex manifold.

We will see another application of Lemma 1 when we show in Section 2 of Chapter II that the twistor space of a spin^h manifold is spin^c.

Since to every symbol there exists a pseudodifferential operator having this symbol as its principal symbol [18, p. 245] we obtain immediately

Corollary. *In the situation of Theorem 1 with H-equivalent G-modules V_1 and V_2 there exists an elliptic $\mathbb{K}$-linear pseudodifferential operator $C^\infty(X, E_1) \to C^\infty(X, E_2)$ of arbitrarily chosen degree.* □

EXAMPLE 1. Let X be an oriented 4-manifold. Then $G = SO(4)$ and $H = SO(3)$. We choose $\mathbb{K} = \mathbb{R}$, $V_1 = \Lambda^1$ (1-forms) and $V_2 = 1 + \Lambda^2_+$ (functions and self-dual 2-forms).

We have to restrict the $SO(4)$-modules V_1 and V_2 to $SO(3)$. $\Lambda^1 = \mathbb{R}^4$ decomposes as an $SO(3)$-module into a trivial line spanned by x_0 and its 3-dimensional orthogonal complement on which $SO(3)$ acts by standard matrix multiplication. Hence $V_1|SO(3) = 1 + \Lambda^1\mathbb{R}^3$. The action of $SO(3)$ on the 3-dimensional space $\Lambda^2_+\mathbb{R}^4$ is nontrivial, hence $\Lambda^2_+\mathbb{R}^4|SO(3) = \Lambda^1\mathbb{R}^3$.

Thus the condition $V_1|H = V_2|H$ of Theorem 1 is satisfied and we conclude that there is an elliptic operator $C^\infty(X, T^*X) \to C^\infty(X, \mathbb{R} \oplus \Lambda^2_+ T^*X)$. Of course, we know one such operator, namely the *half Euler-deRham operator* $\delta + d^+$.

EXAMPLE 2. Let X be a $2m$-dimensional spin manifold. Then $G = Spin(2m)$ and $H = Spin(2m-1)$. We choose $\mathbb{K} = \mathbb{C}$, $V_1 = \Sigma^+$ and $V_2 = \Sigma^-$, the positive and the negative *half spinor representations.* Restriction to H yields the spinor representation of $Spin(2m-1)$, $V_1|H = V_2|H = \Sigma$.

Therefore there exists an elliptic operator $C^\infty(X, \Sigma^+) \to C^\infty(X, \Sigma^-)$ for which we may take the *Dirac operator.*

In both examples we were able to find an elliptic *differential* operator. So the question arises whether this can always be achieved. The answer however is negative.

To demonstrate this we modify Example 1. Let X be an oriented 4-manifold, $G = SO(4)$, $H = SO(3)$, $\mathbb{K} = \mathbb{R}$. We choose $V_1 = 1 + \Lambda^1$ and $V_2 = 2 + \Lambda^2_+$. We have simply added the 1-dimensional trivial representation to both V_1 and V_2. Hence by Theorem 1 there is an elliptic *pseudo*differential operator between the associated bundles.

Using additional symmetry properties of *differential* operators and the Leray-Hirsch Theorem [16] one can show [5] that the existence of an elliptic differential operator $C^\infty(X, \mathbb{R} \oplus T^*X) \to C^\infty(X, \mathbb{R} \oplus \mathbb{R} \oplus \Lambda^2_+ T^*X)$ implies vanishing of the fourth Stiefel-Whitney class, $w_4(X) = 0$. This yields

EXAMPLE 3. There exists an elliptic pseudodifferential operator $C^\infty(\mathbb{CP}^2, \mathbb{R} \oplus T^*\mathbb{CP}^2) \to C^\infty(\mathbb{CP}^2, \mathbb{R} \oplus \mathbb{R} \oplus \Lambda^2_+ T^*\mathbb{CP}^2)$ but there is no elliptic differential operator because $w_4(\mathbb{CP}^2) \neq 0$.

2. K-theoretical formulation

To apply Theorem 1 we need to find G-modules V_1 and V_2 which restrict to the same H-module. To check this condition by decomposing a G-module into irreducible H-summands is, though theoretically possible, usually an unpleasant task in concrete situations. Therefore the following point of view is helpful.

In what follows we will mainly be interested in the index of the operator that we construct. This index depends only on the Chern character of the virtual vector bundle $E_1 - E_2$ and on the differential topology of X. Therefore we should look at the *virtual representation* $V_1 - V_2$ which is an element of the representation ring (character ring) $R(G)$ of G. It is always much simpler to work with virtual representations rather than with actual representations. For the sake of simplicity we restrict ourselves from now on to the case $\mathbb{K} = \mathbb{C}$.

The condition of Theorem 1 says simply that the virtual G-module $V_1 - V_2$ be mapped to 0 under the restriction mapping $R(G) \to R(H)$. Hence we need to compute the kernel $R(G, H)$ of this restriction mapping. Since $R(G)$ is Noetherian $R(G, H)$ is finitely generated. Elliptic operators corresponding to generators of $R(G, H)$ will be called *fundamental.*

If the dimension n of X is odd, then $S^{n-1} = G/H$ is even dimensional and the ranks of G and H coincide. Thus $R(G, H) = 0$ and Theorem 1 does not yield anything interesting. Therefore we will restrict ourselves to the even dimensional case $n = 2m$. In this case the sequence

$$0 \longrightarrow R(G, H) \longrightarrow R(G) \longrightarrow R(H) \longrightarrow 0 \qquad (1)$$

is exact. This can be checked using the classification of transitive and effective Lie group actions on spheres [7, p. 179], [9], [10], [21]. Before we proceed to examples we give a K-theoretical formulation of Theorem 1 which actually works in a slightly more general situation.

Let X be a topological space, let G be a compact Lie group, let $\mathcal{P}$ be a G-principal bundle over X, let $\tau : G \to O(n)$ be an orthogonal representation. We denote the associated Riemannian vector bundle $\mathcal{P} \times_\tau \mathbb{R}^n$ by E. The bundle E replaces the cotangent bundle of X. Again, we assume that G act transitively on $S^{n-1} \subset \mathbb{R}^n$ via τ and denote the isotropy subgroup by H. We say that E has transitive structure group G with isotropy subgroup H. Denote the unit disk bundle of E by DE and the unit sphere bundle by SE. Hence $\partial DE = SE$. Denote all the projections $E \to X$, $DE \to X$, $SE \to X$, and $(DE, SE) \to X$ by π. Then $K^*(E)$, $K^*(DE)$, $K^*(SE)$, and $K^*(DE, SE)$ are modules over $K^*(X)$ via π^*.

Given the G-principal bundle $\mathcal{P}$ we have the homomorphism $\mathrm{assoc}_{\mathcal{P}} : R(G) \to K^0(X)$ which maps a virtual G-module V to its associated virtual vector bundle $\mathcal{P} \times_G V$.

Now Lemma 1 implies that the structure group of the pull-back of $\mathcal{P}$ to SE can be reduced to H. Let $\hat{\mathcal{P}}$ denote this reduction. In particular, we have the map $\mathrm{assoc}_{\hat{\mathcal{P}}} : R(H) \to K^0(SE)$.

The construction in the proof of Theorem 1 associates to G-modules V_1 and V_2 and to an H-isomorphism $V_1 \to V_2$ an element in $K^0(DE, SE)$. Since the space of H-isomorphisms is isomorphic to a product of $GL(k, \mathbb{C})$'s it is connected and the element in $K^0(DE, SE)$ does not depend on the choice of H-isomorphism but only on $V_1 - V_2 \in R(G, H)$. Hence we have a map

$$\mathrm{symb} : R(G, H) \to K^0(DE, SE).$$

If E is the cotangent bundle of a differentiable manifold X, then elements of $K^0(DE, SE)$ are equivalence classes of elliptic symbols.

The K-theoretical formulation of Theorem 1 is

Theorem 2. *Let X be a topological space, let $\pi : E \to X$ be a Riemannian vector bundle of even fiber dimension with transitive structure group G and isotropy subgroup H.*

Then the following diagram is commutative with exact columns.

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
R(G,H) \otimes K^{-1}(X) & \xrightarrow{\mathrm{symb}_{\otimes \pi^*}} & K^{-1}(DE, SE) \\
\downarrow & & \downarrow \\
R(G) \otimes K^{-1}(X) & \xrightarrow{\mathrm{assoc}_{\pi^* P \otimes \pi^*}} & K^{-1}(DE) \\
\downarrow & & \downarrow \\
R(H) \otimes K^{-1}(X) & \xrightarrow{\mathrm{assoc}_{\hat{P} \otimes \pi^*}} & K^{-1}(SE) \\
\downarrow{\scriptstyle 0} & & \downarrow \\
R(G,H) \otimes K^{0}(X) & \xrightarrow{\mathrm{symb}_{\otimes \pi^*}} & K^{0}(DE, SE) \\
\downarrow & & \downarrow \\
R(G) \otimes K^{0}(X) & \xrightarrow{\mathrm{assoc}_{\pi^* P \otimes \pi^*}} & K^{0}(DE) \\
\downarrow & & \downarrow \\
R(H) \otimes K^{0}(X) & \xrightarrow{\mathrm{assoc}_{\hat{P} \otimes \pi^*}} & K^{0}(SE) \\
\downarrow{\scriptstyle 0} & & \downarrow \\
R(G,H) \otimes K^{1}(X) & \xrightarrow{\mathrm{symb}_{\otimes \pi^*}} & K^{1}(DE, SE) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}
$$

The diagram can be extended infinitely by Bott periodicity.

Proof. The right column is the long exact cohomology sequence for K-theory of the pair (DE, SE). Tensoring the exact sequence (1) by $K^*(X)$ yields the left column which is also exact because $R(G, H)$ is torsionfree.

