The Dirac Operator and the Scalar Curvature of Continuously Deformed Algebraic Varieties

Christian Bär and David Bleecker

Abstract. The Bochner-Lichnerowicz formula and the Atiyah-Singer Index Formula for the Dirac operator have been used to find an obstruction (the $\widehat{A}$-genus) to producing metrics of positive scalar curvature on spin manifolds. Here the technique is applied to twisted Dirac operators in order to obtain upper bounds on the minimum of the scalar curvature for Riemannian manifolds which admit certain contractive spin mappings into a fixed Riemannian manifold. The principal application is to obtain such upper bounds for algebraic varieties equipped with arbitrary metrics, which admit contractive maps into $\mathbb{P}^n(\mathbb{C})$ homotopic to inclusions.

1. Introduction

We cannot begin to review all of the remarkable work which has been done in formulating sufficient conditions that a manifold admit a metric with positive scalar curvature, and exploring the consequences. However, we highly recommend the informative articles [**Gro**] and [**St**], and the book [**La-Mi**], especially §4-7 of Part IV. The most basic result is that a compact spin $4k$-manifold with non-zero $\widehat{A}$-genus has no metric of positive scalar curvature. This is an easy consequence of the Bochner-Lichnerowicz formula for the square of Dirac operator and the Atiyah-Singer Index Theorem. There is much more evidence of a connection between spin and scalar curvature. For example in [**Gro-La**] it is shown that every compact, simply-connected manifold of dimension greater than 4 which is *not* spin has a metric of positive scalar curvature. For manifolds (M,g) which have positive scalar curvature $\kappa_g \in C^\infty(M)$, there is the question of just how large $\min(\kappa_g) := \min_{x\in M}(\kappa_g(x)) \in \mathbb{R}$ can be as one varies the metric. Of course, by a contraction of the metric we can make $\min(\kappa_g)$ as large as we like, but suppose that we only consider the set of metrics, say

$$\mathcal{M}(g_0,\Lambda^2) := \left\{ g \in S^2(M) : |\alpha|_g \geq |\alpha|_{g_0} \text{ for all } \alpha \in \Lambda^2(TM) \right\},$$

which do not decrease areas relative to a fixed metric g_0. Some very general results in [**Gro**] are the K-area inequality for spin manifolds (p. 30) and the $K_{\surd}$-area inequality for non-spin manifolds (p. 57) which show that there is a finite upper bound on $\min(\kappa_g)$ for $g \in \mathcal{M}(g_0,\Lambda^2)$. It is desirable to find some reasonably good

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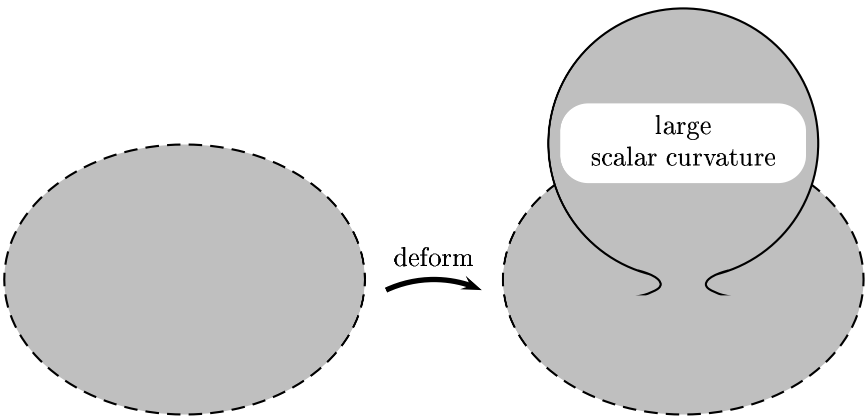
specific upper bounds on $\min(\kappa_g)$ for $g \in \mathcal{M}\left(g_0, \Lambda^2\right)$ in terms of the curvature of g_0, and this will be our main focus. As a starting point, when g_0 is the standard metric on the sphere S^n, there is a sharp result of [**Ll**] which states that $\min(\kappa_g)$ for $g \in \mathcal{M}\left(g_0, \Lambda^2\right)$ is bounded above by $n(n-1) = \kappa_{g_0}$. In other words, a change of metric on S^n, which does not decrease areas anywhere, cannot increase the scalar curvature everywhere. We mention that in [**Mi**] analogous results for certain symmetric spaces are obtained. One of our aims here is to see how close we can come to getting a similar result for complex algebraic varieties (more specifically, complete intersections). As a special case (contained in Corollary 5.5) of our main result, we prove:

SAMPLE RESULT. *Let M be diffeomorphic to a complete intersection of even complex dimension ν in $\mathbb{P}^\mu(\mathbb{C})$ and let $f : M \to \mathbb{P}^\mu(\mathbb{C})$ be a smooth immersion which is continuously homotopic to the inclusion. If κ is the scalar curvature of M with the metric induced via f from the Fubini-Study metric (of holomorphic sectional curvature 4) on $\mathbb{P}^\mu(\mathbb{C})$, then*

$$\min_{x \in M} \kappa(x) \leq 5\nu^2 + 4\nu.$$

The scalar curvature of $\mathbb{P}^\nu(\mathbb{C})$ is $4\nu^2 + 4\nu$, which suggests that this result may not be sharp, but we hope to have at least brought the picture into better focus.

Note that it is not possible to estimate the *maximum* of the scalar curvature from above. It is easy to deform any immersion in such a way that it forms small bubbles with arbitrarily large scalar curvature.



In Section 2, we introduce some concepts and perform a computation which enables us to get a bound on the curvature operator that appears in the Bochner-Lichnerowicz (say B-L) formula for the square of a certain twisted Dirac operator. Since this bound is somewhat geometrically obscure and awkward to compute, we replace it in Section 3 by a bound which is in terms of the conventional irreducible components of the curvature tensor. While this new bound is generally weaker, it does suffice to recapture the result of [**Ll**]. A successful application of the B-L formula requires the existence of a twisted harmonic (or at least nearly harmonic) spinor. Typically, harmonic spinors are produced by applying the Atiyah-Singer Index Theorem, once one knows that the relevant index is nonzero. Thus in Section 4, we compute indices for the relevant twisted Dirac operators, and state some general results. In Section 5, we apply our results to complex algebraic varieties that are complete intersections. Incidentally, we find that much better bounds on the scalar curvature can be obtained by essentially lifting the varieties to S^{2n+1} via the Hopf fibration, even though twisted harmonic spinors on the varieties lift to spinors which are only nearly harmonic. We are indebted to [**Kr**] for this uplifting idea.

It should be mentioned that the technique employed in this paper has also been used successfully to derive upper eigenvalue bounds for Dirac operators, cf. [**Va-Wi, At, Bu, Ba**].

