CHAPTER 1

Preliminaries

1.1. The Spectrum of the Dirac Operator on Compact Manifolds

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1.1.1. Analytic Basics. Let M denote a compact n-dimensional Riemannian spin manifold without boundary. The Dirac operator maps smooth spinor fields to smooth spinor fields,

$$D : C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M).$$

A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of D if there exists a nontrivial spinor field $\varphi \in C^\infty(M, \Sigma M)$ (called *eigenspinor*) such that

$$D\varphi = \lambda\varphi. \tag{1.1}$$

Why are eigenvalues of the Dirac operator interesting? The physical interest stems from the observation that if (1.1) holds, then the time-dependent spinor field $\Phi(t, x) := e^{it\lambda} \cdot \varphi(x)$ satisfies the physical Dirac equation

$$\frac{\partial \Phi}{\partial t} = iD\Phi$$

on the space-time $\mathbb{R} \times M$. Hence λ can be interpreted as the frequency or, equivalently, as the energy of the particle whose wave function is Φ.

The standard theory of self-adjoint elliptic differential operators (see e. g. [**LM89**, Chapter III]) now tells us the following:

- All eigenvalues of D are real, $\lambda \in \mathbb{R}$.
- The eigenvalues form a discrete subset of $\mathbb{R}$, unbounded from above and from below,

$$-\infty \leftarrow \cdots < \lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < \cdots \to \infty.$$

- The multiplicities of the eigenvalues are finite,

$$\dim(E_\lambda) < \infty$$

  where $E_\lambda = \{\varphi \in C^\infty(M, \Sigma M) \mid D\varphi = \lambda\varphi\}$ is the corresponding eigenspace.
- The space of finite linear combinations of eigenspinors $\bigoplus_{j\in\mathbb{Z}} E_{\lambda_j}$ is dense in $C^\infty(M, \Sigma M)$ with respect to the L^2-norm.
- The eigenspaces for different eigenvalues λ and μ are perpendicular, $E_\lambda \perp E_\mu$, with respect to the L^2-scalar product.

Here the L^2-scalar product on the space of all spinor fields is defined by

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle \, dV$$

and the L^2-norm is the corresponding norm, $\|\varphi\| = \sqrt{(\varphi, \varphi)}$. The totality of all eigenvalues of D together with their multiplicities is called the *spectrum* of D on M or the *Dirac spectrum* of M.

1.1.2. Examples for Explicit Computation. Now given a compact Riemannian spin manifold M what is its Dirac spectrum? In general, it is totally hopeless to try to explicitly compute even one single eigenvalue. Only manifolds of a high degree of symmetry are accessible for explicit computation.

1.1.2.1. *The circle.* Already the simplest manifold in question, the circle $M = S^1 = \mathbb{R}/2\pi\mathbb{Z}$, is interesting to consider. Give M the standard (counter-clockwise) orientation. At each point $t \in M$ there is exactly one positively oriented orthonormal tangent basis consisting of the unique positively oriented unit tangent vector. Hence the total space of the oriented frame bundle is again equal to a circle, $P_{SO}(M) = S^1$, the bundle mapping onto M given by the identity.

Now we observe that M has two different spin structures. Recall that a spin structure is a two-fold covering of $P_{SO}(M)$. We can either choose the trivial two-fold covering $P_1 := S^1 \sqcup S^1$ with $P_1 \to P_{SO} = S^1$ given by two copies of the identity or we may choose $P_2 := S^1$ and $P_2 \to P_{SO} = S^1$ given by the mapping $z \mapsto z^2$ in complex notation.

Let us first analyze the situation for the trivial spin structure P_1. In one dimension spinor space is simply $\Sigma_1 = \mathbb{C}$. Clifford multiplication by the positively oriented unit vector is multiplication by i. Associating this to the trivial spin structure yields the trivial complex vector bundle of rank one over S^1. In other words, spinor fields are the same as complex-valued functions. With respect to the standard coordinate t of S^1 the Christoffel symbol of the Levi-Civita connection vanishes and thus the Dirac operator is

$$D = i\frac{d}{dt}.$$

Now the Fourier decomposition of functions on S^1 is exactly the eigenspace decomposition for D, the eigenvalues are $\lambda_k = k$ and the corresponding eigenfunctions are $\varphi_k(t) = e^{-ikt}$. Hence the spectrum is $\mathbb{Z}$, each eigenvalue having multiplicity 1.

In order to discuss the nontrivial spin structure P_2 let us rephrase the observations for the trivial spin structure as follows. Spinor fields were nothing but complex-valued functions on S^1. Equivalently, we can say that spinor fields on S^1 with respect to P_1 are given by periodic complex-valued functions on $\mathbb{R}$ with period 2π. When we replace P_1 by the nontrivial spin structure P_2 the following will change: When we move around $M = S^1$ one time, then we can lift this path continuously to P_1 and we will return to the same point in P_1. But when we do this in P_2 we will return to the opposite point. Thus spinor fields will now correspond to *anti-periodic* complex-valued functions on $\mathbb{R}$:

$$\varphi(t + 2\pi) = -\varphi(t).$$

Nonetheless a Fourier decomposition is also available in this situation, the eigenfunctions are now $\varphi_k(t) = e^{-i(k+1/2)t}$ and the eigenvalues are $\lambda_k = k + 1/2$. Thus the spectrum is $\mathbb{Z} + \frac{1}{2}$, each eigenvalue again having multiplicity 1.

It is remarkable that this simple example already shows that the spectrum of the Dirac operator will in general depend on the choice of spin structure.

Using Fourier analysis one can treat n-dimensional flat tori in a similar fashion, see [**Fri84**].

1.1.2.2. *Homogeneous Spaces.* Now we turn to spaces which have so many symmetries that they look the same at all points. Such space are called *homogeneous.* More precisely, let G be a compact Lie group acting transitively by orientation preserving isometries on our compact n-dimensional Riemannian spin manifold M. Choose $x_0 \in M$. Then we can write M in the form

$$M = G/H$$

where $H \subset G$ is the subgroup of all elements keeping x_0 fixed. The differentials of the isometries in G act on $P_{SO}(M)$. We assume here that this action lifts to the spin structure of M. Then there is a generalization of the Fourier decomposition of the previous section which we now describe.

The idea is to regard the space of spinor fields $C^\infty(M, \Sigma M)$ (or rather its Hilbert space completion with respect to the L^2-scalar product) as a representation space for G and to decompose it into irreducible subrepresentations. Denote the set of all equivalence classes of irreducible unitary representations of G by $\widehat{G}$. For every $\gamma \in \widehat{G}$ denote the corresponding representation space by V_γ. It is a fact that $\dim(V_\gamma) < \infty$. By the so-called *Frobenius reciprocity* there is a finite dimensional space W_γ such that

$$\bigoplus_{\gamma \in \widehat{G}} V_\gamma \otimes W_\gamma$$

embeds densely into $C^\infty(M, \Sigma M)$ as a subrepresentation. The group G acts on a summand $V_\gamma \otimes W_\gamma$ by $\gamma \otimes id$. The Dirac operator commutes with this G-action and leaves the decomposition invariant. It acts on a summand $V_\gamma \otimes W_\gamma$ by $id \otimes D_\gamma$ where the endomorphism D_γ of W_γ can be computed explicitly [**Bär92a**, Prop. 1]. Hence determining the spectrum of D is now the same as computing the eigenvalues of D_γ on W_γ for all $\gamma \in \widehat{G}$. We have thus reduced the problem to finite dimensional linear algebra.