Commutativity of

$$\begin{array}{ccc}
R(G,H)\otimes K^*(X) & \xrightarrow{\text{symb}\otimes\pi^*} & K^*(DE,SE) \\
\downarrow & & \downarrow \\
R(G)\otimes K^*(X) & \xrightarrow{\text{assoc}_{\pi^*P\otimes\pi^*}} & K^*(DE) \\
\downarrow & & \downarrow \\
R(H)\otimes K^*(X) & \xrightarrow{\text{assoc}_{\hat{P}\otimes\pi^*}} & K^*(SE)
\end{array}$$

is obvious from the definitions. Diagram chasing using the fact that $R(H)\otimes K^i(X)\to R(G,H)\otimes K^{i+1}(X)$ and the composition $K^i(DE)\to K^i(SE)\to K^{i+1}(DE,SE)$ are zero yields commutativity of

$$\begin{array}{ccc}
R(H)\otimes K^i(X) & \xrightarrow{\text{assoc}_{\hat{P}\otimes\pi^*}} & K^i(SE) \\
\downarrow{\scriptstyle 0} & & \downarrow \\
R(G,H)\otimes K^{i+1}(X) & \xrightarrow{\text{symb}\otimes\pi^*} & K^{i+1}(DE,SE)\square
\end{array}$$

REMARK. Example 3 in the previous section was obtained from Example 1 by adding a trivial G-module to V_1 and V_2. Thus the differences $V_1 - V_2 \in R(G,H)$ are the same in Example 1 and 3. It is shown in [5, Satz 1.9] that in general every element of $R(G,H)$ can be written as a difference $V_1 - V_2$ of actual G-modules V_1 and V_2 in such a way that there is an elliptic *differential* operator of first order between the associated bundles. Hence we could insist on working with differential operators. But for our purposes elliptic pseudodifferential operators will be sufficient.

3. General integrality theorem

We return to the case where X is an n-dimensional manifold and $E = \mathcal{P}\times_\tau \mathbb{R}^n$ is its cotangent bundle. If X is closed, i.e. compact and without boundary, then any elliptic operator on X is Fredholm and we can compute its index in terms of topological data using the Atiyah-Singer Index Theorem. In particular, this topological expression must be an integer. We are now going to apply this to the operators constructed in the previous sections.

Note that for the following integrality theorem we do not really need the Atiyah-Singer Index Formula, we only need the cohomological formula for the topological index of the elliptic symbol which is an integer by definition.

Theorem 3. *Let X be a closed differentiable manifold of even dimension $n = 2m$. Let G be a compact connected Lie group, let $\tau : G \to SO(2m)$ be an orthogonal representation, let $\mathcal{P}$ be a G-principal bundle over X such that $\mathcal{P} \times_\tau \mathbb{R}^n = T^*X$. In particular, X is oriented. We denote the classifying map of $\mathcal{P}$ by $\Phi_{\mathcal{P}} : X \to BG$. Furthermore, assume that G act transitively on $S^{n-1} \subset \mathbb{R}^n$ via τ and let $H \subset G$ be the isotropy subgroup.*

Then for any $V \in R(G, H)$ and any $W \in K^0(X)$ the rational number

$$\left\{ ch(W) \cdot \Phi^*_{\mathcal{P}} \left(\frac{ch(V)}{\tau^*(e)} \right) \cdot \hat{\mathcal{A}}(TX)^2 \right\} [X]$$

is an integer.

Here $e \in H^{2m}(BSO(2m); \mathbb{Q})$ is the universal Euler class, $ch : R(G) \to H^*(BG; \mathbb{Q})$ is the universal Chern character, and $\hat{\mathcal{A}}(TX)$ is the total $\hat{A}$-class of X.□

The theorem follows directly from the previous discussion and [4, Sect. 2]. Note that the assumption in [4, Prop. 2.17] that $\tau^*(e) \neq 0 \in H^*(BG; \mathbb{Q})$ follows from the transitivity of G.

REMARK. If $V \in RSP(G, H)$, the quaternionic representation group, then $\{\Phi^*_{\mathcal{P}}(\frac{ch(V)}{\tau^*(e)}) \cdot \hat{\mathcal{A}}(TX)^2\}[X]$ is actually an *even* integer because then the elliptic operator is quaternionic, thus kernel and cokernel have even complex dimension.

4. The basic example

We will study many examples in Chapter II but the fundamental example of (even dimensional) spin manifolds deserves some extra attention. Let $G = Spin(2m)$ and let $\tau : Spin(2m) \to SO(2m)$ be the twofold covering map. Then the isotropy subgroup of a point $x_0 \in S^{2m-1}$ is $H = Spin(2m-1)$.

We want to compute generators for the ideal $R(G, H)$. Recall that

$$R(Spin(2m)) = \mathbb{Z}[\Lambda^1, \dots, \Lambda^{m-2}, \Sigma^+, \Sigma^-]$$

and

$$R(Spin(2m-1)) = \mathbb{Z}[\Lambda^1, \dots, \Lambda^{m-2}, \Sigma].$$

Here Λ^k is the $SO(n)$-module of k-forms pulled back to $Spin(n)$ via τ and Σ is the spinor representation. In even dimensions Σ decomposes into the positive and the negative half spin representations, $\Sigma = \Sigma^+ + \Sigma^-$. For facts on representation theory of compact groups see [12], [27]. The restriction homomorphism $R(G) \to$

$R(H)$ is given by

$$\begin{array}{rcl} \Lambda^1 & \longrightarrow & 1+\Lambda^1, \\ \Lambda^2 & \longrightarrow & \Lambda^1+\Lambda^2, \\ & \vdots & \\ \Lambda^{m-2} & \longrightarrow & \Lambda^{m-3}+\Lambda^{m-2}, \\ \Sigma^+ & \longrightarrow & \Sigma, \\ \Sigma^- & \longrightarrow & \Sigma. \end{array}$$

We see that the restriction homomorphism on the Λ^k's is invertible, thus only the spinor representations contribute to the kernel. More precisely, the kernel is generated by one element,

$$R(Spin(2m), Spin(2m-1)) = (\Sigma^+ - \Sigma^-).$$

Hence there is one fundamental operator for even dimensional spin manifolds, the *Dirac operator*.

Theorem. (Atiyah-Hirzebruch [3, Cor. 2])
Let X be a compact spin manifold of even dimension, let $W \in K^0(X)$. Then

$$\{ch(W)\hat{\mathcal{A}}(TX)\}[X]$$

is an integer.

In particular, the $\hat{A}$-genus of a compact spin manifold is an integer.

Furthermore, if $n \equiv 4(8)$, then the $\hat{A}$-genus of a compact spin manifold is divisible by 2.

Proof. The theorem follows from Theorem 3 with $V = \Sigma^+ - \Sigma^-$. A short calculation shows $\Phi^*_{\mathcal{P}}(ch(\Sigma^+ - \Sigma^-)/\tau^*(e)) = \hat{\mathcal{A}}(TX)^{-1}$. □

This integrality theorem is older than the Atiyah-Singer Index Formula and it was one of the hints on the way to its discovery.

5. Thom isomorphism and surjectivity

We have seen that there is a homomorphism

$$\mathrm{symb} \otimes \pi^* : R(G,H) \otimes K^0(X) \longrightarrow K^0(DT^*X, ST^*X)$$

which we can regard as a construction recipe for elliptic symbols naturally associated with the G-structure and twisted with an arbitrary coefficient bundle in $K^0(X)$. Now it is natural to ask how good this construction is, more precisely, can we say anything about surjectivity of this homomorphism ?

In general, we cannot expect the homomorphism symb $\otimes\,\pi^*$ to be surjective simply because the structure group G may not be optimally chosen. For example, on a spin manifold there is the Dirac operator. But if we forget about the spin structure and regard the manifold just as an oriented Riemannian manifold, structure group $G = SO(n)$, then the symbol of the Dirac operator will not be in the image of symb $\otimes\,\pi^*$.

However, we will see that up to a suitable power of 2 our homomorphism is surjective, more precisely

$$\mathrm{symb}\otimes\pi^* : R(G,H)\otimes K^0(X)\otimes\mathbb{Z}[2^{-\alpha}] \longrightarrow K^0(DT^*X, ST^*X)\otimes\mathbb{Z}[2^{-\alpha}]$$

is onto. The number α is a representation theoretical invariant depending on G and τ which we are going to explain next.

Let G be a compact connected Lie group, let $\tau : G \to SO(2m)$ be an orthogonal representation. If τ lifts to a homomorphism $\hat{\tau} : G \to Spin(2m)$, then we set $\alpha(G,\tau) := 0$. Otherwise there exists a connected twofold covering group $\hat{G}$ of G (the fiber product of G and $Spin(2m)$) and $\hat{\tau}$ such that the following diagram commutes.

$$\begin{array}{ccc} \hat{G} & \xrightarrow{\hat{\tau}} & Spin(2m) \\ \downarrow & & \downarrow \\ G & \xrightarrow{\tau} & SO(2m) \end{array}$$

There is a central subgroup $Z \subset \hat{G}$ isomorphic to $\mathbb{Z}_2$ such that $\hat{G}/Z = G$. Denote the nontrivial element in Z by -1.

Since Z lies in the center of $\hat{G}$ Schur's lemma implies that the action of -1 on an irreducible complex $\hat{G}$-module is either trivial or multiplication by -1 depending on whether the module descends to a G-module or not. Hence -1 acts on $R(\hat{G})$ and decomposes it into eigenspaces for the eigenvalues ± 1, $R(\hat{G}) = R^+(\hat{G}) \oplus R^-(\hat{G})$ where $R^+(\hat{G})$ can be identified with $R(G)$.

Let $\dim : R(\hat{G}) \to \mathbb{Z}$ be the homomorphism which maps each virtual module to its dimension. The image of $R^-(\hat{G})$ under dim is an ideal in $\mathbb{Z}$, hence generated by some $k \geq 1$. Now we have $\hat{\tau}^*(\Sigma^+) \in R^-(\hat{G})$ and $\dim(\Sigma^+) = 2^{m-1}$. Thus $k = 2^\alpha$ where $0 \leq \alpha \leq m-1$. We will call $\alpha = \alpha(G,\tau)$ the *surjectivity exponent* of G.