2. Notation and the fundamental estimate

In all of what follows we work within the C^∞ category. Let (M, g_M) and (N, g_N) be compact (and without boundary), oriented Riemannian manifolds of dimensions n and m, respectively, and let $f : M \to N$ be a map. For simplicity, we assume in this section that (M, g_M) and (N, g_N) are equipped with spin structures $P_{Spin(n)}(M) \to P_{SO(n)}(M)$ and $P_{Spin(m)}(N) \to P_{SO(m)}(N)$. However, essentially every result in this section has a natural extension to the case where M and N are not necessarily spinable, but $f : M \to N$ is a spin mapping (i.e., $f^*(w_2(N)) = w_2(M)$). We have more to say about spin mappings in Section 4. For a fine introduction to spin geometry, as well as beautiful, advanced applications, we recommend [**La-Mi**]. Let

$$Cl_{Spin}(M) := P_{Spin(n)}(M) \times_l Cl(n), \tag{2.1}$$

where $Cl(n)$ is the real Clifford algebra for $\mathbb{R}^n$ with the standard inner product and $l : Spin(n) \to End(Cl(n))$ is left multiplication. We define $Cl_{Spin}(N)$ similarly. Let

$$D : \Gamma(Cl_{Spin}(M) \otimes f^*Cl_{Spin}(N)) \hookleftarrow \tag{2.2}$$

be the twisted Dirac operator. For any $\phi \in \Gamma(Cl_{Spin}(M) \otimes f^*Cl_{Spin}(N))$, we have the B-L formula

$$D^2\phi = \nabla^*\nabla\phi + \tfrac{1}{4}\kappa_M\phi + \mathfrak{R}^f\phi. \tag{2.3}$$

Here, ∇ is the covariant derivative operator on $\Gamma(Cl_{Spin}(M) \otimes f^*Cl_{Spin}(N))$ with respect to the connection induced by the spin-lifted Levi-Civita connection for M and the pull-back of the spin-lifted Levi-Civita connection for N; these connections are also used in the definition of D. Moreover,

$$\mathfrak{R}^f \in End(Cl_{Spin}(M) \otimes f^*Cl_{Spin}(N)) \tag{2.4}$$

is given as follows. Let $(e_1, ..., e_n)$ be an orthonormal frame for T_xM and let $(\varepsilon_1, ..., \varepsilon_m)$ be an orthonormal frame for $T_{f(x)}N$. We assume that these frames are *adapted* to f_{*x} in the sense that $(e_1, ..., e_r)$ is a basis for $\ker(f_{*x})^\perp$, and $(\varepsilon_1, ..., \varepsilon_r)$ is a basis for $f_{*x}(T_xM)$. Of course, r is the rank of f_{*x}. By diagonalizing f^*g_N, we *could* assume that $f_{*x}(e_i) = \mu_i\varepsilon_i$, for positive μ_i, $i = 1, ..., r$, but we will *not* assume this now. For a simple element $\sigma \otimes v \in Cl_{Spin}(M)_x \otimes f^*Cl_{Spin}(N)_x$, we have

$$\mathfrak{R}^f(\sigma \otimes v) = \tfrac{1}{2}\sum_{i,j=1}^{r}(e_i \cdot e_j \cdot \sigma) \otimes R^N(f_*e_i, f_*e_j)(v), \tag{2.5}$$

where "$\cdot$" denotes Clifford multiplication, and $R^N(f_*e_i, f_*e_j) \in \mathrm{End}(Cl_{Spin}(N))$ is induced by the curvature of the spin-lifted Levi-Civita connection for N. There is a standard, connection-compatible Riemannian structure $\langle\cdot,\cdot\rangle$ on $Cl_{Spin}(M) \otimes f^*Cl_{Spin}(N)$, which is referred to in the following

PROPOSITION 2.1. *Let* $\phi \in \Gamma(Cl_{Spin}(M) \otimes f^*Cl_{Spin}(N))$, $\phi \not\equiv 0$, *satisfy* $|D\phi|^2 \leq \frac{1}{4}\eta|\phi|^2$, *for some* $\eta \in C^\infty(M)$, *and let* $\rho_f \in C^\infty(M)$ *be a function such that*

$$\langle \mathfrak{R}^f\phi, \phi\rangle \geq -\tfrac{1}{4}\rho_f|\phi|^2. \tag{2.6}$$

Then there is a point $x \in M$ *such that*

$$\kappa_M(x) \leq \eta(x) + \rho_f(x). \tag{2.7}$$

Moreover, if $\kappa_M \geq \eta + \rho_f$ *everywhere, then* $\kappa_M = \eta + \rho_f$ *and* $\nabla\phi = 0$.

PROOF. Taking the L^2 inner product of $D^2\phi = \nabla^*\nabla\phi + \frac{1}{4}\kappa_M\phi + \mathfrak{R}^f\phi$ with ϕ yields (where we denote the volume element on M induced by g by dM)

$$\begin{aligned}\int_M \tfrac{1}{4}\eta|\phi|^2\,dM \geq \int_M |D\phi|^2\,dM &= \int_M \langle D^2\phi,\phi\rangle\,dM \\ &= \int_M \left(\langle\nabla^*\nabla\phi,\phi\rangle + \tfrac{1}{4}\kappa_M|\phi|^2 + \langle\mathfrak{R}^f\phi,\phi\rangle\right)dM \\ &\geq \int_M \left(|\nabla\phi|^2 + \tfrac{1}{4}\kappa_M|\phi|^2 - \tfrac{1}{4}\rho_f|\phi|^2\right)dM. \end{aligned} \tag{2.8}$$

Thus

$$0 \geq \int_M \left(|\nabla\phi|^2 + \tfrac{1}{4}(\kappa_M - \eta - \rho_f)|\phi|^2\right)dM. \tag{2.9}$$

Assume that $\kappa_M \geq \eta + \rho_f$. Then $\nabla\phi = 0$, and consequently $|\phi|$ is a constant. Since $|\phi|$ is a nonzero constant, it then follows that $\kappa_M = \eta + \rho_f$. □

Hence, it is desirable to find a geometrically interesting function ρ_f. To this end, we proceed as follows. One can show that an orthonormal basis for $Cl_{Spin}(M)_x$ is given by elements of the form (where we henceforth drop the dot "$\cdot$" indicating Clifford multiplication, when the context is clear)

$$\sigma_I = e_{i_1}e_{i_2}\cdots e_{i_k}\varphi, \tag{2.10}$$

for suitable $\varphi \in Cl_{Spin}(M)_x$ with $|\varphi| = 1$, where $I = (i_1, \cdots, i_k)$ ranges over increasing multi-indices $(1 \leq i_1 < \cdots < i_k \leq n)$, and for I empty, we define $\sigma_I = \varphi$. Given such $I = (i_1, i_2, \cdots, i_k)$ and a pair (i,j) with $1 \leq i < j \leq n$, there is a unique basis element $\sigma_{I'}$, $I' = (i'_1, i'_2, \cdots, i'_{k'})$, such that

$$|\langle e_j\sigma_I, e_i\sigma_{I'}\rangle| = 1, \tag{2.11}$$

namely $\sigma_{I'} = \pm e_ie_j\sigma_I$, where the sign is chosen so that $\pm e_ie_j\sigma_I$ reduces to a basis element, instead of the negative of a basis element. We define $I(i,j)$ to be the unique increasing multi-index such that

$$\left|\left\langle e_j\sigma_I, e_i\sigma_{I(i,j)}\right\rangle\right| = 1. \tag{2.12}$$