In practice this linear algebra can still be quite hard. Moreover, one has to deal with an infinite number of finite dimensional eigenvalue problems, one for each $\gamma \in \widehat{G}$. This can be carried out only if one finds some uniform pattern in all the endomorphisms D_γ. Formulas get a lot simpler if G/H is a symmetric space. The Dirac spectrum for many of them is known by now. In the following table we collect the compact Riemannian spin manifolds for which the Dirac spectrum has been computed. Not all of them are symmetric but they are all locally homogeneous. Some of them (spheres and complex projective spaces) have been studied with various different methods.

$\mathbb{R}^n/\Gamma$	flat tori	**[Fri84]**
$\mathbb{R}^3/\Gamma$	3-dim. Bieberbach manifolds	**[Pfä00]**
S^n	spheres of constant curvature	**[Sul79]**, **[Bär96a]**, **[Tra93]**, **[CH96]**
S^n/Γ	spherical space forms	**[Bär96a]**
S^{2m+1}	spheres with Berger metrics	**[Hit74]** for $m = 1$ **[Bär96b]** for general m
$S^3/\mathbb{Z}_k$	3-dim. lens spaces with Berger metric	**[Bär92a]**
G	simply connected compact Lie groups	**[Feg87]**
$\mathbb{C}P^{2m-1}$	complex projective spaces	**[CFG89, CFG94]**, **[SS93]**,**[AB98]**
$\mathbb{H}P^m$	quaternionic projective spaces	**[Bun91b]** for $m = 2$ **[Mil92]** for general m
$Gr_2(\mathbb{R}^{2m})$	certain real Grassmannians	**[Str80b]** for $m = 3$ **[Str80a]** for general m
$Gr_{2p}(\mathbb{R}^{2m})$	certain real Grassmannians	**[See97]**
$Gr_2(\mathbb{C}^{m+2})$	certain complex Grassmannians	**[Mil98]**
$G_2/SO(4)$		**[See97, See99]**
H^3/Γ	3-dim. Heisenberg manifolds	**[AB98]**

TABLE 1

1.1.3. Eigenvalue Estimates. Even if it is not possible to explicitly compute the Dirac spectrum of a manifold M one may still hope to get some control on the eigenvalues in terms of geometric data. For large classes of manifolds one can at least give geometric estimates. Estimating the modulus of an eigenvalue from above or from below turns out to be quite different stories.

1.1.3.1. *Lower Bounds.* We will start by estimating the Dirac eigenvalues from below on manifolds with positive scalar curvature. A naïve bound can be obtained as follows: Recall the Schrödinger-Lichnerowicz formula for the square of the Dirac operator

$$D^2 = \nabla^*\nabla + \frac{\mathrm{Scal}}{4}. \tag{1.2}$$

Now assume $\mathrm{Scal} \geq S$ for some positive constant S and let $D\varphi = \lambda\varphi$. We compute

$$\begin{aligned} \lambda^2(\varphi,\varphi) &= (D^2\varphi,\varphi) = \left(\left(\nabla^*\nabla + \frac{\mathrm{Scal}}{4}\right)\varphi,\varphi\right) \\ &\geq (\nabla\varphi,\nabla\varphi) + \frac{S}{4}(\varphi,\varphi) \geq \frac{S}{4}(\varphi,\varphi). \end{aligned}$$

Therefore each eigenvalue λ of the Dirac operator must satisfy

$$\lambda^2 \geq \frac{S}{4}.$$

What is naïve about this estimate? It is not optimal in sense that there are no manifolds where equality in the estimate is attained. The reason is that we wasted too much when we estimated $(\nabla\varphi,\nabla\varphi)$ by 0. The right approach was found by

Friedrich in [**Fri80**]. One defines a new connection for spinors by

$$\widehat{\nabla}_X\varphi := \nabla_X\varphi + \frac{\lambda}{n}X\cdot\varphi.$$

Now one computes

$$\widehat{\nabla}^*\widehat{\nabla} \;=\; \nabla^*\nabla - 2\frac{\lambda}{n}D + \frac{\lambda^2}{n} \;=\; D^2 - \frac{\mathrm{Scal}}{4} - 2\frac{\lambda}{n}D + \frac{\lambda^2}{n}.$$

If φ is an eigenspinor for the eigenvalue λ, then

$$\begin{aligned} 0 &\le (\widehat{\nabla}\varphi,\widehat{\nabla}\varphi) \;=\; (\widehat{\nabla}^*\widehat{\nabla}\varphi,\varphi) \\ &= \lambda^2(\varphi,\varphi) - \frac{1}{4}(\mathrm{Scal}\,\varphi,\varphi) - 2\frac{\lambda^2}{n}(\varphi,\varphi) + \frac{\lambda^2}{n}(\varphi,\varphi) \\ &\le \left(\frac{n-1}{n}\lambda^2 - \frac{S}{4}\right)(\varphi,\varphi) \end{aligned}$$

and therefore

$$\lambda^2 \;\ge\; \frac{n}{4(n-1)}S.$$

This estimate is sharp because equality is attained e. g. for spheres. In case M is a Kähler manifold or quaternionic-Kähler it is possible to further improve the estimate. The basic idea is still the same; one tries to find refined Schrödinger-Lichnerowicz formulas in order to loose less in the estimate. Technically things become a lot more complicated. We summarize the results:

THEOREM 1.1. *Let M be a compact Riemannian spin manifold of (real) dimension n. Suppose the scalar curvature satisfies Scal $\ge S > 0$. Then all Dirac eigenvalues λ satisfy*

$$\lambda^2 \;\ge\; c_n\frac{S}{4}$$

where

a) (Friedrich [**Fri80**]) *In general,*
$$c_n = \frac{n}{n-1}.$$
b) (Kirchberg [**Kir86, Kir90**]) *If M is Kähler, then*
$$c_n = \begin{cases} \frac{n+2}{n}, & \text{if } \frac{n}{2} \text{ is odd} \\ \frac{n}{n-2}, & \text{if } \frac{n}{2} \text{ is even.} \end{cases}$$
c) (Kramer, Weingart, Semmelmann [**KSW99**]) *If M is quaternionic-Kähler, then*
$$c_n = \frac{n+12}{n+8}.$$

The estimate is sharp in all cases. Recalling the proof of the estimate in the general case we see that if a Dirac eigenvalue satisfies $\lambda^2 = \frac{n}{4(n-1)}S$, then the corresponding eigenspinor must satisfy $\widehat{\nabla}\varphi = 0$, or equivalently,

$$\nabla_X\varphi \;=\; -\frac{\lambda}{n}X\cdot\varphi$$

for all tangent vectors X. Such spinor fields are called *Killing spinors.* The Killing spinor equation is overdetermined, thus a generic manifold will not have nontrivial Killing spinors. The simplest example of a manifold with Killing spinors is the sphere S^n with the standard metric of constant curvature. In even dimensions $n \neq 6$ this is the only example. In the other dimensions there are different geometric types of manifolds admitting nontrivial Killing spinors. A complete classification of those types has been achieved by Bär in [**Bär93**] after previous work on construction methods for examples and partial classification results in [**FG85, CGLS86, Hij86a, Fra87, FK88, FK89, FK90, Gru90, BFGK91**].