Theorem 4. *Let X be a compact CW-complex, let $\pi : E \to X$ be a Riemannian vector bundle of fiber dimension $2m$ with compact connected transitive structure group G and isotropy subgroup H.*

Then the map

$$R(G,H)\otimes K^*(X)\otimes\mathbb{Z}[\tfrac{1}{2}] \longrightarrow K^*(DE,SE)\otimes\mathbb{Z}[\tfrac{1}{2}]$$

is surjective. If the surjectivity exponent vanishes, $\alpha = 0$, then the assertion holds also without inverting 2, i.e.

$$R(G,H)\otimes K^*(X) \longrightarrow K^*(DE,SE)$$

is surjective.

EXAMPLE. Let $G = U(m)$ and let $\tau : U(m) \to SO(2m)$ be the standard inclusion. Then $\hat{U}(m)$ has a one-dimensional representation $(\Lambda^{0,m})^{\frac{1}{2}}$ whose square yields the module of $(0,m)$-forms $\Lambda^{0,m}$. Thus $(\Lambda^{0,m})^{\frac{1}{2}} \in R^-(\hat{U}(m))$ and $\alpha(U(m)) = 0$ even though $\tau : U(m) \to SO(2m)$ does not lift to $Spin(2m)$. Hence if X is an almost complex manifold, then every elliptic symbol class comes from $R(U(m), U(m-1))$ twisted by coefficients in $K^0(X)$.

At the end of Chapter II we will present Table 2 containing the surjectivity exponents for many groups G.

Proof of Theorem 4. If $\tau : G \to SO(2m)$ lifts to $Spin(2m)$ put $\sigma := \hat{\tau}^*(\Sigma^+ - \Sigma^-) \in R(G,H)$. Otherwise, pick $V \in R^-(\hat{G})$ of dimension 2^α and put $\sigma := \hat{\tau}^*(\Sigma^+ - \Sigma^-) \otimes V \in R^+(\hat{G}) \cong R(G)$. We will show

CLAIM: The map $K^*(X) \otimes \mathbb{Z}[2^{-\alpha}] \longrightarrow K^*(DE, SE) \otimes \mathbb{Z}[2^{-\alpha}]$ mapping $W \in K^*(X)$ to $\pi^*W \cdot \mathrm{symb}(\sigma)$ is an isomorphism.

We start with the case that the bundle E is trivial, $E = X \times \mathbb{R}^{2m}$. Then the Thom space DE/SE is homeomorphic to $X \times S^{2m}/X \times \{\text{point}\}$. The long exact sequence for K-theory of the pair $(X \times S^{2m}, X \times \{\text{point}\})$ yields

$$K^{i-1}(X \times S^{2m}) \longrightarrow K^{i-1}(X) \longrightarrow K^i(DE, SE) \longrightarrow K^i(X \times S^{2m}) \longrightarrow K^i(X) \tag{2}$$

The Künneth formula for K-theory [1] yields an isomorphism

$$K^*(X) \otimes K^0(S^{2m}) \longrightarrow K^*(X \times S^{2m})$$

whose composition with the restriction mapping $K^*(X \times S^{2m}) \to K^*(X)$ is simply $\mathrm{id} \otimes \dim$. The map $\mathrm{id} \otimes \dim : K^*(X) \otimes K^0(S^{2m}) \to K^*(X)$ is onto and its kernel is $K^*(X) \otimes \tilde{K}^0(S^{2m})$. Thus sequence (2) breaks up to

$$0 \longrightarrow K^i(DE, SE) \longrightarrow K^i(X) \otimes \tilde{K}^0(S^{2m}) \longrightarrow 0.$$

It is well known that $\tilde{K}^0(S^{2m}) = K^0(D^{2m}, S^{2m-1}) \cong \mathbb{Z}$ is generated by $\mathrm{symb}(\Sigma^+ - \Sigma^-)$, see [2]. Hence $\mathrm{symb}(\sigma)$ generates $K^0(D^{2m}, S^{2m-1}) \otimes \mathbb{Z}[2^{-\alpha}]$. Thus the claim is proved if E is a trivial bundle.

Now let X be a compact CW-complex. We prove the claim by induction on the number of cells of X.

Let X be obtained from X' by attaching one k-cell. We assume that the claim be true for X'. We can cover X by two compact sets A and B such that

(i) The interior of A and B still cover X, i.e. $\dot{A} \cup \dot{B} = X$.

(ii) A is contained in the interior of the new k-cell.

(iii) B contains X' and the inclusion mapping $X' \hookrightarrow B$ is a homotopy equivalence.

By (ii) the bundle E is trivial over A and $A \cap B$. Hence the claim is true for A and $A \cap B$.

By (iii) and the induction hypothesis the claim is also true for B.

Now we use the Mayer-Vietoris sequence for K-theory [17]

$$\cdots \longrightarrow K^i(X) \longrightarrow K^i(A) \oplus K^i(B) \longrightarrow K^i(A \cap B) \longrightarrow K^{i+1}(X) \longrightarrow \cdots$$

Tensoring by $\mathbb{Z}[2^{-\alpha}]$ again yields an exact sequence because $\mathbb{Z}[2^{-\alpha}]$ is a flat $\mathbb{Z}$-module. We obtain the following commutative diagram with exact columns.

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\begin{array}{c}(K^{i-1}(A) \oplus K^{i-1}(B)) \\ \otimes \mathbb{Z}[2^{-\alpha}]\end{array} & \longrightarrow & \begin{array}{c}(K^{i-1}(DE|_A, SE|_A) \oplus K^{i-1}(DE|_B, SE|_B)) \\ \otimes \mathbb{Z}[2^{-\alpha}]\end{array} \\
\downarrow & & \downarrow \\
K^{i-1}(A \cap B) \otimes \mathbb{Z}[2^{-\alpha}] & \longrightarrow & K^{i-1}(DE|_{A\cap B}, SE|_{A\cap B}) \otimes \mathbb{Z}[2^{-\alpha}] \\
\downarrow & & \downarrow \\
K^i(X) \otimes \mathbb{Z}[2^{-\alpha}] & \longrightarrow & K^i(DE, SE) \otimes \mathbb{Z}[2^{-\alpha}] \\
\downarrow & & \downarrow \\
\begin{array}{c}(K^{i}(A) \oplus K^{i}(B)) \\ \otimes \mathbb{Z}[2^{-\alpha}]\end{array} & \longrightarrow & \begin{array}{c}(K^{i}(DE|_A, SE|_A) \oplus K^{i}(DE|_B, SE|_B)) \\ \otimes \mathbb{Z}[2^{-\alpha}]\end{array} \\
\downarrow & & \downarrow \\
K^{i}(A \cap B) \otimes \mathbb{Z}[2^{-\alpha}] & \longrightarrow & K^{i}(DE|_{A\cap B}, SE|_{A\cap B}) \otimes \mathbb{Z}[2^{-\alpha}] \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

We know that the outer four horizontal arrows are isomorphisms, hence by the 5-lemma we conclude that so is the middle one.

Since $\mathbb{Z}[2^{-\alpha}] = \mathbb{Z}$ if $\alpha = 0$ and $\mathbb{Z}[2^{-\alpha}] = \mathbb{Z}[\frac{1}{2}]$ if $\alpha > 0$, the theorem follows. □

REMARK. What we have actually shown in the proof of Theorem 4 is the fact that $K^*(DE, SE) \otimes \mathbb{Z}[2^{-\alpha}]$ is a free module over $K^*(X) \otimes \mathbb{Z}[2^{-\alpha}]$ of rank one with generator symb(σ). For certain special groups G with $\alpha = 0$, i.e. $G = U(m)$, this can be found in the literature and is then called *Thom isomorphism* in K-theory.

II. Examples

We compute the ideal $R(G,H)$ for the most important structure groups G. For the most classical examples like (oriented) Riemannian manifolds ($G = O(n), G = SO(n)$), almost complex manifolds ($G = U(m)$), spin manifolds ($G = Spin(n)$), and spin^c manifolds ($G = Spin^c(n)$) the corresponding fundamental elliptic operators (Euler-deRham operator, signature operator, Cauchy-Riemann operator, Dirac operator) have been found over the years.

For some less well studied groups like $G = Spin^h(n)$ and $G = Sp(q) \cdot Sp(1)$ (almost quaternionic manifolds) we use our method to systematically determine the fundamental elliptic operators. In particular, we obtain new integrality results for such manifolds. As an application, we show that the projective Cayley plane does not admit an almost quaternionic structure. In the case of spin^h manifolds we digress somewhat from our path and include a short study of the corresponding twistor space.

In the end we summarize the results in Table 2.

1. Some classical examples

In this section we compute $R(G,H)$ for some well studied structure groups G. For these groups we cannot expect to find new elliptic operators but it is still interesting to see how they fit into our framework.

We start with $G = SO(2m), m \geq 2$, corresponding to $2m$-dimensional **oriented Riemannian manifolds**. We have already seen in the previous chapter that $R(Spin(2m)) = \mathbb{Z}[\Lambda^1, \ldots, \Lambda^{m-2}, \Sigma^+, \Sigma^-]$ and $R(Spin(2m), Spin(2m-1)) = (\Sigma^+ - \Sigma^-)$. Now $R(SO(2m))$ can be identified with the subring of $R(Spin(2m))$ consisting of those polynomials which are even in Σ^+ and Σ^-. Thus we obtain

$$\begin{aligned} R(SO(2m), SO(2m-1)) &= R(SO(2m)) \cap R(Spin(2m), Spin(2m-1)) \\ &= \{(\Sigma^+ - \Sigma^-) \cdot \psi \mid \psi \text{ is odd in } \Sigma^\pm\} \\ &= ((\Sigma^+ - \Sigma^-) \cdot \Sigma^+, (\Sigma^+ - \Sigma^-) \cdot \Sigma^-) \\ &= (1 - \Lambda^1 \pm \cdots + (-1)^m \Lambda^m_+, \\ &\quad\; 1 - \Lambda^1 \pm \cdots + (-1)^m \Lambda^m_-). \end{aligned}$$

Hence we get two fundamental operators, two *half Euler-deRham operators* the sum of which is just the ordinary *Euler-deRham operator* corresponding to $1 - \Lambda^1 \pm \cdots - \Lambda^{2m-1} + 1$ and the difference is the *signature operator* corresponding to $\Lambda^m_+ - \Lambda^m_-$.