We use similar notation for $Cl_{Spin}(N)_{f(x)}$, say $v_J := \varepsilon_{j_1}\varepsilon_{j_2}...\varepsilon_{j_k}\psi$ for some fixed $\psi \in Cl_{Spin}(N)_{f(x)}$ with $|\psi| = 1$, $J := (j_1, j_2, \cdots, j_k)$, $1 \le j_1 < j_2 < \cdots < j_k \le m$, and

$$\left|\left\langle \varepsilon_k v_J, \varepsilon_h v_{J(h,k)} \right\rangle\right| = 1. \tag{2.13}$$

Let $f_{*pi} := \langle f_*(e_i), \varepsilon_p\rangle$, and note that $f_{*pi} = 0$ for $i > r$ or $p > r$. By definition of the action of the Lie algebra of Spin(m) on $Cl(m)$, we have

$$\begin{aligned}
\mathfrak{R}^f(\sigma \otimes v) &= \tfrac{1}{2}\sum_{i,j=1}^{r}(e_ie_j\sigma)\otimes R^N(f_*e_i, f_*e_j)(v)\\
&= \tfrac{1}{2}\sum_{i,j=1}^{r}(e_ie_j\sigma)\otimes \sum_{p,q=1}^{r} R^N(f_{*pi}\varepsilon_p, f_{*qj}\varepsilon_q)(v)\\
&= \tfrac{1}{2}\sum_{i,j=1}^{r}(e_ie_j\sigma)\otimes \sum_{h<k}^{m}\sum_{p,q=1}^{r} f_{*pi}f_{*qj}R^N_{hkpq}\left(-\tfrac{1}{2}\right)\varepsilon_h\varepsilon_k v\\
&= -\tfrac{1}{2}\sum_{i<j}^{r}\sum_{h<k}^{m}(e_ie_j\sigma)\otimes \sum_{p,q=1}^{r} R^N_{hkpq}f_{*pi}f_{*qj}\varepsilon_h\varepsilon_k v. \qquad (2.14)
\end{aligned}$$

For $\phi = \sum_{I,J} a_{IJ}(\sigma_I \otimes v_J) \in Cl_{Spin}(M)_x \otimes f^*Cl_{Spin}(N)_x$, we then have

$$\begin{aligned}
2\left\langle \mathfrak{R}^f\phi, \phi\right\rangle &= 2\left\langle \mathfrak{R}^f\left(\sum_{I,J} a_{IJ}(\sigma_I\otimes v_J)\right), \sum_{I',J'} a_{I'J'}(\sigma_{I'}\otimes v_{J'})\right\rangle\\
&= -\left\langle \sum_{I,J} a_{IJ}\sum_{i<j}^{r}\sum_{h<k}^{m}\left(\sum_{p,q=1}^{r} R^N_{hkpq}f_{*pi}f_{*qj}\right)(e_ie_j\sigma_I)\otimes(\varepsilon_h\varepsilon_k v_J),\right.\\
&\qquad \left.\sum_{I',J'} a_{I'J'}(\sigma_{I'}\otimes v_{J'})\right\rangle\\
&= -\sum_{I,J,I',J'}\sum_{i<j}^{r}\sum_{h<k}^{m} a_{IJ}a_{I'J'}\left(\sum_{p,q=1}^{r} R^N_{hkpq}f_{*pi}f_{*qj}\right)\cdot\\
&\qquad \cdot\langle e_j\sigma_I, e_i\sigma_{I'}\rangle\langle\varepsilon_k v_J, \varepsilon_h v_{J'}\rangle\\
&= -\sum_{I,J}\sum_{i<j}^{r}\sum_{h<k}^{m}\left(\sum_{p,q=1}^{r} R^N_{hkpq}f_{*pi}f_{*qj}\right) a_{IJ}a_{I(i,j)J(h,k)}\cdot\\
&\qquad \cdot(\pm 1)\left\langle e_j\sigma_I, e_i\sigma_{I(i,j)}\right\rangle\left\langle\varepsilon_k v_J, \varepsilon_h v_{J(h,k)}\right\rangle\\
&\ge -\sum_{i<j}^{r}\sum_{h<k}^{m}\left|\sum_{p,q=1}^{r} R^N_{hkpq}f_{*pi}f_{*qj}\right|\sum_{I,J}|a_{IJ}|\left|a_{I(i,j)J(h,k)}\right|\\
&\ge -\sum_{i<j}^{r}\sum_{h<k}^{m}\left(\left|\sum_{p,q=1}^{r} R^N_{hkpq}f_{*pi}f_{*qj}\right|\sum_{I,J}\tfrac{1}{2}\left(|a_{IJ}|^2 + \left|a_{I(i,j)J(h,k)}\right|^2\right)\right)\\
&= -\sum_{i<j}^{r}\sum_{h<k}^{m}\left(\left|\sum_{p,q=1}^{r} R^N_{hkpq}f_{*pi}f_{*qj}\right|\right)\sum_{I,J}|a_{IJ}|^2 \qquad (2.15)
\end{aligned}$$

The expression $\sum_{i<j}^{r}\sum_{h<k}^{m}\left(\left|\sum_{p,q=1}^{r} R^N_{hkpq} f_{*pi} f_{*qj}\right|\right)$ depends on the choice of adapted bases $e_1, \cdots, e_n$ and $\varepsilon_1, \cdots, \varepsilon_m$. Since this choice has been arbitrary, we have

$$\langle \mathfrak{R}^f \phi, \phi \rangle \geq -\tfrac{1}{2} \min_{\mathcal{B}} \left(\sum_{i<j}^{r} \sum_{h<k}^{m} \left| \sum_{p,q=1}^{r} R^N_{hkpq} f_{*pi} f_{*qj} \right| \right) |\phi|^2 \tag{2.16}$$

where the min is taken over the set $\mathcal{B}$ of all choices of adapted bases $e_1, \cdots, e_n$ and $\varepsilon_1, \cdots, \varepsilon_m$. We remark that for an integer $q > 0$, $Cl\,(q)$ regarded as a module over itself splits into a number of irreducible $Cl\,(q)$ modules (spinor representation spaces $\Sigma\,(q)$). For q even, the $\Sigma\,(q)$ decompose further into $Spin(q)$ submodules $\Sigma\,(q)^{\pm}$. In particular, since the operator D (or D^2) and the curvature operator $\mathfrak{R}^f$ respect these decompositions, we may consider the related operators