Complex-projective space $\mathbb{CP}^{\frac{n}{2}}$ is a Kähler manifold and it is spin if its complex dimension $\frac{n}{2}$ is odd. In this case it indeed provides an example where equality is attained in case b). If $\frac{n}{2} \equiv 1 \mod 4$, then this is the only example but if $\frac{n}{2} \equiv 3 \mod 4$, then equality is attained if and only if M is the twistor space of a quaternionic-Kähler manifold. For example, $\mathbb{CP}^{\frac{n}{2}}$ is the twistor space of quaternionic-projective space $\mathbb{HP}^{\frac{n}{4}-\frac{1}{2}}$.

If $\frac{n}{2}$ is even, then complex-projective space is not spin and this is the reason why the case distinction is necessary for Kähler manifolds. The manifolds for which equality holds in the estimate are then the twisted products of T^2 and twistor spaces. These results were achieved by Moroianu [**Mor95, Mor99**] after preliminary work by Kirchberg, Lichnerowicz, and Friedrich [**Kir88, Lic90, Fri93**].

For quaternionic-Kähler manifolds Kramer, Semmelmann, and Weingart have shown [**KSW98**] that equality is attained only for quaternionic-projective space $\mathbb{HP}^{\frac{n}{4}}$.

The estimates obtained so far yield nontrivial results only if the manifold has strictly positive scalar curvature. One may suspect that if the scalar curvature is negative on a small part of the manifold and sufficiently large on the rest one should still be able to obtain a positive lower bound. In a way this is true. To formulate the result suppose $n \geq 3$ and denote the *Yamabe operator* acting on smooth functions on M by

$$Y = 4\frac{n-1}{n-2}\Delta + \mathrm{Scal}.$$

Here Δ is the usual Laplace-Beltrami operator. Let $\mu_1(Y)$ be the smallest eigenvalue of Y.

THEOREM 1.2 (Hijazi [**Hij86b**]). *Let M be a compact Riemannian spin manifold of dimension $n \geq 3$. Then all Dirac eigenvalues λ of M satisfy*

$$\lambda^2 \geq \frac{n}{n-1}\frac{\mu_1(Y)}{4}.$$

Note that $\mu_1(Y) \geq \min_M \mathrm{Scal}$ so that Theorem 1.2 implies Friedrich's estimate. But $\mu_1(Y)$ is still positive if there is a little bit of negative scalar curvature. The equality case turns out to be the same as in Friedrich's case; equality holds in Hijazi's estimate if and only if the corresponding eigenspinor is a Killing spinor.

Hijazi's original proof combined Friedrich's modification of the spinor connection with a clever conformal change of the metric. There is now a completely different proof available based on a so-called refined Kato inequality [**CGH00**, Prop. 3.4]. This is a general principle allowing to estimate spectral data for operators acting on sections in a vector bundle (here D^2) against spectral data of a comparison operator acting on functions (here Y).

There is one manifold where one can give a nontrivial estimate for the Dirac eigenvalues without any geometric assumption. This is the 2-dimensional sphere.

THEOREM 1.3 (Bär [**Bär91, Bär92b**]). *Let $M = S^2$ be equipped with any Riemannian metric. Then all Dirac eigenvalues satisfy*

$$\lambda^2 \quad \geq \quad \frac{4\pi}{\text{area}(M)}.$$

Equality holds for the smallest eigenvalue if and only if M has constant curvature.

This estimate was conjectured by Lott in [**Lot86**] where he had shown that for some positive constant C the estimate $\lambda^2 \geq C/\text{area}(M)$ must hold. It turned out that Theorem 1.3 can also be deduced from Hijazi's results, see [**Hij91**]. Interestingly, there is a similar estimate for the Laplace operator on the 2-sphere going in the opposite direction. Hersch [**Her70**] proved that the first positive Laplace eigenvalue $\mu_1(\Delta)$ of S^2 equipped with any Riemannian metric satisfies

$$\mu_1(\Delta) \quad \leq \quad \frac{8\pi}{\text{area}(M)}.$$

It is believed that an estimate like in Theorem 1.3 is impossible for any manifold of dimension $n \geq 3$. More precisely, it is conjectured that given any compact spin manifold M of dimension $n \geq 3$ and given a spin structure on M, then one can find a Riemannian metric such that 0 is a Dirac eigenvalue. Eigenspinors for the eigenvalue 0 are called *harmonic spinors*. This conjecture is known to be true in dimensions $n \equiv 0, 1, 3, 7 \mod 8$ [**Hit74, Bär96b**]. Moreover, one knows that S^n carries metrics with harmonic spinors also in dimensions $n \equiv 0 \mod 4$ [**See01**, Thm. 3.27].

The situation for surfaces of higher genus is more subtle. The spin structure is then no longer unique. It turns out that there are two types of spin structures on surfaces which can be distinguished by the so-called *Arf-invariant*. For one type (Arf $= -1$) one has metrics with harmonic spinors. Hence a result like Theorem 1.3 is out of question. For the other type (Arf $= 1$) there is a lower estimate similar to the one in Theorem 1.3 but it is not sharp [**AB02**, Thms. 5.1 and 6.1].

Let us now sketch the proof of Theorem 1.3.

PROOF. We modify the spinor connection once more. Let $f : M \to \mathbb{R}$ be a smooth function. We set $\widetilde{\nabla}_X\varphi := \nabla_X\varphi + \frac{\lambda}{2}X \cdot \varphi - X \cdot \nabla_{\text{grad}f}\varphi - 2\langle\text{grad}f, X\rangle\varphi$. Let K be the Gauss curvature of M. An elementary but tedious calculation yields

$$\begin{aligned} \widetilde{\nabla}^*(e^{-2f}\widetilde{\nabla}\varphi) \quad &= \quad e^{-2f}\Big\{D^2 - \frac{K}{2} - \lambda D + \frac{\lambda^2}{2} + \Delta f - 2\,\text{grad}f \cdot D + \lambda\,\text{grad}f \cdot \\ &\qquad -2\left[\nabla_{\text{grad}f} + (\nabla_{\text{grad}f})^*\right]\Big\}\varphi. \end{aligned} \tag{1.3}$$