Let us take a look at even dimensional **not necessarily orientable Riemannian manifolds**, i.e. $G = O(2m)$. It is well known [12] that

$$R(O(n)) = \mathbb{Z}[\Lambda^1, \ldots, \Lambda^n]/I_n$$

where I_n is generated by $\Lambda^k\Lambda^n - \Lambda^{n-k}$, $k = 1, \ldots n$. The restriction mapping $R(O(2m)) \to R(O(2m-1))$ is given by

$$\begin{array}{rcl} \Lambda^1 & \to & 1 + \Lambda^1, \\ \Lambda^2 & \to & \Lambda^1 + \Lambda^2, \\ & \vdots & \\ \Lambda^{2m-1} & \to & \Lambda^{2m-2} + \Lambda^{2m-1}, \\ \Lambda^{2m} & \to & \Lambda^{2m-1}. \end{array}$$

Since the relation ideal I_{2m} is mapped onto I_{2m-1} one sees easily that $R(O(2m), O(2m-1))$ is generated by $1 - \Lambda^1 \pm \cdots + \Lambda^{2m}$. Hence the *Euler-deRham operator* is the fundamental operator.

The next classical example is given by **almost complex manifolds**, i.e. $G = U(m)$. The representation ring is

$$R(U(m)) = \mathbb{Z}[\Lambda^{0,1}, \ldots, \Lambda^{0,m}, \Lambda^{m,0}]/(\Lambda^{0,m}\Lambda^{m,0} - 1)$$

where $\Lambda^{p,q}$ denotes the $U(m)$-module of (p,q)-forms. Since $\tau : U(m) \to SO(2m)$ is the standard imbedding, H is just $U(m-1)$. A similar calculation as in the previous example yields

$$R(U(m), U(m-1)) = (1 - \Lambda^{0,1} \pm \cdots + (-1)^m\Lambda^{0,m}).$$

Again, there is one fundamental operator, the *Cauchy-Riemann operator.*

As a generalization of almost complex manifolds as well as of spin manifolds we can look at **spin^c manifolds**. The group $Spin^c(n)$ is defined by

$$Spin^c(n) = \frac{Spin(n) \times U(1)}{\{(1,1), (-1,-1)\}}.$$

We have the commutative diagram

$$\begin{array}{ccc} Spin(n) \times U(1) & \xrightarrow{\hat{\tau}} & Spin(n) \\ \downarrow & & \downarrow \\ Spin^c(n) & \xrightarrow{\tau} & SO(n). \end{array}$$

Hence if $G = Spin^c(n)$, then $H = Spin^c(n-1)$. For $\hat{G} = Spin(2m) \times U(1)$ and $\hat{H} = Spin(2m-1) \times U(1)$ we have $R(\hat{G}) = \mathbb{Z}[\Lambda^1, \ldots, \Lambda^{m-2}, \Sigma^+, \Sigma^-, z, \bar{z}]/(z\bar{z} - 1)$ and $R(\hat{H}) = \mathbb{Z}[\Lambda^1, \ldots, \Lambda^{m-2}, \Sigma, z, \bar{z}]/(z\bar{z} - 1)$. Here z denotes the standard representation of $U(1)$. The restriction mapping is given by

$$\begin{array}{rcl} \Lambda^1 & \to & 1 + \Lambda^1, \\ \Lambda^2 & \to & \Lambda^1 + \Lambda^2, \\ & \vdots & \\ \Lambda^{m-2} & \to & \Lambda^{m-3} + \Lambda^{m-2}, \\ \Sigma^+ & \to & \Sigma, \\ \Sigma^- & \to & \Sigma, \\ z & \to & z, \\ \bar{z} & \to & \bar{z}. \end{array}$$

Hence we obtain $R(\hat{G}, \hat{H}) = (\Sigma^+ - \Sigma^-)$. Since $(-1, -1)$ acts trivially on the Λ^k's and via multiplication by -1 on $\Sigma^\pm$, z, and $\bar{z}$ we can identify $R(G)$ with the subring of $R(\hat{G})$ consisting of those polynomials which are even in $\Sigma^\pm$, z, and $\bar{z}$. This yields four generators for $R(G, H)$, namely

$$R(G,H) = ((\Sigma^+ - \Sigma^-) \cdot z, (\Sigma^+ - \Sigma^-) \cdot \bar{z}, (\Sigma^+ - \Sigma^-) \cdot \Sigma^+, (\Sigma^+ - \Sigma^-) \cdot \Sigma^-).$$

Now we note that we can express all generators by the first one, namely $(\Sigma^+ - \Sigma^-) \cdot \bar{z} = (\Sigma^+ - \Sigma^-) \cdot z \cdot (\bar{z}^2)$ and $(\Sigma^+ - \Sigma^-) \cdot \Sigma^\pm = (\Sigma^+ - \Sigma^-) \cdot z \cdot (\bar{z}\Sigma^\pm)$. Thus

$$R(G,H) = ((\Sigma^+ - \Sigma^-) \cdot z).$$

Hence we obtain one *twisted Dirac operator* as the fundamental operator. We note that since the $\hat{G}$-module z is 1-dimensional the surjectivity exponent is zero, $\alpha(Spin^c(2m), \tau) = 0$.

The corresponding integrality theorem is in this case

Theorem. (Atiyah-Hirzebruch [3, Cor. 1])
Let X be a compact spin^c manifold of even dimension, let $W \in K^0(X)$, let $c \in H^2(X;\mathbb{Z})$ such that $c \equiv w_2(X)$ mod 2. Then

$$\{ch(W)e^{c/2}\hat{\mathcal{A}}(TX)\}[X]$$

is an integer.

Proof. The theorem follows from Theorem 3 with $V = (\Sigma^+ - \Sigma^-) \cdot z$. The condition $c \equiv w_2(X)$ mod 2 insures that c is the first Chern class of the canonical line bundle for some spin^c structure. In other words, a spin^c structure $\mathcal{P}$ can be chosen so that $c = c_1(\mathcal{P} \times_{Spin^c(n)} z^2)$. $\square$

It turns out that for those connected compact groups G for which $\tau : G \to SO(2m)$ lifts to $\hat{\tau} : G \to Spin(2m)$ the ideal $R(G,H)$ has one generator, namely $\hat{\tau}^*(\Sigma^+ - \Sigma^-)$. Hence in these cases the Dirac operator is the fundamental operator. This applies in particular to simply connected groups G such as $SU(m)$ (almost complex manifolds with vanishing first Chern class), $Sp(q)$, and exotic $Spin(7)$ or $Spin(9)$.

2. Spin^h manifolds

We call an n-dimensional differentiable manifold X *spin*^{h} if it has structure group

$$G = Spin^h(n) = \frac{Spin(n) \times Sp(1)}{\{(1,1),(-1,-1)\}}.$$

In other words, $Spin^h(n)$ is the quaternionic analogue to $Spin^c(n)$, we simply replace the $U(1)$-factor by an $Sp(1)$-factor. The orthogonal representation τ is induced by projection $\hat{\tau}$ onto the $Spin(n)$-factor, i.e. the following diagram commutes.

$$\begin{array}{ccc} Spin(n) \times Sp(1) & \xrightarrow{\hat{\tau}} & Spin(n) \\ \downarrow & & \downarrow \\ Spin^h(n) & \xrightarrow{\tau} & SO(n) \end{array}$$

Spin^h manifolds form a very big class of manifolds because all spin^c manifolds and all almost quaternionic manifolds are spin^h. The exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin^h(n) \longrightarrow SO(n) \times SO(3) \longrightarrow 0 \qquad (3)$$

shows that spin^h manifolds carry a *canonical* $SO(3)$*-bundle* E. This $SO(3)$-bundle is analogous to the canonical line bundle of spin^c manifolds.

Sequence (3) yields the exact cohomology sequence

$$H^1(X; Spin^h(n)) \longrightarrow H^1(X; SO(n)) \oplus H^1(X; SO(3)) \xrightarrow{w_2} H^2(X; \mathbb{Z}_2) \qquad (4)$$

which shows that the condition that an $SO(3)$-bundle E be canonical for some spin^h structure is exactly

$$w_2(E) = w_2(X). \qquad (5)$$

To obtain elliptic operators on even dimensional spin^h manifolds we compute $R(Spin^h(2m), Spin^h(2m-1))$. Put $G = Spin^h(2m), H = Spin^h(2m-1), \hat{G} = Spin(2m) \times Sp(1)$, and $\hat{H} = Spin(2m-1) \times Sp(1)$. We know

$$R(\hat{G}) = \mathbb{Z}[\Lambda^1, \ldots, \Lambda^{m-2}, \Sigma^+, \Sigma^-, \rho]$$

and

$$R(\hat{H}) = \mathbb{Z}[\Lambda^1, \ldots, \Lambda^{m-2}, \Sigma, \rho]$$

where ρ is the standard representation of $Sp(1) = SU(2)$ on $\mathbb{C}^2$. As in the case of spin^c manifolds we see easily that $R(\hat{G}, \hat{H}) = (\Sigma^+ - \Sigma^-)$ and since $R(G)$ is the subring of $R(\hat{G})$ consisting of those polynomials which are even in $\Sigma^\pm$ and ρ we get

$$\begin{aligned} R(G,H) &= ((\Sigma^+ - \Sigma^-)\rho, (\Sigma^+ - \Sigma^-)\Sigma^+, (\Sigma^+ - \Sigma^-)\Sigma^-) \\ &= ((\Sigma^+ - \Sigma^-)\rho, 1 - \Lambda^1 \pm \cdots + (-1)^m \Lambda^m_+, \\ &\quad 1 - \Lambda^1 \pm \cdots + (-1)^m \Lambda^m_-). \end{aligned}$$

We thus obtain three fundamental operators, two *half Euler-deRham operators* and one *twisted Dirac operator.*

Now we can formulate the corresponding integrality theorem.