(2.17)

$$\begin{array}{ll} D : \Gamma\left(\Sigma\,(M) \otimes f^*\Sigma\,(N)\right) \hookleftarrow & n \text{ and } m \text{ odd} \\ D : \Gamma\left(\Sigma\,(M)^+ \otimes f^*\Sigma\,(N)\right) \to \Gamma\left(\Sigma\,(M)^- \otimes f^*\Sigma\,(N)\right) & n \text{ even and } m \text{ odd} \\ D : \Gamma\left(\Sigma\,(M)^+ \otimes f^*\Sigma\,(N)^{\pm}\right) \to \Gamma\left(\Sigma\,(M)^- \otimes f^*\Sigma\,(N)^{\pm}\right) & n \text{ and } m \text{ even} \end{array}$$

Then (2.3), (2.8) and (2.16) hold for ϕ in the domain of any of the above related operators. A possible choice for ρ_f in (2.6) is

$$\widetilde{\rho}_f\,(x) := 2 \min_{\mathcal{B}} \left(\sum_{i<j}^{r} \sum_{h<k}^{m} \left| \sum_{p,q=1}^{r} R^N_{hkpq} f_{*pi} f_{*qj} \right| \right). \tag{2.18}$$

In applications, we will sometimes find it necessary to further twist $Cl_{Spin}\,(M) \otimes f^*Cl_{Spin}\,(N)$ by a complex line bundle L over M equipped with a Hermitian metric and a compatible connection ∇^L with curvature $F^L \in \Omega^2\,(M, i\mathbb{R})$. We then have a Dirac operator

$$D^L : \Gamma\left(L \otimes Cl_{Spin}\,(M) \otimes f^*Cl_{Spin}\,(N)\right) \hookleftarrow \tag{2.19}$$

and a corresponding B-L formula

$$\left(D^L\right)^2 \phi = \left(\nabla^L \otimes \nabla\right)^* \left(\nabla^L \otimes \nabla\right) \phi + \tfrac{1}{4}\kappa_M \phi + \left(I \otimes \mathfrak{R}^f\right)\phi + \mathfrak{R}^L \phi, \tag{2.20}$$

where on a simple element $\phi = \tau \otimes (\sigma \otimes v) \in L \otimes \left(Cl_{Spin}\,(M) \otimes f^*Cl_{Spin}\,(N)\right)$

(2.21)

$$\left(I \otimes \mathfrak{R}^f\right)\phi + \mathfrak{R}^L \phi = \tau \otimes \mathfrak{R}^f\,(\sigma \otimes v) + \tfrac{1}{2} \sum_{i,j=1}^{n} F^L\,(e_i, e_j)\,(\tau) \otimes \left((e_i e_j \sigma) \otimes v\right).$$

(see [**La-Mi**], p. 164). In the notation of (2.10), locally for $\phi = \sum_{I,J} a_{IJ}\,\tau \otimes (\sigma_I \otimes v_J)$ with $|\tau| = 1$, one checks

$$\langle \mathfrak{R}^L \phi, \phi \rangle \geq -\tfrac{1}{2} \left(\sum_{i,j=1}^{n} \left| F^L\,(e_i, e_j) \right| \right) |\phi|^2. \tag{2.22}$$

For the set $\mathcal{B}$ of all orthonormal bases of T_xM, we define

$$\rho_L\,(x) := \min_{\mathcal{B}} \left(2 \sum_{i,j=1}^{n} \left| F^L\,(e_i, e_j) \right| \right). \tag{2.23}$$

Then there is an obvious L-twisted extension of Proposition 2.1, namely

PROPOSITION 2.2. *Let* $\phi \in \Gamma\left(L \otimes Cl_{Spin}\left(M\right) \otimes f^{*}Cl_{Spin}\left(N\right)\right)$, $\phi \not\equiv 0$, *satisfy* $\left|D^{L}\phi\right|^{2} \leq \frac{1}{4}\eta\left|\phi\right|^{2}$, *for some* $\eta \in C^{\infty}\left(M\right)$, *and let* $\rho_{f} \in C^{\infty}\left(M\right)$ *be a function such that*

$$\left\langle \mathfrak{R}^{f}\phi, \phi\right\rangle \geq -\tfrac{1}{4}\rho_{f}\left|\phi\right|^{2}. \tag{2.24}$$

Then there is a point $x \in M$ *such that*

$$\kappa_{M}\left(x\right) \leq \eta\left(x\right) + \rho_{f}\left(x\right) + \rho_{L}\left(x\right). \tag{2.25}$$

Moreover, if $\kappa_{M} \geq \eta+\rho_{f}+\rho_{L}$ *everywhere, then* $\kappa_{M} = \eta+\rho_{f}+\rho_{L}$ *and* $\left(\nabla^{L}\otimes\nabla\right)\phi = 0$.

3. Alternative forms

There are other expressions which serve as a suitable ρ_{f}, which we now describe. The full Riemann curvature tensor of N at $y \in N$ can be regarded as a symmetric operator (known as the curvature operator of (N, g_{N}) at y)

$$R_{y}^{N} : \Lambda^{2}\left(T_{y}N\right) \to \Lambda^{2}\left(T_{y}N\right) \text{ via}$$

$$g_{N}\left(R^{N}\left(X\wedge Y\right), Z\wedge W\right) = g_{N}\left(R^{N}\left(X,Y\right)W, Z\right) \tag{3.1}$$

in terms of the usual $R^{N} \in \Lambda^{2}\left(N, End\left(TN\right)\right)$. Moreover, $f_{*x} : T_{x}M \to T_{f(x)}N$ induces

$$\Lambda^{2}\left(f_{*x}\right) : \Lambda^{2}\left(T_{x}M\right) \to \Lambda^{2}\left(T_{f(x)}N\right). \tag{3.2}$$

The matrix elements of $R_{f(x)}^{N} \circ \Lambda^{2}\left(f_{*x}\right) \in \mathrm{Hom}\left(\Lambda^{2}\left(T_{x}M\right), \Lambda^{2}\left(T_{f(x)}N\right)\right)$ with respect to the bases $\{e_{i}\wedge e_{j} : 1 \leq i < j \leq n\}$ and $\{\varepsilon_{h}\wedge\varepsilon_{k} : 1 \leq h < k \leq m\}$ are denoted by

$$\left(R_{f(x)}^{N} \circ \Lambda^{2}\left(f_{*x}\right)\right)_{hk,ij} := \left\langle \varepsilon_{h}\wedge\varepsilon_{k}, R_{f(x)}^{N} \circ \Lambda^{2}\left(f_{*x}\right)\left(e_{i}\wedge e_{j}\right)\right\rangle. \tag{3.3}$$