Now let λ be a Dirac eigenvalue and φ a corresponding eigenspinor, $D\varphi = \lambda\varphi$. By a partial integration and by (1.3) we obtain

$$\begin{aligned} 0 \quad \leq \quad & \|e^{-f}\widetilde{\nabla}\varphi\|^2 \;=\; (\widetilde{\nabla}^* e^{-2f}\widetilde{\nabla}\varphi, \varphi) \\ = \quad & \int_M e^{-2f}\Big\{\lambda^2|\varphi|^2 - \frac{K}{2}|\varphi|^2 - \lambda^2|\varphi|^2 + \frac{\lambda^2}{2}|\varphi|^2 + (\Delta f)|\varphi|^2 - 2\lambda\langle \mathrm{grad} f\cdot\varphi, \varphi\rangle \\ & +\lambda\langle \mathrm{grad} f\cdot\varphi,\varphi\rangle - 2\langle[\nabla_{\mathrm{grad} f} + (\nabla_{\mathrm{grad} f})^*]\varphi,\varphi\rangle\Big\} dV \\ = \quad & \int_M e^{-2f}\Big\{\frac{\lambda^2}{2}|\varphi|^2 - \frac{K}{2}|\varphi|^2 + (\Delta f)|\varphi|^2 - \lambda\langle \mathrm{grad} f\cdot\varphi,\varphi\rangle \\ (1.4) \qquad & -2\langle[\nabla_{\mathrm{grad} f} + (\nabla_{\mathrm{grad} f})^*]\varphi,\varphi\rangle\Big\} dV. \end{aligned}$$

For the last term we get

$$\begin{aligned} \int_M e^{-2f}\langle[\nabla_{\mathrm{grad} f} + (\nabla_{\mathrm{grad} f})^*]\varphi,\varphi\rangle dV \quad &= \quad (\nabla_{\mathrm{grad} f}\varphi, e^{-2f}\varphi) + ((\nabla_{\mathrm{grad} f})^*\varphi, e^{-2f}\varphi) \\ &= \quad (\nabla_{\mathrm{grad} f}\varphi, e^{-2f}\varphi) + (\varphi, \nabla_{\mathrm{grad} f}(e^{-2f}\varphi)) \\ &= \quad \int_M \langle \mathrm{grad} f, \mathrm{grad}\langle\varphi, e^{-2f}\varphi\rangle\rangle dV \\ (1.5) \qquad\qquad &= \quad \int_M (\Delta f) e^{-2f}|\varphi|^2 dV. \end{aligned}$$

Since the term $\langle \mathrm{grad} f\cdot\varphi,\varphi\rangle = -\langle\varphi, \mathrm{grad} f\cdot\varphi\rangle = \overline{\langle \mathrm{grad} f\cdot\varphi,\varphi\rangle}$ is purely imaginary and all other terms are real we conclude from (1.4) and (1.5)

$$0 \leq \int_M e^{-2f}\Big\{\frac{\lambda^2}{2}|\varphi|^2 - \frac{K}{2}|\varphi|^2 - (\Delta f)|\varphi|^2\Big\} dV. \tag{1.6}$$

Now we will make an optimal choice for the function f. Define the function

$$h(x) := \frac{K(x)}{2} - \frac{1}{2\,\mathrm{area}(M)}\int_M K(y)\, dV(y)$$

and observe

$$\int_M h(x) dV(x) = 0.$$

This means that h is perpendicular to the constant functions with respect to the L^2-scalar product. The constant functions form the kernel of the Laplace operator, a self-adjoint differential operator. Thus h (and hence $-h$) lies in the image of the Laplace operator, i. e. there exists a smooth function f such that

$$\Delta f = -h.$$

Plugging this into (1.6) yields by the Gauss-Bonnet theorem

$$\begin{aligned} 0 \quad \leq \quad & \int_M e^{-2f}\Big\{\frac{\lambda^2}{2} - \frac{1}{2\,\mathrm{area}(M)}\int_M K(y)\, dV(y)\Big\}\ |\varphi|^2\ dV \\ = \quad & \frac{1}{2}\int_M e^{-2f}\Big\{\lambda^2 - \frac{2\pi\chi(M)}{\mathrm{area}(M)}\Big\}\ |\varphi|^2\ dV \\ = \quad & \frac{1}{2}\int_M e^{-2f}\Big\{\lambda^2 - \frac{4\pi}{\mathrm{area}(M)}\Big\}\ |\varphi|^2\ dV \end{aligned}$$

We have proved the estimate

$$\lambda^2 \geq \frac{4\pi}{\text{area}(M)}.$$

It remains to discuss the equality case. If the curvature K is constant, then one easily checks that equality holds. Conversely, if there is an eigenvalue satisfying

$$\lambda^2 = \frac{4\pi}{\text{area}(M)},$$

then the previous proof shows that the corresponding eigenspinor φ must satisfy $\widetilde{\nabla}_X\varphi = 0$, i. e.

$$\nabla_X\varphi = -\frac{\lambda}{2}X\cdot\varphi - \text{grad}f\cdot X\cdot\varphi.$$

Using this one easily computes for the spinorial curvature

$$R^\Sigma(e_1,e_2)\varphi = (\lambda(e_1(f)e_2 - e_2(f)e_1) + (-\lambda^2/2 + \Delta f)e_1\cdot e_2)\cdot\varphi$$

where e_1, e_2 denotes an orthonormal basis of the tangent plane. On the other hand, using the usual formula relating spinorial curvature to the standard Riemann curvature tensor we obtain

$$R^\Sigma(e_1,e_2)\varphi = -\frac{1}{2}Ke_1\cdot e_2\cdot\varphi.$$

Combining these two equations we get

$$[\lambda(e_1(f)e_2 - e_2(f)e_1) + (K/2 - \lambda^2/2 + \Delta f)e_1\cdot e_2]\cdot\varphi = 0. \tag{1.7}$$

Taking the (pointwise) scalar product with $e_2\cdot\varphi$ yields

$$\begin{aligned} 0 &= \langle[\lambda(e_1(f)e_2 - e_2(f)e_1) + (K/2 - \lambda^2/2 + \Delta f)e_1\cdot e_2]\cdot\varphi, e_2\cdot\varphi\rangle \\ &= \lambda e_1(f)|\varphi|^2 + \lambda e_2(f)\langle\varphi, e_1\cdot e_2\cdot\varphi\rangle + 0, \end{aligned}$$

hence

$$e_1(f)|\varphi|^2 + e_2(f)\langle\varphi, e_1\cdot e_2\cdot\varphi\rangle = 0.$$

Similarly, multiplying with $e_1\cdot\varphi$ we see

$$e_1(f)\langle\varphi, e_1\cdot e_2\cdot\varphi\rangle - e_2(f)|\varphi|^2 = 0.$$

In matrix notation

$$\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}|\varphi|^2 & \langle\varphi, e_1\cdot e_2\cdot\varphi\rangle\\ \langle\varphi, e_1\cdot e_2\cdot\varphi\rangle & -|\varphi|^2\end{pmatrix}\begin{pmatrix}e_1(f)\\e_2(f)\end{pmatrix}$$

Since $\det\begin{pmatrix}|\varphi|^2 & \langle\varphi, e_1\cdot e_2\cdot\varphi\rangle\\ \langle\varphi, e_1\cdot e_2\cdot\varphi\rangle & -|\varphi|^2\end{pmatrix} = -(|\varphi|^4 + \langle\varphi, e_1\cdot e_2\cdot\varphi\rangle^2)$ is nonzero wherever φ does not vanish (hence on an open dense subset of M) we conclude that the gradient of f vanishes everywhere. Thus $\Delta f = 0$ and by multiplying (1.7) with $e_1\cdot e_2\cdot\varphi$ we obtain

$$K/2 - \lambda^2/2 = 0.$$

Hence the curvature is constant and the proof is complete.