Theorem 5. *Let X be a compact spin^h manifold of dimension $n = 2m$ with canonical $SO(3)$-bundle E. Let $W \in K^0(X)$ and let $p_1(E) \in H^4(X;\mathbb{Z})$ be the first Pontrjagin class of E.*

Then $p_1(E) \equiv w_2(M)^2$ mod 2 and the rational number

$$2\left\{ ch(W) \cosh\left(\frac{\sqrt{p_1(E)}}{2}\right) \hat{\mathcal{A}}(TX)\right\}[X]$$

is an integer.

In particular,

$$2\left\{ \cosh\left(\frac{\sqrt{p_1(E)}}{2}\right) \hat{\mathcal{A}}(TX)\right\}[X]$$

is an integer.

Proof. Since the reduction of $p_1(E)$ modulo 2 is $w_2(E)^2$, the first part of the assertion follows from (5). We now apply Theorem 3 with $V = (\Sigma^+ - \Sigma^-)\cdot\rho$. The only thing left to do is to compute $ch(\rho)$.

Denote the global weights of the $Sp(1)$-module ρ by z_0 and z_0^{-1} say. Accordingly, we write for the Chern character

$$ch(\rho) = e^{x_0} + e^{-x_0} = 2\cosh(x_0).$$

The complexification of E is associated to the module $\rho^2 - 1$ having the weights z_0^2, z_0^{-2} and 1. Therefore

$$c(E \otimes \mathbb{C}) = (1 + 2x_0)(1 - 2x_0) = 1 - 4x_0^2.$$

Hence the first Pontrjagin class is given by

$$p_1(E) = 4x_0^2,$$

which implies

$$ch(\rho) = 2\cosh\left(\frac{\sqrt{p_1(E)}}{2}\right).$$

Theorem 5 now follows from Theorem 3. □

Note that $\cosh(x)$ is an even power series so that $\cosh\left(\frac{\sqrt{p_1(E)}}{2}\right)$ is actually a power series in $p_1(E)$.

Theorem 5 is a special case of K. H. Mayer's integrality theorem [20] to which we will return in the third chapter. The following two conclusions are immediate.

Corollary 1. *Let X be a compact spin^h manifold of dimension $n = 2m$ with canonical $SO(3)$-bundle E. If the first Pontrjagin class $p_1(E)$ of E is a torsion class, then $2\hat{A}(X)$ is an integer.*□

Corollary 2. *Let X be a compact spin^h manifold of dimension $n = 2m$ whose forth Betti number vanishes, $b_4(X;\mathbb{R}) = 0$. Then $2\hat{A}(X)$ is an integer.*□

One can use Theorem 5 to derive divisibility properties of $SO(3)$-bundles.

EXAMPLE 1. We consider the $4q$-dimensional manifold $X = \underbrace{S^4 \times \cdots \times S^4}_{q \text{ factors}}$. Let E be an arbitrary $SO(3)$-bundle over X.

Then the characteristic number $\{p_1(E)^q\}[X]$ is divisible by $2^{2q-1} \cdot (2q)!$

The proof is as follows. Since trivially $w_2(E) = 0 = w_2(X)$, E is canonical for some spin^h structure. We note that $\hat{\mathcal{A}}(X) = \hat{\mathcal{A}}(S^4)^q = 1$ and we know that the following expression is an integer:

$$\begin{aligned} 2\left\{\cosh\left(\frac{\sqrt{p_1(E)}}{2}\right)\right\}[X] &= 2\left\{\frac{(\sqrt{p_1(E)}/2)^{2q}}{(2q)!}\right\}[X] \\ &= \frac{1}{2^{2q-1}\cdot(2q)!}\{p_1(E)^q\}[X]. \square \end{aligned}$$

In the case $q = 1$ this means that the Pontrjagin number of any $SO(3)$-bundle over S^4 is divisible by 4. It is true in general that the Pontrjagin number of an $SO(3)$-bundle E over a 4-manifold with $w_2(E) = 0$ is divisible by 4 because E

can be lifted to an $SU(2)$-bundle F and $p_1(E) = -4c_2(F)$, compare [13, App. E].

EXAMPLE 2. Let E be an arbitrary $SO(3)$-bundle over $X = \mathbb{CP}^2$.

Then either $w_2(E) = 0$ *and* $p_1(E)[X] \equiv 0(4)$ *or* $w_2(E) = \tilde{a}$ *and* $p_1(E)[X] \equiv 1(4)$.

Here $\tilde{a}$ is the mod2-reduction of the generator $a \in H^2(X;\mathbb{Z})$.

The case $w_2(E) = 0$ follows from the remark at the end of Example 1. If $w_2(E) = \tilde{a} = w_2(X)$, then E is canonical for some spin^h structure and we can again apply Theorem 5. The following expression must be an integer:

$$\begin{aligned} 2\left\{\cosh(\sqrt{p_1(E)}/2)\hat{\mathcal{A}}(X)\right\}[X] &= 2\left\{\left(1+\frac{p_1(E)}{8}\right)\left(1-\frac{a^2}{8}\right)\right\}[X] \\ &= \frac{1}{4}\left(p_1(E)[X]-1\right).\square \end{aligned}$$

Spin^h manifolds form the natural class of spaces containing all spin^c manifolds as well as the almost quaternionic manifolds just like spin^c manifolds contain all spin and all almost complex manifolds. We have seen that spin^h manifolds share one property with spin^c manifolds, there is an integrality theorem. There are also vanishing theorems for spin^h manifolds [5] [23] analogous to Hitchin's vanishing theorem for spin^c manifolds [15]. To conclude this section we study one property of spin^h manifold which they inherit from almost quaternionic manifolds, namely they have a twistor space.

DEFINITION. Let X be a spin^h manifold with canonical $SO(3)$-bundle E. Then the unit sphere bundle $Z \subset E$ is called *twistor space* of X.

Proposition. *The twistor space of a spin^h manifold is a spin^c manifold.*

Proof. Let $\pi : Z \to M$ be the projection of the twistor space Z to X, let $\mathcal{P}$ be the $Spin^h(n)$-principal bundle over X. The tangent bundle of Z decomposes into $TZ = \pi^*TX \oplus V$, where V is the vertical bundle along the fibers in Z. The fibers are $S^2 = \mathbb{CP}^1$, hence V is an $SO(2)$-bundle. V has a "square root", the complex dual of the tautological Hopf bundle along the fibers. In other words, V is associated to the $U(1)$-principal bundle of the Hopf bundle to the representation z^2.

By Lemma 1 the structure group of $\pi^*\mathcal{P}$ can be reduced to $Spin^c(n)$. Hence the structure group of Z can be reduced to $Spin^c(n) \times U(1)$. Composing the canonical embedding $Spin^c(n) \hookrightarrow Spin^c(n+2)$ with the map $U(1) = Spin(2) \hookrightarrow Spin(n+2) \hookrightarrow Spin(n+2) \times U(1) \to Spin^c(n+2)$ we obtain the commutative diagram

$$\begin{array}{ccc} Spin^c(n) \times Spin(2) & \longrightarrow & Spin^c(n+2) \\ \downarrow & & \downarrow \\ SO(n) \times SO(2) & \hookrightarrow & SO(n+2) \end{array}$$

This yields the spin^c structure on Z. □

The more structure there is on the original manifold X the more structure we can expect on the twistor space. We collect the proposition and some results by Bérard Bergery [7] and Salamon [24] in the following table.

X	twistor space Z
spin^h	spin^c
almost quaternionic	almost complex
quaternionic	complex
quaternionic-Kähler with $Ric \neq 0$	complex contact structure
quaternionic-Kähler with $Ric > 0$	Kähler-Einstein

Tab. 1

Spin^h manifolds have been studied independently by Nagase [22] [23]. He calls them spin^q manifolds.

3. Almost quaternionic manifolds

We are now going to study a class of manifolds which are a quaternionic analogue of almost complex manifolds. We start with some definitions, compare [8], [24], [25].

A $4q$-dimensional manifold with structure group $GL(q,\mathbb{H})\cdot Sp(1) \subset GL(4q,\mathbb{R})$ is called *almost quaternionic*. Such a manifold is characterized by *local* existence of almost complex structures I, J, and $K = IJ = -JI$ such that the $SO(3)$-bundle E spanned by I, J, and K is defined *globally*. E is then the *canonical* $SO(3)$*-bundle*. One can always reduce the structure group $GL(q,\mathbb{H}) \cdot Sp(1)$ to the maximal compact subgroup $Sp(q) \cdot Sp(1) = Sp(q) \times Sp(1)/\mathbb{Z}_2$, $\mathbb{Z}_2 = \{(1,1),(-1,-1)\}$. In other words, one can always choose a quaternionic-hermitian metric.

If the $GL(q,\mathbb{H})Sp(1)$-principal bundle carries a torsionfree connection, then the manifold is called *quaternionic*. If even the $Sp(q)\cdot Sp(1)$-bundle carries a torsionfree connection which then is the Levi-Civita connection of the quaternionic-hermitian metric, then the manifold is called *quaternionic-Kähler*. In other words, a manifold is quaternionic-Kähler if and only if its *holonomy* is contained in $Sp(q) \cdot Sp(1)$.

We will only be concerned with almost quaternionic manifolds, special metrics or connections play no role.

Lemma 2. *Let X be an almost quaternionic manifold of dimension $n = 4q$. Then X is spin^h and if q is even, then X is spin.*

Proof. For any q there is the commutative diagram

$$\begin{array}{ccc} Sp(q) \times Sp(1) & \xrightarrow{\phi} & Spin(n) \\ \downarrow & & \downarrow \\ Sp(q) \cdot Sp(1) & \subset & SO(n) \end{array}$$

For q even $\phi(-1,-1) = 1$ holds, i.e. the embedding $Sp(q) \cdot Sp(1) \subset SO(n)$ lifts to an embedding $Sp(q) \cdot Sp(1) \subset Spin(n)$.