Then

$$\begin{aligned}\sum_{i<j}^{r}\sum_{h<k}^{m}\left|\sum_{p,q=1}^{r} R_{hkpq}^{N} f_{*pi} f_{*qj}\right| &= \sum_{i<j}^{r}\sum_{h<k}^{m}\left|\left\langle \varepsilon_{h}\wedge\varepsilon_{k}, R_{f(x)}^{N} \circ \Lambda^{2}\left(f_{*x}\right)\left(e_{i}\wedge e_{j}\right)\right\rangle\right| \\ &= \sum_{i<j}^{r}\sum_{h<k}^{m}\left|\left(R_{f(x)}^{N} \circ \Lambda^{2}\left(f_{*x}\right)\right)_{hk,ij}\right|.\end{aligned} \tag{3.4}$$

By Schwarz's inequality,

$$\begin{aligned}&\left(\sum_{i<j}^{r}\sum_{h<k}^{m}\left|\left(R_{f(x)}^{N} \circ \Lambda^{2}\left(f_{*x}\right)\right)_{hk,ij}\right|\right)^{2} \\ &\leq \tfrac{1}{4}r\left(r-1\right)m\left(m-1\right)\sum_{i<j}^{r}\sum_{h<k}^{m}\left|\left(R_{f(x)}^{N} \circ \Lambda^{2}\left(f_{*x}\right)\right)_{hk,ij}\right|^{2}.\end{aligned} \tag{3.5}$$

To proceed further we use the elementary

LEMMA 3.1. *Let V and W be inner product spaces of dimensions n and m respectively, and let $L : V \to W$ be a linear map of rank r. Then there are orthonormal bases $v_1, ..., v_n$ of V and $w_1, ..., w_m$ of W and positive $\alpha_i \in \mathbb{R}$ $(1 \le i \le r)$, such that $Lv_i = \alpha_i w_i$ $(1 \le i \le r)$, while $Lv_i = 0$ for $r+1 \le i \le n$. The α_i are the positive eigenvalues of $\sqrt{L^*L} \in End(V)$.*

For the proof one simply chooses $v_1, ..., v_n$ to be an orthonormal basis of eigenvectors of the symmetric operator L^*L with eigenvalues α_i^2, one puts $w_i = \frac{1}{\alpha_i} Lv_i$ for $1 \le i \le r$ and one arbitrarily extends $w_1, ..., w_r$ to an orthonormal basis of W.

Thus, assuming that $\Lambda^2(f_{*x})$ is contractive, there are orthonormal bases $w_1, ..., w_{r(r-1)/2}$ and $\omega_1, ..., \omega_{r(r-1)/2}$ of $\Lambda^2\left(\ker(f_{*x})^{\perp}\right)$ and $\Lambda^2(\text{image}(f_{*x}))$, such that

$$\Lambda^2(f_{*x})(w_a) = \lambda_a \omega_a \ \text{ for } 1 \le a \le \tfrac{1}{2} r(r-1), \tag{3.6}$$

where $0 < \lambda_a \le 1$. We extend these bases to orthonormal bases for $\Lambda^2(T_xM)$ and $\Lambda^2\left(T_{f(x)}N\right)$. The norm of $R^N_{f(x)} \circ \Lambda^2(f_{*x}) \in \text{Hom}\left(\Lambda^2(T_xM), \Lambda^2\left(T_{f(x)}N\right)\right)$ relative to the inner product induced by $g_M(x)$ and $g_N(f(x))$ is the same whether it is expressed using the pair of orthonormal bases $(\{e_i \wedge e_j\}, \{\varepsilon_i \wedge \epsilon_j\})$ or the pair $(\{w_a\}, \{\omega_b\})$. Using this fact

$$\sum_{i<j}^{r} \sum_{h<k}^{m} \left| \left(R^N_{f(x)} \circ \Lambda^2(f_{*x}) \right)_{hk,ij} \right|^2$$

$$\begin{aligned}
&= \sum_{a=1}^{r(r-1)/2} \sum_{b=1}^{m(m-1)/2} \left| \left\langle \omega_b, \left(R^N_{f(x)} \circ \Lambda^2(f_{*x}) \right)(w_a) \right\rangle \right|^2 \\
&= \sum_{a=1}^{r(r-1)/2} \sum_{b=1}^{m(m-1)/2} \left| \left\langle \omega_b, R^N_{f(x)}(\lambda_a \omega_a) \right\rangle \right|^2 \\
&\le \sum_{a=1}^{r(r-1)/2} \sum_{b=1}^{m(m-1)/2} \left| \left\langle \omega_b, R^N_{f(x)}(\omega_a) \right\rangle \right|^2 \\
&\le \sum_{a,b=1}^{m(m-1)/2} \left| \left\langle \omega_b, R^N_{f(x)}(\omega_a) \right\rangle \right|^2 \\
&= \sum_{i<j,h<k}^{m} \left| R^N_{hk,ij} \right|^2_{f(x)}.
\end{aligned} \tag{3.7}$$

Combining (3.5) and (3.7), we obtain (under the assumption that $\Lambda^2(f_{*x})$ is contractive),

$$\begin{aligned}
&\sum_{i<j}^{r}\sum_{h<k}^{m}\left|\left(R^N_{f(x)}\circ\Lambda^2(f_{*x})\right)_{hk,ij}\right| \\
&\leq \tfrac{1}{2}\sqrt{r(r-1)m(m-1)}\left(\sum_{i<j}^{r}\sum_{h<k}^{m}\left|\left(R^N_{f(x)}\circ\Lambda^2(f_{*x})\right)_{hk,ij}\right|^2\right)^{\frac{1}{2}} \\
&\leq \tfrac{1}{2}\sqrt{r(r-1)m(m-1)}\left(\tfrac{1}{4}\sum_{i,j,h,k=1}^{m}\left|R^N_{hkij}\right|^2_{f(x)}\right)^{\frac{1}{2}} \\
&\leq \tfrac{1}{4}\sqrt{r(r-1)m(m-1)}\left|R^N\right|_{f(x)}, \qquad (3.8)
\end{aligned}$$

where

$$\left|R^N\right| := \left(\sum_{i,j,h,k=1}^{m}\left|R^N_{hkij}\right|^2\right)^{\frac{1}{2}} \qquad (3.9)$$

is the usual norm of the Riemann curvature tensor R^N relative to g_N. We get the best results by applying (3.8) to the trace-free part of R^N rather than to R^N itself. More precisely, as in Section 1.G of [**Be**], decompose the curvature operator $R^N_{f(x)}$ into a multiple of the identity and trace-free part which further decomposes into a trace-free Ricci part and a Weyl part, say