Given this estimate on the 2-sphere and the fact $4\pi = 2\pi\chi(S^2) = \frac{1}{2}\int_{S^2}\text{Scal}\,dV$ one might be tempted to conjecture a more general estimate on n-dimensional manifolds of the form

$$\lambda^2 \geq \frac{n}{4(n-1)}\frac{\int_M \text{Scal}\,dV}{\text{vol}(M)}.$$

However such an estimate is not possible as is shown by the following theorem.

THEOREM 1.4 (Ammann-Bär [**AB00**]). *Let M be a closed spin manifold of dimension $n \geq 3$. Then there exist constants $0 < C_1 \leq C_2 \leq C_3 \leq \dots$ and there exist Riemannian metrics $g_1, g_2, g_3, \dots$ on M such that*

$$\lambda_k(g_j)^2 \leq C_k$$

for all k and j while

$$\frac{\int_M Scal_{g_j} dV_{g_j}}{\mathrm{vol}_{g_j}(M)} \stackrel{j\to\infty}{\longrightarrow} \infty.$$

The proof is based on the *variational characterization* of the eigenvalues. In general, we have

$$\lambda_k^2 = \inf_{\mathrm{V}} \mathrm{Sup}_{\varphi \in \mathrm{V}\setminus\{0\}} \frac{(\mathrm{D}\varphi, \mathrm{D}\varphi)}{||\varphi||^2}$$

where the infimum is taken over all k-dimensional vector subspaces $V \subset C^\infty(M, \Sigma M)$. This characterization follows easily from the properties of the spectrum as described in the beginning of this section. Let us now prove Theorem 1.4.

PROOF. We start by choosing a Riemannian metric g_0 on M such that (M, g) contains an embedded Euclidean ball B of radius 1.

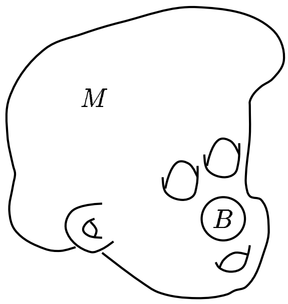

FIG. 1

Write the Euclidean ball B as a union of two annuli and one smaller ball, $B = A_1 \cup A_2 \cup A_3$, where $A_1 = \{x \in \mathbb{R}^n \mid 2/3 \leq |x| \leq 1\}$, $A_2 = \{x \in \mathbb{R}^n \mid 1/3 \leq |x| \leq 2/3\}$ and $A_3 = \{x \in \mathbb{R}^n \mid |x| \leq 1/3\}$. Now fix two parameters $0 < r < 1$ and $L > 0$. Choose a Riemannian metric $g_{r,L}$ on M with the following properties:

- $g_{r,L}$ coincides with g_0 on $M - B$
- $g_{r,L}$ is independent of L on A_1 and on A_3
- $(A_2, g_{r,L})$ is isometric to $S^{n-1}(r) \times [0, L]$ with the product metric where $S^{n-1}(r)$ denotes the round sphere of constant sectional curvature $1/r^2$.

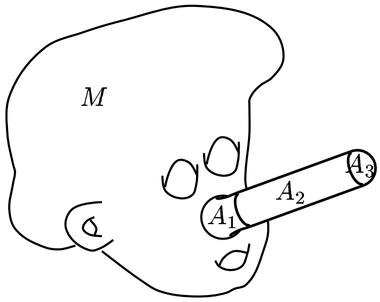

FIG. 2

Heuristically, there is a nose of radius r and length L growing out of the ball. For this reason we call these metrics *Pinocchio metrics.*

Claim 1. *The k^{th} eigenvalue $\lambda_k(r,L)^2$ of the square $D^2_{g_{r,L}}$ of the Dirac operator (w.r.t. the metric $g_{r,L}$) is bounded from above by a constant $C_k > 0$ independent of r and L.*

The proof is very simple. Choose a k-dimensional vector space V_k of spinors ψ on M vanishing on B. Then $\psi \in V_k$ can be considered a spinor for all metrics $g_{r,L}$. We plug it into the Rayleigh quotient for the Dirac operator to get

$$\begin{aligned}\lambda_k(r,L)^2 &\leq \operatorname{Sup}_{\psi\in V_k,\psi\neq 0}\frac{\int_M \langle D^2_{g_{r,L}}\psi,\psi\rangle_{g_{r,L}}\,dV}{\int_M\langle\psi,\psi\rangle_{g_{r,L}}\,dV}\\ &= \operatorname{Sup}_{\psi\in V_k,\psi\neq 0}\frac{\int_M \langle D^2_{g_0}\psi,\psi\rangle_{g_0}\,dV}{\int_M\langle\psi,\psi\rangle_{g_0}\,dV}\\ &=: C_k.\end{aligned}$$

Claim 2. *The normalized total scalar curvature is unbounded from above for $r \in (0,1)$ and $L \in (0,\infty)$.*

Let ω_k denote the volume of the k-dimensional unit sphere. We compute

$$\begin{aligned}\frac{\int_M \mathrm{Scal}_{g_{r,L}}\,dV}{vol_{g_{r,L}}(M)} &= \frac{\int_{(M\setminus B)\cup A_1\cup A_3}\mathrm{Scal}_{g_{r,L}}\,dV + \int_{A_2}\mathrm{Scal}_{g_{r,L}}\,dV}{vol_{g_{r,L}}((M\setminus B)\cup A_1\cup A_3) + vol_{g_{r,L}}(A_2)}\\ &= \frac{\int_{(M\setminus B)\cup A_1\cup A_3}\mathrm{Scal}_{g_{r,L}}\,dV + L\cdot\frac{(n-1)(n-2)}{r^2}\cdot r^{n-1}\cdot\omega_{n-1}}{vol_{g_{r,L}}((M\setminus B)\cup A_1\cup A_3) + L\cdot r^{n-1}\cdot\omega_{n-1}}\\ &\longrightarrow \frac{(n-1)(n-2)}{r^2}\end{aligned}$$

for $L\to\infty$ because on $M\setminus A_2$ the metric $g_{r,L}$ does not depend on L by construction. Hence

$$\operatorname{Sup}_{L>0}\frac{\int_M \mathrm{Scal}_{g_{r,L}}\,dV}{vol_{g_{r,L}}(M)} \geq \frac{(n-1)(n-2)}{r^2}$$

and therefore

$$\operatorname{Sup}_{L>0,\ 0<r<1}\frac{\int_M \mathrm{Scal}_{g_{r,L}}\,dV}{vol_{g_{r,L}}(M)} = \infty.$$

This proves the theorem.