If q is odd, then $\phi(-1,-1) = -1$. Thus the embedding

$$\Phi = \phi \times pr_2 : Sp(q) \times Sp(1) \hookrightarrow Spin(n) \times Sp(1),$$

satisfies $\Phi(-1,-1) = (-1,-1)$. Hence Φ induces an embedding

$$\bar{\Phi} : Sp(q) \cdot Sp(1) \hookrightarrow Spin(n) \cdot Sp(1).$$

This proves the lemma. □

To find elliptic operators for almost quaternionic manifolds we have to compute $R(G, H)$ where $G = Sp(q) \cdot Sp(1)$ and $H = Sp(q-1) \cdot Sp(1)$. Set $\hat{G} = Sp(q) \times Sp(1)$ and $\hat{H} = Sp(q-1) \times Sp(1)$. In the first section we mentioned that $R(Sp(q), Sp(q-1)) = (\Sigma^+ - \Sigma^-)$ where $\Sigma^\pm$ denotes restriction of the half-spin representations via the inclusion $Sp(q) \subset Spin(4q)$. Thus

$$R(\hat{G}, \hat{H}) = (\Sigma^+ - \Sigma^-).$$

The representation rings are

$$\begin{aligned} R(\hat{G}) &= \mathbb{Z}[\Lambda^{1,0}, \ldots, \Lambda^{q,0}, \rho] \\ R(\hat{H}) &= \mathbb{Z}[\Lambda^{1,0}, \ldots, \Lambda^{q-1,0}, \rho] \end{aligned}$$

where $\Lambda^{k,l}$ is the restriction of the $SU(2q)$-module of (k,l)-forms to $Sp(q)$ and ρ is the standard 2-dimensional representation of $Sp(1) = SU(2)$.

Expressed in these generators $\Sigma^+ - \Sigma^-$ can be written as

$$\Sigma^+ - \Sigma^- = \sum_{k=0}^{q} p_k(\rho) \Lambda^{q-k,0} \tag{6}$$

where p_k are polynomials defined recursively by

$$\begin{aligned} p_0(t) &= 1, \\ p_1(t) &= -t, \\ p_2(t) &= t^2 - 2, \\ p_k(t) &= -p_{k-1}(t)t - p_{k-2}(t), k \geq 3. \end{aligned}$$

To pass to the group $Sp(q) \cdot Sp(1) = Sp(q) \times Sp(1)/\{(1,1),(-1,-1)\}$ we have to look at the action of $(-1,-1)$. It is easy to see that $(-1,-1)$ acts on $\Lambda^{k,0}$ via multiplication by $(-1)^k$ and on ρ by -1. If we give $\Lambda^{k,0}$ degree k and ρ degree -1, then $R(G)$ consists of all polynomials in $R(\hat{G})$ of even degree. Equation (6) and a simple induction show that $\Sigma^+ - \Sigma^-$ has even degree if q is even and odd degree if q is odd. This implies

$$R(G,H) = \begin{cases} (\Sigma^+ - \Sigma^-), & \text{if } q \text{ is even} \\ \left.\begin{array}{l} ((\Sigma^+ - \Sigma^-) \cdot \Lambda^{1,0}, (\Sigma^+ - \Sigma^-) \cdot \Lambda^{3,0}, \ldots, \\ (\Sigma^+ - \Sigma^-) \cdot \Lambda^{q,0}, (\Sigma^+ - \Sigma^-) \cdot \rho), \end{array}\right\} & \text{if } q \text{ is odd} \end{cases} \tag{7}$$

We see that if q is even, then almost quaternionic manifolds are automatically spin and the *Dirac operator* is the fundamental operator and if q is odd the situation is more complicated. In the latter case we obtain $\frac{q+3}{2}$ fundamental operators. Moreover, it is clear that we have for the surjectivity exponent

$$\alpha = \begin{cases} 0, & \text{if } q \text{ is even} \\ 1, & \text{if } q \text{ is odd} \end{cases}$$

To formulate the integrality theorem for almost quaternionic manifolds we need one more notation. For a topological space X let $\rho_k : H^{2*}(X;\mathbb{Q}) \to H^{2*}(X;\mathbb{Q})$ be the *Adams operation*, i.e. ρ_k is multiplication by k^m on $H^{2m}(X;\mathbb{Q})$. It can be shown by induction [5, Lemma 3.7] that for any complex vector bundle E over X

$$ch(\Lambda^k E) = \frac{1}{k}\sum_{\mu=1}^{k}(-1)^{\mu+1}\rho_\mu(ch(E)) \cdot ch(\Lambda^{k-\mu}E)$$

holds. Hence if we define for any mixed cohomology class $x \in H^{2*}(X;\mathbb{Q})$

$$\begin{aligned} \lambda^0 x &= 1, \\ \lambda^1 x &= x, \\ \lambda^k x &= \frac{1}{k}\sum_{\mu=1}^{k}(-1)^{\mu+1}\rho_\mu(x) \cdot \lambda^{k-\mu}x, k \geq 2 \end{aligned}$$

then

$$ch(\Lambda^k E) = \lambda^k ch(E).$$

Now we turn to the integrality theorem for almost quaternionic manifolds. We begin with the case that q is odd because for q even we are in the class of spin manifolds.

Theorem 6. *Let X be a closed almost quaternionic manifold of dimension $n = 4q$ with q odd. Let E be the canonical $SO(3)$-bundle with first Pontrjagin class $p_1(E) \in H^4(X;\mathbb{Z})$. Let $W \in K^0(X)$.*

Then the following rational numbers are integers:

$$2\left\{ch(W)\cosh(\sqrt{p_1(E)}/2)\hat{\mathcal{A}}(TX)\right\}[X],$$

$$\left\{ch(W)\lambda^k\left(\frac{ch(TX\otimes\mathbb{C})}{2\cosh(\sqrt{p_1(E)}/2)}\right)\hat{\mathcal{A}}(TX)\right\}[X], k = 1,3,\ldots,q.$$

Proof. This follows from Theorem 3 with $V = (\Sigma^+ - \Sigma^-)\cdot\rho$ or $V = (\Sigma^+ - \Sigma^-)\cdot\Lambda^{k,0}$ resp. We have seen in the previous section on spin^h manifolds that the first expression is an integer because

$$ch(\rho) = 2\cosh(\sqrt{p_1(E)}/2).$$

It remains to compute $ch(\Lambda^{k,0})$. The tangent bundle is induced by τ where $\tau\otimes\mathbb{C} = \Lambda^{1,0}\rho$. Hence

$$ch(\Lambda^{1,0}) = \frac{ch(\tau\otimes\mathbb{C})}{ch(\rho)}$$

and for the higher powers

$$ch(\Lambda^{k,0}) = ch(\Lambda^k(\Lambda^{1,0})) = \lambda^k ch(\Lambda^{1,0}) = \lambda^k\left(\frac{ch(\tau\otimes\mathbb{C})}{ch(\rho)}\right).\square$$

Corollary 1. *Let X be a closed almost quaternionic manifold of dimension $n = 4q$ with q odd. Let E be the the canonical $SO(3)$-bundle whose first Pontrjagin class $p_1(E) \in H^4(X;\mathbb{Z})$ be a torsion class.*

Then $2\hat{A}(X)$ is an integer as well as the numbers

$$\left\{\lambda^k\left(\frac{1}{2}ch(TX\otimes\mathbb{C})\right)\hat{\mathcal{A}}(TX)\right\}[X], k = 1,3,\ldots,q.\square$$

Corollary 2. *Let X be a closed almost quaternionic manifold of dimension $n = 4q$ with q odd. Let the fourth Betti number vanish, $b_4(X;\mathbb{R}) = 0$.*

Then $2\hat{A}(X)$ *is an integer as well as the numbers*

$$\left\{\lambda^k\left(\frac{1}{2}ch(TX\otimes\mathbb{C})\right)\hat{\mathcal{A}}(TX)\right\}[X], k=1,3,\ldots,q.\Box$$

A proof analogous to the one of Theorem 6 yields

Theorem 7. *Let* X *be a closed almost quaternionic manifold of dimension* $n=4q$ *with* q *even. Let* E *be the canonical* $SO(3)$*-bundle with first Pontrjagin class* $p_1(E)\in H^4(X;\mathbb{Z})$. *Let* $W\in K^0(X)$.

Then the $\hat{A}$*-genus of* X *and the following rational numbers are integers:*

$$\left\{ch(W)\lambda^k\left(\frac{ch(TX\otimes\mathbb{C})}{2\cosh(\sqrt{p_1(E)}/2)}\right)\hat{\mathcal{A}}(TX)\right\}[X], k=2,4,\ldots,q.\Box$$

EXAMPLE. In [11] it has been shown that the projective Cayley plane does not admit an almost complex structure. We use Theorem 7 to show that *the projective Cayley plane* $CaP^2=F_4/Spin(9)$ *does not admit an almost quaternionic structure.*

To start we express the Chern character of the complexified tangent bundle of an arbitrary 16-dimensional manifold X in terms of its Pontrajagin classes.

$$\begin{aligned} ch(TX\otimes\mathbb{C}) &= 16+p_1+\frac{1}{12}(p_1^2-2p_2) \\ &+\frac{1}{2^3\cdot 3^2\cdot 5}(p_1^3-3p_1p_2+3p_3) \\ &+\frac{1}{2^6\cdot 3^2\cdot 5\cdot 7}(p_1^4-4p_1^2p_2+4p_1p_3+2p_2^2-4p_4). \end{aligned}$$

This can be shown using standard calculations with characteristic classes. For $X=CaP^2$ cohomology and the Pontrjagin class are known [11], namely there is an element $u\in H^8(CaP^2;\mathbb{Z})$ such that

$$\begin{aligned} H^0(CaP^2;\mathbb{Z}) &= \mathbb{Z}, \\ H^8(CaP^2;\mathbb{Z}) &= \mathbb{Z}\cdot u, \\ H^{16}(CaP^2;\mathbb{Z}) &= \mathbb{Z}\cdot u^2, \\ H^j(CaP^2;\mathbb{Z}) &= 0, \text{ otherwise}, \\ p &= 1+6u+39u^2. \end{aligned}$$

From this we get

$$ch(TCaP^2\otimes\mathbb{C})=16-u+\frac{1}{2^4\cdot 3\cdot 5}u^2.$$

Using the recursive formula for λ^2 we obtain

$$\lambda^2(\frac{1}{2}ch(TCaP^2 \otimes \mathbb{C})) = 28 - \frac{1}{8}u^2.$$

Since CaP^2 is spin and has a metric of positive curvature the $\hat{A}$-genus of CaP^2 must vanish. Hence the total $\hat{A}$-class is of the form

$$\hat{\mathcal{A}}(CaP^2) = 1 + \hat{\mathcal{A}}_2,$$

where $\hat{\mathcal{A}}_2 \in H^8(CaP^2; \mathbb{Q})$. Thus

$$\begin{aligned} \lambda^2(\frac{1}{2}ch(TCaP^2 \otimes \mathbb{C})) \cdot \hat{\mathcal{A}}(CaP^2) &= (28 - \frac{1}{8}u^2)(1 + \hat{\mathcal{A}}_2) \\ &= 28 + 28\hat{\mathcal{A}}_2 - \frac{1}{8}u^2. \end{aligned}$$

Integration yields

$$\{\lambda^2(\frac{1}{2}ch(TCaP^2 \otimes \mathbb{C})) \cdot \hat{\mathcal{A}}(CaP^2)\}[CaP^2] = -\frac{1}{8}.$$

If CaP^2 had an almost quaternionic structure, then by Theorem 7

$$\{\lambda^2(\frac{1}{2}ch(TCaP^2 \otimes \mathbb{C})) \cdot \hat{\mathcal{A}}(CaP^2)\}[CaP^2]$$

would have to be an integer, a contradiction.