(3.10)

$$R^N_{f(x)} = \frac{\kappa_N(f(x))}{m(m-1)}I^N_{f(x)} + F^N_{f(x)} = \frac{\kappa_N(f(x))}{m(m-1)}I^N_{f(x)} + TFRicci^N_{f(x)} + W^N_{f(x)}.$$

Then,

$$\begin{aligned}
\sum_{i<j}^{r}\sum_{h<k}^{m}\left|\sum_{p,q=1}^{r}R^N_{hkpq}f_{*pi}f_{*qj}\right| &= \sum_{i<j}^{r}\sum_{h<k}^{m}\left|\left\langle\varepsilon_h\wedge\varepsilon_k, R^N_{f(x)}\circ\Lambda^2(f_{*x})(e_i\wedge e_j)\right\rangle\right| \\
&\leq \sum_{i<j}^{r}\sum_{h<k}^{m}\left|\left\langle\varepsilon_h\wedge\varepsilon_k, \left(\frac{\kappa_N(f(x))}{m(m-1)}I^N_{f(x)}\right)\circ\Lambda^2(f_{*x})(e_i\wedge e_j)\right\rangle\right| \\
&\quad+\sum_{i<j}^{r}\sum_{h<k}^{m}\left|\left\langle\varepsilon_h\wedge\varepsilon_k, F^N_{f(x)}\circ\Lambda^2(f_{*x})(e_i\wedge e_j)\right\rangle\right| \\
&\leq \frac{\frac{1}{2}r(r-1)}{m(m-1)}\left|\kappa_N(f(x))\right| + \sum_{i<j}^{r}\sum_{h<k}^{m}\left|\left(F^N_{f(x)}\circ\Lambda^2(f_{*x})\right)_{hk,ij}\right| \\
&\leq \frac{1}{2}\frac{r(r-1)}{m(m-1)}\left|\kappa_N(f(x))\right| + \tfrac{1}{4}\sqrt{r_x(r_x-1)m(m-1)}\left|F^N_{f(x)}\right|. \qquad (3.11)
\end{aligned}$$

We then get the following more elegant (but possibly weaker) choice for ρ_f

$$\rho_f(x) = \frac{r_x(r_x-1)}{m(m-1)}\left|\kappa_N(f(x))\right| + \tfrac{1}{2}\sqrt{r_x(r_x-1)m(m-1)}\left|F^N_{f(x)}\right|, \qquad (3.12)$$

where r_x (previously r) is the rank of f_{*x}. In view of Proposition 2.1 we have

THEOREM 3.2. *Let (M, g_M) and (N, g_N) be Riemannian spin manifolds of dimensions n and m respectively. Let $f : M \to N$ be a smooth area-nonincreasing map, such that there is a nonzero harmonic section for any of the Dirac operators in (2.17). If r_x $(\leq \min(n,m))$ denotes the rank of f_* at x, then there is a point $x \in M$ such that*

$$\kappa_M(x) \leq \frac{r_x(r_x-1)}{m(m-1)} |\kappa_N(f(x))| + \tfrac{1}{2}\sqrt{r_x(r_x-1)m(m-1)} \left| F^N_{f(x)} \right|, \tag{3.13}$$

where $\left| F^N_{f(x)} \right|$ is the norm of the trace-free part of Riemann curvature tensor of (N, g_N) at $f(x)$ with respect to g_N.

REMARK 3.3. *If (N, g_N) is Einstein, $F^N = W^N$. When (N, g_N) is conformally flat or* $\dim N \leq 3$, *we have $F^N_{f(x)} = TFRicci^N_{f(x)}$. If (N, g_N) is the standard sphere, then $F^N = 0$ and we obtain the sharp estimate of Llarull.*

REMARK 3.4. *If we did not perform the splitting (3.10), then in place of (3.13) we obtain the apparently less applicable result*

$$\kappa_M(x) \leq \tfrac{1}{2}\sqrt{r_x(r_x-1)m(m-1)} \left| R^N_{f(x)} \right|, \tag{3.14}$$

where $\left| R^N_{f(x)} \right|$ is the norm of the Riemann curvature tensor of (N, g_N) at $f(x)$ with respect to g_N.

4. Index computations

In order that the preceding theorems apply, one needs to guarantee the existence of nonzero Dirac harmonic sections for the operators in (2.17). In certain cases, this can be done via the Atiyah-Singer Index Theorem for twisted Dirac operators. However, we will first indicate how to get results for non-spin manifolds by considering operators which are only locally twisted Dirac operators. We continue to assume that M and N are compact and orientable. If $U \subseteq M$ is a coordinate ball in M, such that $f(U)$ is contained in a coordinate ball V in N, then the bundles $\Sigma(U)$ and $f^*\Sigma(V)|_U$ exist even if M and N are not spin. If transition functions for local bundles $\Sigma(U) \otimes f^*\Sigma(V)|_U$ can be found so that there is a global bundle E_f with $E_f|_U = \Sigma(U) \otimes f^*\Sigma(V)|_U$, then the locally defined Dirac operators

$$D_U : \Gamma(\Sigma(U) \otimes f^*\Sigma(V)|_U) \to \Gamma(\Sigma(U) \otimes f^*\Sigma(V)|_U) \tag{4.1}$$

may be used to define a global first-order elliptic differential operator

$$D : \Gamma(E_f) \to \Gamma(E_f). \tag{4.2}$$

It is not difficult to verify that E_f exists if $w_2(M) = f^*w_2(N)$, and in this case we call f a **spin mapping**. Note that E_f is just the spin bundle associated with a spin structure for the Riemannian bundle $TM \oplus f^*TN$. Furthermore, if $n :=$ dim M is even, there is a splitting $E_f = E_f^+ \oplus E_f^-$, such that

$$\left. E_f^\pm \right|_U = \Sigma(U)^\pm \otimes f^*\Sigma(V)|_U. \tag{4.3}$$

This is a consequence of the assumption that M and N are orientable. If M and N were both nonorientable and $w_1(M) = f^*w_1(N)$, as well as $w_2(M) = f^*w_2(N)$, then E_f would still exist but a splitting $E_f = E_f^+ \oplus E_f^-$ satisfying (4.3) would not exist. Returning to the case at hand, we have

$$D^+ : \Gamma\left(E_f^+\right) \to \Gamma\left(E_f^-\right) \tag{4.4}$$

Since the n-form (local index form) whose integral is the index of D may be computed locally, the index of D^+ will be given by the same formula as in the case where M and N are spin. When M and N are spin, we have

$$\operatorname{index}\left(D^+\right) = \left(\widehat{\mathbf{A}}\left(M\right) f^*\left(ch\left(\Sigma\left(N\right)\right)\right)\right)[M] \tag{4.5}$$