Another very nice lower eigenvalue estimate can be achieved for manifolds bounding manifolds of nonnegative scalar curvature.

THEOREM 1.5 (Hijazi-Montiel-Zhang [**HMZ00**, Thm. 6]). *Let N be an $(n+1)$-dimensional compact Riemannian spin manifold with boundary $M = \partial N$. Suppose the scalar curvature of N is nonnegative,* $\mathrm{Scal}_N \geq 0$ *and the mean curvature of M (w. r. t. the inner normal) is nonnegative, $H \geq 0$. Then all nonnegative eigenvalues of the Dirac operator on M satisfy*

$$\lambda \geq \frac{n}{2} \mathrm{inf_M H}.$$

This estimate is sharp for the standard sphere bounding a Euclidean ball. Let us sketch the proof of the theorem.

PROOF. From the Schrödinger-Lichnerowicz formula (1.2) applied to the manifold N and the assumption $\mathrm{Scal}_N \geq 0$ we get for each spinor ψ on N

$$\int_N \langle D_N^2 \psi, \psi \rangle \, dV \geq \int_N \langle \nabla^* \nabla \psi, \psi \rangle \, dV.$$

Now one does a partial integration on both sides to remove the second derivatives. The resulting boundary terms can be arranged in such a manner that one obtains

$$\int_M \left(\langle D_M \psi, \psi \rangle - \frac{nH}{2} |\psi|^2 \right) dA \geq -\frac{n}{n+1} \int_N |D_N \psi|^2 \, dV$$

Then one argues that given an eigenspinor φ on M for the eigenvalue λ one can solve the boundary value problem

$$\begin{cases} D_N \psi = 0 & \text{on } N, \\ \pi_+ \psi = \varphi & \text{on } M. \end{cases}$$

Here $\pi_+ : C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M)$ denotes the projection onto the subspace generated by the D_M-eigenspaces for nonnegative eigenvalues. The theorem then follows immediately.

A slightly improved version of this estimate can be found in [**HMZ02**].

1.1.3.2. *Upper Bounds.* Theorem 1.4 already gives upper bounds on the Dirac eigenvalues for certain metrics to be constructed. Here is a geometric upper bound for hypersurfaces in Euclidean space.

THEOREM 1.6 (Bär [**Bär98**]). *Let M be a closed oriented immersed hypersurface in $\mathbb{R}^{n+1}$. We give M the induced Riemannian metric and spin structure. Let H be the mean curvature of M. Then there are $2^{[n/2]}$ Dirac eigenvalues (counted with multiplicity) satisfying*

$$\lambda^2 \leq \frac{n^2}{4} \frac{\int_M H^2 \, dV}{vol(M)}.$$

The proof is again based on the variational characterization of the eigenvalues. One inserts restrictions of parallel spinors on $\mathbb{R}^{n+1}$ to M into the Rayleigh quotient $\frac{(D\varphi, D\varphi)}{\|\varphi\|^2}$. The general formula

$$D^M \varphi = -\nu \cdot D^{\mathbb{R}^{n+1}} \varphi + \frac{n}{2} H \varphi - \nabla^{\mathbb{R}^{n+1}} \varphi$$

relating the Dirac operators D^M on M and $D^{\mathbb{R}^{n+1}}$ on $\mathbb{R}^{n+1}$ yields for a parallel φ

$$D^M\varphi = \frac{n}{2}H\varphi$$

and therefore

$$\frac{(D^M\varphi, D^M\varphi)}{\|\varphi\|^2} = \frac{n^2}{4}\frac{\int_M H^2|\varphi|^2 dV}{\int_M |\varphi|^2 dV} = \frac{n^2}{4}\frac{\int_M H^2\, dV}{vol(M)}$$

since $|\varphi|$ is constant for parallel φ. This proves the theorem up to some technical details concerning the identification of spinors on $\mathbb{R}^{n+1}$ with spinors on M.

The estimate in Theorem 1.6 is sharp in the sense that equality is attained for M a sphere of any radius in $\mathbb{R}^{n+1}$. It is an open question whether or not there are other hypersurfaces M in $\mathbb{R}^{n+1}$ for which equality holds. If one assumes that the mean curvature H is constant, then the lower bound in Theorem 1.5 and the upper bound in Theorem 1.6 agree. One can then deduce the classical Alexandrov Theorem stating that the only embedded compact hypersurfaces in Euclidean space of constant mean curvature are the round spheres, see [**HMZ00**, Thm. 7].

There are similar results for hypersurfaces of spheres and hyperbolic spaces, see [**Bär98, Gin01**].

1.1.3.3. *Further results.* More involved Dirac eigenvalue bounds can e. g. be found in [**Bär91, Bär92c, Amm00a, Amm00b, HZ01a, HZ01b, FK02, Kir02**]. There are also Dirac eigenvalue bounds for compact manifolds with boundary [**HMZ01, HMR01**].

The spectrum of the Dirac operator does not fully determine the underlying manifold. This can be seen from the existence of *Dirac isospectral manifolds*, i. e. pairs of nonisometric Riemannian spin manifolds having the same Dirac spectrum. Known examples are certain flat tori, certain spherical space forms [**Bär96a**, Thm. 5] and certain nilmanifolds. In [**AB98**, Thm. 5.6] a 1-parameter family of Riemannian metrics is constructed on a 7-dimensional nilmanifold such that the Dirac spectrum is constant (as a function of the parameter) for some spin structures while it changes for the other spin structures.

1.2. The Spectrum of the Dirac Operator on Open Manifolds

From now on let M denote an n-dimensional *complete* noncompact Riemannian spin manifold.

1.2.1. Analytic Basics. The spectral theory is now more involved than in the compact case. Again, a complex number λ is called an *eigenvalue* of the Dirac operator D, if there exists a nonzero $\varphi \in C^\infty(M, \Sigma M) \cap L^2(M, \Sigma M)$ satisfying

$$D\varphi = \lambda\varphi.$$

Note that in the compact case smooth spinors are automatically square integrable while here we additionally impose it. The set of eigenvalues together with their multiplicities is called the *point spectrum* $\mathrm{spec}_p(D)$ of D. Again, eigenvalues are necessarily real and eigenspaces E_λ for different eigenvalues are mutually perpendicular. But the multiplicity $\dim E_\lambda$ of an eigenvalue λ may now be infinite, the point spectrum need no longer be discrete, and $\bigoplus_{\lambda\in\mathrm{spec}_p(D)} E_\lambda$ need no longer be dense in $C^\infty(M,\Sigma M)\cap L^2(M,\Sigma M)$.