In this examples it helps that $H^4(CaP^2; \mathbb{Z}) = 0$ so that we need not worry about the possible values of the first Pontrjagin class of the canonical $SO(3)$-bundle.

REMARK. For *quaternionic* manifolds some of the twisted Dirac operators have already been studied because some of them can be expanded into elliptic complexes similar to the deRham complex for the Euler-deRham operator and the Dolbeault complex for the Cauchy-Riemann operator, see [26], [6].

4. Table

To finish this chapter we collect a few examples of possible structure groups in the table below. The notation has been explained in the previous sections. In particular, α denotes the surjectivity exponent.

G	$\tau \otimes \mathbb{C}$	n	α	generators of $R(G,H)$	fundam. operators
$SO(2m)$	Λ^1	$2m$	$m-1$	$1-\Lambda^1 \pm \cdots + (-1)^m \Lambda^m_+$, $1-\Lambda^1 \pm \cdots + (-1)^m \Lambda^m_-$	two half Euler-deRham op.s
$O(2m)$	Λ^1	$2m$	-	$1-\Lambda^1 \pm \cdots + \Lambda^{2m}$	Euler-deRham op.
$Spin(2m)$	Λ^1	$2m$	0	$\Sigma^+ - \Sigma^-$	Dirac op.
$Spin^c(2m)$	Λ^1	$2m$	0	$(\Sigma^+ - \Sigma^-) \cdot z$	twisted Dirac op.
$Spin^h(2m)$	Λ^1	$2m$	1	$1-\Lambda^1 \pm \cdots + (-1)^m \Lambda^m_+$, $1-\Lambda^1 \pm \cdots + (-1)^m \Lambda^m_-$, $(\Sigma^+ - \Sigma^-) \cdot \rho$	two half Euler-deRham op.s, twisted Dirac op.
$U(m)$	$\Lambda^{1,0} + \Lambda^{0,1}$	$2m$	0	$1-\Lambda^{1,0} \pm \cdots + (-1)^m \Lambda^{m,0}$	Cauchy-Riemann op.
$SU(m)$	$\Lambda^{1,0} + \Lambda^{m-1,0}$	$2m$	0	$1-\Lambda^{1,0} \pm \cdots$ $+(-1)^{m-1}\Lambda^{m-1,0} + (-1)^m$ $= \Sigma^+ - \Sigma^-$	Cauchy-Riemann op. $\cong$ Dirac op.
$Sp(q)Sp(1)$, q even	$\Lambda^{1,0} \cdot \rho$	$4q$	0	$\Sigma^+ - \Sigma^-$	Dirac op.
$Sp(q)Sp(1)$, q odd	$\Lambda^{1,0} \cdot \rho$	$4q$	1	$(\Sigma^+ - \Sigma^-) \cdot \rho$, $(\Sigma^+ - \Sigma^-) \cdot \Lambda^{1,0}$, $(\Sigma^+ - \Sigma^-) \cdot \Lambda^{3,0}$, $\vdots$ $(\Sigma^+ - \Sigma^-) \cdot \Lambda^{q,0}$	$\frac{q+3}{2}$ twisted Dirac op.s
$Sp(q)U(1)$, q even	$\Lambda^{1,0} \cdot (z+\bar{z})$	$4q$	0	$\Sigma^+ - \Sigma^-$	Dirac op.
$Sp(q)U(1)$, q odd	$\Lambda^{1,0} \cdot (z+\bar{z})$	$4q$	0	$(\Sigma^+ - \Sigma^-) \cdot z$	twisted Dirac op.
$Sp(q)$	$2\Lambda^{1,0}$	$4q$	0	$\Sigma^+ - \Sigma^-$	Dirac op.

Tab. 2

III. Immersions

We apply the integrality results of the previous sections to immersion problems. The point is that immersions (e.g. into spin manifolds) yield reductions of the structure group. Then we can apply our construction method for elliptic operators and obtain integrality of certain topological expressions. This yields lower bounds on the codimension. As a special case we obtain a classical theorem of K. H. Mayer.

1. The integrality theorem

Immersions with certain properties yield structure groups for the manifolds under consideration from which we can obtain elliptic symbols and corresponding integrality theorems. This leads to lower bounds for the codimension. We consider closed manifolds with transitive G_{TX}-structure which can be immersed into a spin manifold such that the normal bundle carries a G_ν-structure. From this we deduce integrality of certain topological expressions.

More precisely, let $G_{TX} \subset SO(n)$, $n = 2m$, and $G_\nu \subset SO(k)$ be connected Lie subgroups and let G_{TX} act transitively on $S^{n-1} \subset \mathbb{R}^n$. There is no transitivity assumption on G_ν. For $x_0 \in S^{n-1}$ let $H_{TX} \subset G_{TX}$ denote the isotropy subgroup. We consider the preimages of these groups under the twofold covering mappings $\pi_1 : Spin(n) \to SO(n)$ and $\pi_2 : Spin(k) \to SO(k)$ and we obtain $\hat{G}_{TX} = \pi_1^{-1}(G_{TX}), \hat{H}_{TX} = \pi_1^{-1}(H_{TX})$, and $\hat{G}_\nu = \pi_2^{-1}(G_\nu)$. The two central elements $\pm 1 \in Spin(n)$ or $Spin(k)$ are also contained in $\hat{G}_{TX}$, $\hat{H}_{TX}$, and $\hat{G}_\nu$.

The main result of this section is

Theorem 8. *Let X be an n-dimensional closed manifold with a transitive G_{TX}-structure, $n = 2m$ even. Let X be immersed into an $(n+k)$-dimensional spin manifold Y, e.g. $Y = \mathbb{R}^{n+k}$, such that the normal bundle ν carries a G_ν-structure. Let $\Phi_{TX} : X \to BG_{TX}$ and $\Phi_\nu : X \to BG_\nu$ be the classifying maps for the tangent and the normal bundle.*

Let $\sigma \in R(\hat{G}_{TX}, \hat{H}_{TX})$ and $V \in R(\hat{G}_\nu)$ such that $(-1,-1)$ acts trivially on $\sigma \cdot V$. Then

$$\left\{ \Phi_\nu^* \left((\pi_2^*)^{-1} ch(V) \right) \cdot \Phi_{TX}^* \left(\frac{(\pi_1^*)^{-1} ch(\sigma)}{e|BG_{TX}} \right) \cdot \hat{\mathcal{A}}(TX)^2 \right\} [X]$$

is an integer.

As in Theorem 3, $e \in H^{2m}(BSO(2m);\mathbb{Q})$ is the universal Euler class, $ch : R(G) \to H^*(BG;\mathbb{Q})$ is the universal Chern character, and $\hat{\mathcal{A}}(TX)$ is the total $\hat{A}$-class of X.

Proof. X has a G_{TX}-structure and thus also a $G_{TX} \times G_\nu$-structure. Since X is immersed in a spin manifold this $G_{TX} \times G_\nu$-structure can be lifted to a G-structure where G is the preimage of $G_{TX} \times G_\nu \subset SO(n+k)$ in $Spin(n+k)$.

We have $G = \frac{\hat{G}_{TX} \times \hat{G}_\nu}{\mathbb{Z}_2}$ where $\mathbb{Z}_2 = \{(1,1),(-1,-1)\}$. Since $\mathbb{Z}_2$ acts trivially on $\sigma \cdot V$ by assumption, $\sigma \cdot V$ can be regarded as an element of $R(G)$. Because of $\sigma \in R(\hat{G}_{TX}, \hat{H}_{TX})$ the virtual representation $\sigma \cdot V$ is actually contained in $R(G,H)$ where $H = \frac{\hat{H}_{TX} \times \hat{G}_\nu}{\mathbb{Z}_2}$ is the isotropy subgroup of G.

Theorem 3 yields the assertion. □

In the following two tables we list a few examples. We can consider G_{TX} and G_ν separately and combine them arbitrarily. We choose the virtual modules σ and V such that $-1 \in \hat{G}_{TX}$ and $-1 \in \hat{G}_\nu$ act via multiplication by -1. Then $\mathbb{Z}_2$ acts trivially on $\sigma \cdot V$.