Now the total class $\widehat{\mathbf{A}}\left(M\right) \in H^*\left(M, Q\right)$ makes sense even when M is not spin, but we need to express $ch\left(\Sigma\left(N\right)\right)$ in a form which still makes sense when N is not spin. According to [**Gi**], p. 244, for $m = 2\mu$ even, the Chern character of the spin bundle $\Sigma\left(N\right)$ is obtained as follows. Expand

(4.6)
$$\prod_{i=1}^{\mu}\left(e^{x_i/2} + e^{-x_i/2}\right) = 2^\mu \prod_{i=1}^{\mu} \cosh\left(x_i/2\right) = 2^\mu + ch_1\left(\sigma_1\right) + ch_2\left(\sigma_1, \sigma_2\right) + \cdots ,$$

where σ_i is the i-th elementary symmetric polynomial in $x_1^2, ..., x_n^2$. Then replace each σ_i by the Pontrjagin class $p_i(N)$. The result can be expressed in terms of the Hirzebruch polynomial $\widehat{\mathbf{L}}\left(N\right)$ and $\widehat{\mathbf{A}}\left(N\right)$. The defining power series for $\widehat{\mathbf{A}}$ is $\frac{x/2}{\sinh(x/2)}$, while that for $\widehat{\mathbf{L}}$ is $\frac{x/2}{\tanh(x/2)}$. Since

$$\frac{\frac{x/2}{\tanh(x/2)}}{\frac{x/2}{\sinh(x/2)}} = \frac{\sinh\left(x/2\right)}{\tanh\left(x/2\right)} = \cosh\left(x/2\right), \tag{4.7}$$

we have

$$ch\left(\Sigma\left(N\right)\right) = 2^\mu \frac{\widehat{\mathbf{L}}\left(N\right)}{\widehat{\mathbf{A}}\left(N\right)}. \tag{4.8}$$

Thus, when $n = \dim M$ is even, so that the bundle E_f exists and splits ($E_f = E_f^+ \oplus E_f^-$) as in (4.3), we have

$$\begin{aligned} \operatorname{index}\left(D^+\right) &= \left(\widehat{\mathbf{A}}\left(M\right) f^*\left(2^\mu \frac{\widehat{\mathbf{L}}\left(N\right)}{\widehat{\mathbf{A}}\left(N\right)}\right)\right)[M] \\ &= 2^\mu \left(f^*\left(\widehat{\mathbf{L}}\left(N\right)\right) \frac{\widehat{\mathbf{A}}\left(M\right)}{f^*\left(\widehat{\mathbf{A}}\left(N\right)\right)}\right)[M]. \end{aligned} \tag{4.9}$$

When M and N are both spin and even-dimensional, in place of $\Sigma\left(N\right)$ one could consider the half-spinor bundles $\Sigma\left(N\right)^+$ and $\Sigma\left(N\right)^-$ and the Dirac operators

$$D^{+,\pm} : \Gamma\left(\Sigma\left(M\right)^+ \otimes f^*\Sigma\left(N\right)^\pm\right) \to \Gamma\left(\Sigma\left(M\right)^- \otimes f^*\Sigma\left(N\right)^\pm\right). \tag{4.10}$$

In fact, if M and N are not necessarily spin but f is a spin mapping, we have global bundles $E^{+,\pm}$ and $D^{+,\pm} : \Gamma\left(E^{+,\pm}\right) \to \Gamma\left(E^{-,\pm}\right)$ which are locally of the form (4.10). If $\nu_N \in H^m(N,\mathbb{Z})$ denotes the generating orientation class of N and $\chi(N)$ the Euler characteristic of N, then (see [**Gi**], p. 244)

$$\begin{aligned}\operatorname{index}\left(D^{+,+}\right) - \operatorname{index}\left(D^{+,-}\right) &= (-1)^\mu \left(\widehat{\mathbf{A}}(M)\, f^*\left(\chi(N)\,\nu_N\right)\right)[M] \\ &= (-1)^\mu \chi(N) \left(\widehat{\mathbf{A}}(M)_{n-m} f^*(\nu_N)\right)[M] = (-1)^\mu \chi(N)\ \widehat{\mathbf{A}}\text{-}\deg(f). \qquad (4.11)\end{aligned}$$

Here, the so-called $\widehat{\mathbf{A}}$**-degree of** f

$$\widehat{\mathbf{A}}\text{-}\deg(f) := \left(\widehat{\mathbf{A}}(M)_{n-m} f^*(\nu_N)\right)[M] \qquad (4.12)$$

is introduced in [**La-Mi**], p. 309. It is just the ordinary degree of f when $n = m$. We have

$$\begin{aligned}\operatorname{index}\left(D^{+,\pm}\right) &= 2^{\mu-1}\left(f^*\left(\widehat{\mathbf{L}}(N)\right)\frac{\widehat{\mathbf{A}}(M)}{f^*\left(\widehat{\mathbf{A}}(N)\right)}\right)[M] \\ &\quad \pm\tfrac{1}{2}(-1)^\mu \chi(N)\ \widehat{\mathbf{A}}\text{-}\deg(f). \qquad (4.13)\end{aligned}$$

Thus, we have

Theorem 4.1. *Let (M,g_M) and (N,g_N) be compact, orientable Riemannian manifolds of dimensions n and m respectively, with n even. Let $f: M \to N$ be a smooth area-nonincreasing spin mapping, such that either*

$$\left(f^*\left(\widehat{\mathbf{L}}(N)\right)\frac{\widehat{\mathbf{A}}(M)}{f^*\left(\widehat{\mathbf{A}}(N)\right)}\right)[M] \neq 0, \qquad (4.14)$$

or

$$\chi(N)\ \widehat{\mathbf{A}}\text{-}\deg(f) \neq 0. \qquad (4.15)$$

If r_x denotes the rank of f_* at x, then there is a point $x \in M$ such that, in the notation of Theorem 3.2,

$$\kappa_M(x) \le \frac{r_x(r_x-1)}{m(m-1)}\left|\kappa_N(f(x))\right| + \tfrac{1}{2}\sqrt{r_x(r_x-1)\,m(m-1)}\left|F^N_{f(x)}\right|, \qquad (4.16)$$

Remark 4.2. *The inequality (4.16) may be replaced by*

$$\kappa_M(x) \le \tfrac{1}{2}\sqrt{r_x(r_x-1)\,m(m-1)}\left|R^N_{f(x)}\right|, \qquad (4.17)$$

where $\left|R^N_{f(x)}\right|$ is the norm of the Riemann curvature tensor of (N,g_N) at $f(x)$ with respect to g_N.