We say that a number $\lambda\in\mathbb{C}$ lies in the *essential spectrum* $\mathrm{spec}_e(D)$ if there exists a sequence φ_j of smooth spinors with compact support which are orthonormal with respect to $(\cdot,\cdot)_{L^2}$ such that

$$\|(D-\lambda)\varphi_j\|_{L^2}\to 0$$

for $j\to\infty$. The φ_j can be regarded as approximate eigenspinors for D. Such a sequence is called a *Weyl sequence* for λ. We have $\mathrm{spec}_e(D)\subset\mathbb{R}$.

The point spectrum together with the essential spectrum form the spectrum $\mathrm{spec}(D)$ of D. Note that the point spectrum and the essential spectrum need not be disjoint. For example if λ is an eigenvalue of infinite multiplicity, then it lies in both the point spectrum and in the essential spectrum. Here an orthonormal basis of the infinite dimensional eigenspace can be used as a Weyl sequence. If one removes all essential spectrum (e. g. all eigenvalues of infinite multiplicity) from the point spectrum, then one is left with the *discrete spectrum*,

$$\mathrm{spec}_d(D) := \mathrm{spec}_p(D)\setminus\mathrm{spec}_e(D).$$

Removing all eigenvalues from the essential spectrum yields the *continuous spectrum*,

$$\mathrm{spec}_c(D) := \mathrm{spec}_e(D)\setminus\mathrm{spec}_p(D).$$

Hence we have two disjoint decompositions of the spectrum,

$$\mathrm{spec}(D)=\mathrm{spec}_d(D)\sqcup\mathrm{spec}_e(D)=\mathrm{spec}_p(D)\sqcup\mathrm{spec}_c(D).$$

Given a compact subset $K\subset M$ and $\lambda\in\mathrm{spec}_e(D)$ one can choose a Weyl sequence φ_j for λ such that $K\cap\mathrm{supp}(\varphi_j)=\emptyset$. This shows that $\mathrm{spec}_e(D)$ is unaffected by changes of the manifold in compact sets.

THEOREM 1.7 (Decomposition Principle). *Let M_1 and M_2 be two complete Riemannian spin manifolds, let $K_i\subset M_i$ be compact. Suppose there is a spin structure preserving isometry between $M_1\setminus K_1$ and $M_2\setminus K_2$. Then the Dirac operators on M_1 and on M_2 have the same essential spectrum.*

Note that M_1 and M_2 need not even be homeomorphic. This robustness of $\mathrm{spec}_e(D)$ contrasts strongly with the behavior of the discrete spectrum. Eigenvalues are very sensitive to changes of the geometry of the manifold.

1.2.2. Examples for Explicit Computation. The Dirac operator has been studied much less on noncompact manifolds than on compact ones. So we do not have many examples of noncompact manifolds for which we know the Dirac spectrum explicitly.

1.2.2.1. *Euclidean space.* The Dirac operator on Euclidean space is a differential operator with constant coefficients. The classical method to determine the spectrum of such operators is to apply Fourier transformation turning the differential operator into a multiplication operator. The spectrum can then be read off easily.

THEOREM 1.8. *Let $M = \mathbb{R}^n$ with the Euclidean metric. Then the Dirac operator has no eigenvalues, $spec_p(D) = \emptyset$, and $spec_c(D) = \mathbb{R}$.*

We give an elementary direct proof in the 1-dimensional case, $n = 1$. Recall from Section 1.1.2.1 that spinors on $M = \mathbb{R}$ can be identified with complex-valued functions and that the Dirac operator is then given by $D = i\frac{d}{dt}$. Suppose $\lambda \in \mathbb{R}$ and $\varphi \in C^\infty(\mathbb{R}, \mathbb{C})$ such that

$$i\frac{d}{dt}\varphi = \lambda\varphi.$$

This ordinary differential equation has the general solution $\varphi(t) = C \cdot e^{-i\lambda t}$, $C \in \mathbb{C}$. Therefore $|\varphi| \equiv |C|$ and φ cannot be square-integrable unless it is identically zero. This proves $\mathrm{spec}_p(D) = \emptyset$.

Fix $\lambda \in \mathbb{R}$. Choose a nonnegative smooth function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{supp}(\chi) \subset [-2, 2]$ and $\chi \equiv 1$ on $[-1, 1]$. For $j, k \in \mathbb{N}$ set $\psi_{j,k}(t) := \chi(\frac{t}{j} - k) \cdot e^{-i\lambda t}$. From $|\psi_{j,k}(t)| = \chi(\frac{t}{j} - k)$ we see

$$\|\psi_{j,k}\|^2_{L^2} = \int_{-\infty}^{\infty} \chi(\tfrac{t}{j} - k)^2\, dt = j \cdot \int_{-\infty}^{\infty} \chi(u)^2\, du.$$

Since $\mathrm{supp}(\psi_{j,k}) \subset [j(k-2), j(k+2)]$ we can choose $k = k(j)$ such that $\psi_{j,k(j)}$ have mutually disjoint support. In particular, they are orthogonal with respect to the L^2-scalar product. Thus $\varphi_j := \left(j \cdot \int_{-\infty}^{\infty} \chi(u)^2\, du\right)^{-1/2} \psi_{j,k(j)}$ are orthonormal.

Moreover, $|(D - \lambda)\psi_{j,k}| = |\frac{i}{j}\chi'(\frac{t}{j} - k)e^{-i\lambda t}| = \frac{1}{j}|\chi'(\frac{t}{j} - k)|$ yields

$$\|(D - \lambda)\psi_{j,k}\|^2_{L^2} = \frac{1}{j^2}\int_{-\infty}^{\infty} \chi'(\tfrac{t}{j} - k)^2\, dt = \frac{1}{j}\int_{-\infty}^{\infty} \chi'(u)^2\, du.$$

Hence

$$\|(D - \lambda)\varphi_j\|^2_{L^2} = \frac{\int_{-\infty}^{\infty} \chi'(u)^2\, du}{j^2 \cdot \int_{-\infty}^{\infty} \chi(u)^2\, du} \to 0$$

for $j \to \infty$. Thus φ_j is a Weyl sequence for λ. Since $\lambda \in \mathbb{R}$ was arbitrary this shows $\mathrm{spec}_e(D) = \mathbb{R}$ and since the point spectrum is empty also $\mathrm{spec}_c(D) = \mathbb{R}$.

In the physically motivated literatur the Dirac operator with potential, $D + V$, has attracted much attention where the potential V is a Hermitian endomorphism field of $\Sigma\mathbb{R}^n$, see e. g. [**Tha92**, Sec. 4.7] and the references therein.

1.2.2.2. *Hyperbolic space.* In [**Bun91a**] the spectrum of the Dirac operator on real hyperbolic space $\mathbb{R}H^n$ is determined regarding $\mathbb{R}H^n$ as a homogeneous space, $\mathbb{R}H^n = \mathrm{SO}^+(n, 1)/\mathrm{SO}(n)$ where $\mathrm{SO}^+(n, 1)$ denotes the group of orientation- and timeorientation preserving Lorentz transformations. One can then use representation theoretic methods similar to the ones in Section 1.1.2.2. These methods are technically somewhat involved and in fact there is an incorrect statement about the eigenvalue 0 in [**Bun91a**]. A much simpler and more geometric computation of the Dirac spectrum on $\mathbb{R}H^n$ uses the warped product structure, see e. g. [**Bai97**]. The result is

THEOREM 1.9. *Let $M = \mathbb{R}H^n$ be real hyperbolic space. Then the Dirac operator has no eigenvalues, $spec_p(D) = \emptyset$, and $spec_c(D) = \mathbb{R}$.*

So the result is the same as for Euclidean space.