G_{TX}	σ	$\Phi^*_{TX}\left(\frac{(\pi_1^*)^{-1}ch(\sigma)}{e\|BG_{TX}} \cdot \hat{\mathcal{A}}^2\right)$
$SO(n)$	$\Sigma^+ - \Sigma^-$	$\hat{\mathcal{A}}(TX)$
$U(m)$	$(1 - \Lambda^{1,0} \pm \cdots) \cdot (\Lambda^{m,0})^{1/2}$	$(-1)^m e^{c_1(TX)/2} \cdot \mathcal{TD}(TX)$
$Sp(q)Sp(1)$, q even	$(\Sigma^+ - \Sigma^-) \cdot \rho$	$2cosh(\sqrt{p_1(E)}/2) \cdot \hat{\mathcal{A}}(TX)$
	$(\Sigma^+ - \Sigma^-) \cdot \Lambda^{j,0}$, j odd	$\lambda^j\left(\frac{ch(TX \otimes \mathbb{C})}{2cosh(\sqrt{p_1(E)}/2)}\right) \cdot \hat{\mathcal{A}}(TX)$
$Sp(q)Sp(1)$, q odd	$\Sigma^+ - \Sigma^-$	$\hat{\mathcal{A}}(TX)$
	$(\Sigma^+ - \Sigma^-) \cdot \Lambda^{j,0}$, j even	$\lambda^j\left(\frac{ch(TX \otimes \mathbb{C})}{2cosh(\sqrt{p_1(E)}/2)}\right) \cdot \hat{\mathcal{A}}(TX)$
$Sp(q)U(1)$, q even	$(\Sigma^+ - \Sigma^-) \cdot z$	$e^{c_1(L)/2} \cdot \hat{\mathcal{A}}(TX)$
$Sp(q)U(1)$, q odd	$\Sigma^+ - \Sigma^-$	$\hat{\mathcal{A}}(TX)$

Tab. 3

Some explanations: L denotes the canonical $U(1)$-bundle, E the canonical $SO(3)$-bundle, c_1 the first Chern class, p_1 the first Pontrjagin class, and $\mathcal{TD}$ the total *Todd class*. The group $\hat{U}(m)$ is a twofold cover of $U(m)$. It has a module whose square is $\Lambda^{m,0}$. We denote it by $(\Lambda^{m,0})^{1/2}$.

The structure group G_ν of the normal bundle need not act transitively on the unit sphere. Thus many choices of G_ν are possible. For the sake of simplicity we restrict ourselves to the cases $G_\nu = SO(k)$ and $G_\nu = U(l)$.

G_ν	V	$\Phi_\nu^*\left((\pi_2^*)^{-1}ch(V)\right)$
$SO(k)$, $k=2l$ even	$\Sigma^+ - \Sigma^-$	$e(\nu)\cdot\hat{\mathcal{A}}(\nu)^{-1}$
	$\Sigma^+ + \Sigma^-$	$2^l \cdot \mathcal{M}(\nu)$
$SO(k)$, $k=2l+1$ odd	Σ	$2^l \cdot \mathcal{M}(\nu)$
$U(l)$	$(\Lambda^{l,0})^{1/2}$	$e^{c_1(\nu)/2}$

Tab. 4

Here $\mathcal{M}(\nu)$ is the multiplicative class for the power series $cosh(x/2)$, i.e. if we write the Pontrjagin class $p(\nu)$ formally as $p(\nu) = \prod_{j=1}^{l}(1+x_j^2)$, then

$$\mathcal{M}(\nu) = \prod_{j=1}^{l} cosh(x_j/2). \tag{8}$$

Let us call $\mathcal{M}$ the *Mayer class.*

2. Mayer's theorem

As an example we combine $G_{TX} = SO(n)$ with $G_\nu = SO(k)$ and we obtain

Theorem 9. (K.H. Mayer [20, Satz 3.2])
Let X be an n-dimensional closed oriented manifold, $n = 2m$ even, which can be immersed in an $(n+k)$-dimensional spin manifold with normal bundle ν.

If $k = 2l$ is even, then the following expressions are integers:

$$\left\{e(\nu)\hat{\mathcal{A}}(\nu)^{-1}\hat{\mathcal{A}}(TX)\right\}[X]$$

and

$$2^l\left\{\mathcal{M}(\nu)\hat{\mathcal{A}}(TX)\right\}[X].$$

If $k = 2l+1$ is odd, then

$$2^l\left\{\mathcal{M}(\nu)\hat{\mathcal{A}}(TX)\right\}[X]$$

is an integer. □

Of course, one can still twist by arbitrary coefficient bundles and one can examine for which n and k the resulting elliptic operators are quaternionic, thus improving the integrality result by a factor 2.

If the target space is Euclidean space $\mathbb{R}^{n+k}$, then $\mathcal{M}(TX)\cdot\mathcal{M}(\nu) = \mathcal{M}(TX \oplus \nu) = \mathcal{M}(T\mathbb{R}^{n+k}|_X) = 1$ because the Mayer class $\mathcal{M}$ is multiplicative. Thus $\mathcal{M}(\nu) = \mathcal{M}(TX)^{-1}$ and integrality of $2^l\left\{\mathcal{M}(TX)^{-1}\hat{\mathcal{A}}(TX)\right\}[X]$ yields a lower bound on l (and hence on the codimension k) in terms of Pontrjagin numbers of X. In [20] one can find applications to immersions of projective spaces into Euclidean space.

3. Immersions with special structure

We have seen that Mayer's theorem follows from Theorem 8 by combing $G_{TX} = SO(n)$ and $G_\nu = SO(k)$. By imposing additional structure on X and/or ν one can obtain many more integrality results of this kind. For example, combining $G_{TX} = Sp(q)Sp(1)$, q even, and $G_\nu = U(l)$ yields

Theorem 10. *Let X be a $4q$-dimensional compact almost-quaternionic manifold, q even, immersed into a $(4q+2l)$-dimensional spin manifold. Assume that the normal bundle ν carries a complex structure. Let E be the canonical $SO(3)$-bundle of X.*

Then the following expressions are integers:

$$2\left\{e^{c_1(\nu)/2}cosh(\sqrt{p_1(E)}/2)\hat{\mathcal{A}}(TX)\right\}[X],$$

$$\left\{e^{c_1(\nu)/2}\lambda^j\left(\frac{ch(TX\otimes\mathbb{C})}{2cosh(\sqrt{p_1(E)}/2)}\right)\hat{\mathcal{A}}(TX)\right\}[X], j \text{ odd.}\square$$

Of course, many more combinations of G_{TX} and G_ν are possible. With this method one can also study immersions into other kinds of manifolds rather than spin manifolds. For example, one can study immersions into spin^c manifolds. Then one has to regard modules of the preimages of G_{TX} and G_ν in $Spin^c(n \text{ or } k)$. In the case of $G_{TX} = SO(n)$ and $G_\nu = SO(k)$ one obtains K.H. Mayer's second integrality theorem [20, Satz 3.1]. Again, many other choices for G_{TX} and G_ν are possible.

This technique allows to derive topological restrictions against immersions for such manifolds for which one is able to explicitely compute the characteristic numbers involved. In particular, it should give interesting non-immersion results for many homogeneous spaces.

References

[1] M. F. Atiyah, *Vector bundles and the Künneth formula*, Topology **1** (1962), *245-248*

[2] M. F. Atiyah, R. Bott, A. Shapiro, *Clifford modules*, Topology **3** (1964), *3-38*

[3] M. F. Atiyah, F. Hirzebruch, *Riemann-Roch theorems for differentiable manifolds*, Bull. AMS **65** (1959), *276-281*

[4] M. F. Atiyah, I. M. Singer, *The index of elliptic operators III*, Ann. of Math. **87** (1968), *546-604*

[5] C. Bär, *Elliptische Operatoren und Darstellungstheorie kompakter Gruppen*, Bonner Math. Schr. **248** (1993)

[6] R. J. Baston, *Quaternionic complexes*, J. of Geom. and Phys. **8** (1992), *29-52*

[7] A. L. Besse, *Einstein manifolds*, Springer-Verlag, Berlin, Heidelberg 1987

[8] E. Bonan, *Sur les G-structures de type quaternionien*, Cahiers Top. et Géom. Diff. **9** (1967), *389-463*

[9] A. Borel, *Some remarks about Lie groups transitive on spheres and tori*, Bull. AMS **55** (1949), *580-587*

[10] A. Borel, *Le plan projectif des octaves et les sphères comme espaces homogènes*, C. R. Acad. Sc. Paris **230** (1950), *1378-1380*

[11] A. Borel, F. Hirzebruch, *Characteristic classes and homogeneous spaces I*, Am. J. Math. **80** (1958), *458-538*

[12] T. Bröcker, T. tom Dieck, *Representations of compact Lie groups*, Springer-Verlag, New York 1985

[13] D. S. Freed, K. K. Uhlenbeck, *Instantons and four-manifolds*, Springer-Verlag, New York, Berlin, Heidelberg, 2. Aufl. 1991

[14] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag, New York, Heidelberg, Berlin 1978

[15] N. Hitchin, *Harmonic Spinors*, Adv. Math. **14** (1974), *1-55*

[16] D. Husemoller, *Fibre bundles*, Springer-Verlag, New York, Heidelberg, Berlin 1966

[17] M. Karoubi, *K-Theory, An Introduction*, Springer-Verlag, Berlin 1978

[18] H. B. Lawson, M.-L. Michelsohn, *Spin geometry*, Princeton University Press, Princeton 1989

[19] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris **257** (1963), *7-9*

[20] K. H. Mayer, *Elliptische Differentialoperatoren und Ganzzahligkeitssätze für charakteristische Zahlen*, Topology **4** (1965), *295-313*

[21] D. Montgomery, H. Samelson, *Transformation groups of spheres*, Ann. of Math. **44** (1943), *454-470*

[22] M. Nagase, *Spin^q structures*, J. Math. Soc. Japan **47** (1995), *93-119*

[23] M. Nagase, *Spin^q, Twistor and Spin^c*, Preprint Saitama University 1993

[24] S. M. Salamon, *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), *143-171*

[25] S. M. Salamon, *Differential geometry of quaternionic manifolds*, Ann. Sc. Éc. Norm. Sup. **19** (1986), *31-55*

[26] S. M. Salamon, *Index theory and quaternionic Kähler manifolds*, Preprint 1992

[27] J. Tits, *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen*, Springer-Verlag, Berlin, Heidelberg, New York 1967

Author's address:

Mathematisches Institut
Universität Freiburg
Hebelstr. 29
79104 Freiburg
Germany

e-mail: `baer@arcade.mathematik.uni-freiburg.de`