1.2.2.3. *Further examples.* Both Euclidean and real hyperbolic space are examples of a class of manifolds called Riemannian symmetric spaces. These spaces are classified and in principle they are particularly well-suited for representation theoretic methods.

The point spectrum of these spaces is understood. Goette and Semmelmann show in [**GS02**] that the point spectrum of the Dirac operator on a Riemannian symmetric space of noncompact type is either empty or $\{0\}$. They give different characterizations for when the eigenvalue 0 appears.

The continuous spectrum has been determined by Camporesi and Pedon in [**CP02**] for hyperbolic spaces (Riemannian symmetric spaces of noncompact type and rank 1). The result says that if M is a hyperbolic space, then

$$\mathrm{spec}_c(D) = \begin{cases} \mathbb{R}, & \text{if } M \in \{\mathbb{R}H^n, \mathbb{H}H^n, \mathbb{O}H^2\} \\ \mathbb{R}, & \text{if } M = \mathbb{C}H^n \text{ with } n \text{ odd} \\ (-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty), & \text{if } M = \mathbb{C}H^n \text{ with } n \text{ even} \end{cases}$$

The exceptional case $M = \mathbb{C}H^n$ with even complex dimension n is precisely the case when 0 is an eigenvalue.

1.2.3. Qualitative Results. As for compact manifolds an explicit computation of the eigenvalues of the Dirac operator is possible only in exceptional cases. Due to its robustness the essential spectrum is more accessible. As an illustration let us look at *hyperbolic manifolds* which are, by definition, complete Riemannian manifolds with constant sectional curvature -1. Such manifolds are always of the form $M = \mathbb{R}H^n/\Gamma$ for some suitable discrete group Γ of isometries of real hyperbolic space. There is up to now no hyperbolic manifold of finite volume for which one can compute the point spectrum neither for the Dirac operator nor for any other geometric elliptic operator like the Laplace-Beltrami operator.

Nevertheless, one can say something about the essential spectrum. For the sake of simplicity, let us concentrate on the 3-dimensional case. A 3-dimensional hyperbolic spin manifold of finite volume is known to be decomposable in the form

$$M = M_0 \sqcup E_1 \sqcup \cdots \sqcup E_k$$

where M_0 is compact with boundary and the E_j are the *cusps*. They have the form $E_j = (0, \infty) \times T^2$ with the Riemannian metric $g = dt^2 + e^{-2t} \cdot g_{\text{flat}}$ where t denotes the variable in $[0, \infty)$ and g_{flat} is a flat metric on the 2-torus T^2.

All topological information is contained in M_0 but by the decomposition principle it is irrelevant for the essential spectrum. Hence the essential spectrum can be determined by looking the cusps alone and they are very simple and given completely explicitly. One gets

THEOREM 1.10 (Bär [**Bär00**]). *Every complete oriented hyperbolic manifold M of dimension 3 and finite volume has a spin structure for which*

$$spec(D) = spec_d(D).$$

Some also have a spin structure such that

$$spec(D) = \mathbb{R}.$$

There are no other possibilities.

In other words, either $\mathrm{spec}_e(D) = \emptyset$ or $\mathrm{spec}_e(D) = \mathbb{R}$ and it is the spin structure which is responsible for the presence or the absence of essential spectrum.

One of the main sources for the construction of examples of 3-dimensional hyperbolic manifolds of finite volume is as follows: Take a knot or link L in $\mathbb{R}^3$. Add one point to $\mathbb{R}^3$ to obtain S^3 and regard L as sitting in the 3-sphere, $L \subset S^3$. It can be shown that for "most" links the manifold $M := S^3 \setminus L$ can be given a hyperbolic metric of finite volume. Each component of the link corresponds to one cusp. For such an M there is a simple criterion for whether or not there exists a spin structure with $\mathrm{spec}(D) = \mathbb{R}$.

THEOREM 1.11 (Bär [**Bär00**, Thm. 4]). *Let $L \subset S^3$ be a link, let $M = S^3 \setminus L$ carry a hyperbolic metric of finite volume.*

If the linking number of all pairs of components (L_i, L_j) of L is even,

$$Lk(L_i, L_j) \equiv 0 \ mod\ 2,$$

$i \neq j$, then the spectrum of the Dirac operator on M is discrete for all spin structures,

$$spec(D) = spec_d(D).$$

If there exist two components L_i and L_j of L, $i \neq j$, with odd linking number, then M has a spin structure such that the spectrum of the Dirac operator satisfies

$$spec(D) = \mathbb{R}.$$

Determining linking numbers modulo 2 is equivalent to counting overcrossings modulo 2 in planar link diagrams, hence extremely simple.

The complements of the following links possess a hyperbolic structure of finite volume. All linking numbers are even. Count e. g. the overcrossings of blue over red. Hence the Dirac spectrum on those hyperbolic manifolds is discrete for all spin structures.

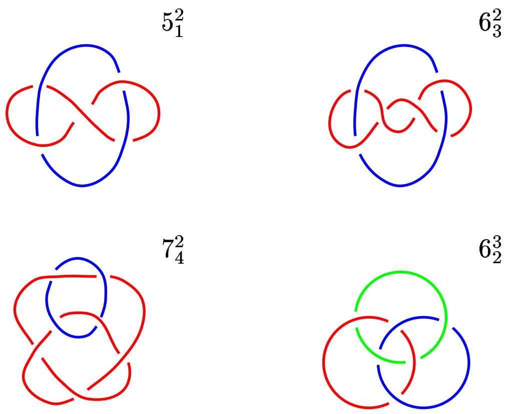

FIG. 3

This example includes the Whitehead link (5^2_1) and the Borromeo rings (6^3_2).

The complements of the following links possess a hyperbolic structure of finite volume. There are odd linking numbers. Hence those hyperbolic manifolds have a spin structure for which the Dirac spectrum is the whole real line.

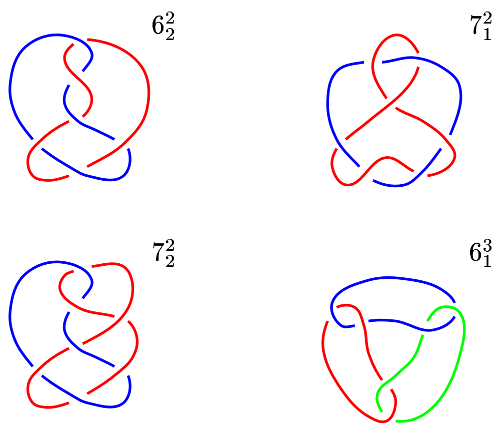

FIG. 4

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