Since the identity map of an orientable manifold is a spin mapping, we have

Corollary 4.3. *Let M be a compact, orientable manifold of even dimension n and let g_0 and g_1 be Riemannian metrics on M, such that the identity $(M,g_1) \to (M,g_0)$ is area-nonincreasing. Suppose that either*

$$sign(M) = \mathbf{L}(M)[M] \neq 0, \qquad (4.18)$$

where $sign(M)$ is the signature of M, or

$$\chi(M) \neq 0. \qquad (4.19)$$

If $\kappa_1(x)$ *denotes the scalar curvature of* (M, g_1) *at* $x \in M$ *and* F_x^0 *is the trace-free part of Riemann curvature tensor* R_x^0 *of* (M, g_0) *at* x*, then there is a point* $x \in M$ *such that*

$$\kappa_1(x) \le |\kappa_0(x)| + \tfrac{1}{2} n (n-1) \left|F_x^0\right|. \tag{4.20}$$

In particular,

$$\min \kappa_1 \le \max \left(|\kappa_0| + \tfrac{1}{2} n (n-1) \left|F^0\right|\right). \tag{4.21}$$

so that there is an upper bound on $\min \kappa_1$ *over all metrics* g_1 *on* M *which give each subsurface no smaller area than some fixed metric* g_0 *does.*

REMARK 4.4. *Alternatively, under the assumptions, there is a point* $x \in M$*, such that*

$$\kappa_1(x) \le \tfrac{1}{2} n (n-1) \left|R_x^0\right|. \tag{4.22}$$

5. Applications

Our aim is to obtain upper bounds on the minimum of the scalar curvature κ_g for a wide class of smooth Riemannian manifolds (M, g) which are diffeomorphic to smooth algebraic varieties (more specifically, complete intersections) (V, g_0) via length-nonincreasing diffeomorphisms $(M, g) \to (V, g_0)$, where g_0 is the metric induced on V from the standard Fubini-Study metric on the ambient $\mathbb{P}^\mu(\mathbb{C})$. Actually, we do somewhat more, but it is best to directly consult the main result, Theorem 5.4.

A **complete intersection** is a nonsingular algebraic variety in $\mathbb{P}^\mu(\mathbb{C})$, which is the *transverse* intersection of nonsingular hypersurfaces defined by homogeneous polynomials. We denote by $V^{\mu-r}(a_1, \cdots, a_r)$ a complete intersection in $\mathbb{P}^\mu(\mathbb{C})$, which is defined by homogeneous polynomials of degrees $a_1, \cdots, a_r$. Let $M = V^{\mu-r}(a_1, \cdots, a_r)$ with an arbitrary Riemannian metric g_M, and let $f : M \to \mathbb{P}^\mu(\mathbb{C})$ be a smooth map homotopic to the inclusion $\iota : V^{\mu-r}(a_1, \cdots, a_r) \subseteq \mathbb{P}^\mu(\mathbb{C})$. Our first goal will be to show

LEMMA 5.1. *The map* $f : M \to \mathbb{P}^\mu(\mathbb{C})$ *is a spin mapping if* $|a| := a_1 + \cdots + a_r$ *is even.*

PROOF. We may assume that f is the inclusion $V = V^{\mu-r}(a_1, \cdots, a_r) \subseteq \mathbb{P}^\mu(\mathbb{C})$, since the notion of spin mapping is invariant under homotopy. Let $\mathbf{x}$ be the standard generator of $H^2(\mathbb{P}^\mu(\mathbb{C}), \mathbb{Z})$, namely $\mathbf{x} = \mathbf{c}_1(\xi^*) = -\mathbf{c}_1(\xi)$ where ξ is the tautological line bundle over $\mathbb{P}^\mu(\mathbb{C})$. It is well-known [**Hi**, p. 159] that

$$\mathbf{c}(TV) = (1+\mathbf{x})^{\mu+1} \prod_{i=1}^{r} (1 + a_i \mathbf{x})^{-1} = 1 + (\mu + 1 - |a|)\,\mathbf{x} + \dots. \tag{5.1}$$

Since $c_1(V^{\mu-r}(a_1, \cdots, a_r)) = (\mu + 1 - |a|)\,\mathbf{x}$ and $c_1(\mathbb{P}^\mu(\mathbb{C})) = (\mu+1)\,\mathbf{x}$,

$$\iota^* w_2(\mathbb{P}^\mu(\mathbb{C})) = [(\mu+1)\,\mathbf{x}]_2 = [(\mu+1-|a|)\,\mathbf{x}]_2 = w_2\left(V^{\mu-r}(a_1, \cdots, a_r)\right) \tag{5.2}$$

if $|a|$ is even. Hence, the inclusion $\iota : V^{\mu-r}(a_1, \cdots, a_r) \subseteq \mathbb{P}^\mu(\mathbb{C})$ is a spin mapping if $|a|$ is even. □

For the time being we assume that $|a|$ is even, so that $f : M \to \mathbb{P}^\mu(\mathbb{C})$ is a spin mapping. Recall that we then have a global first-order elliptic differential operator

$$D : \Gamma(E_f) \to \Gamma(E_f) \tag{5.3}$$

which is locally of the form

$$D_{U_\alpha} : \Gamma\left(\Sigma(U_\alpha) \otimes f^*\Sigma(V_\alpha)|_{U_\alpha}\right) \hookleftarrow \tag{5.4}$$

where $\{U_\alpha\}$ is a covering of M and $\{V_\alpha\}$ is a covering of $\mathbb{P}^\mu(\mathbb{C})$ with $f(U_\alpha) \subseteq V_\alpha$. We let SM be the pull-back via f of the Hopf bundle $S^{2\mu+1} \to \mathbb{P}^\mu(\mathbb{C})$. The connection 1-form θ pulls back, via $f_S : SM \to S^{2\mu+1}$ covering $f : M \to \mathbb{P}^\mu(\mathbb{C})$, to a connection $\theta^f := f_S^*\theta$ on SM. If $\pi_M : SM \to M$ denotes the projection, we give SM the Riemannian metric

$$g_{SM} := \pi_M^* g_M + (-i\theta)^2. \tag{5.5}$$

If $f : M \to \mathbb{P}^\mu(\mathbb{C})$ does not increase lengths, the mapping $f_S : SM \to S^{2\mu+1}$ does not increase areas. We can locally pull back $\Sigma(U_\alpha)$ to $\pi_M{}^*\Sigma(U_\alpha)$ on $\pi_M{}^{-1}(U_\alpha)$, and we can use $\pi : S^{2\mu+1} \to \mathbb{P}^\mu(\mathbb{C})$ to pull back $\Sigma(V_\alpha)$ to $\pi^*\Sigma(V_\alpha)$ on $\pi^{-1}(V_\alpha)$. Recall that the two irreducible modules $\Sigma(\mathbb{R}^{2n+1})$ of $\mathbb{C}l(\mathbb{R}^{2n+1})$ are obtained from the unique irreducible modules $\Sigma(\mathbb{R}^{2n})$ of $\mathbb{C}l(\mathbb{R}^{2n})$ by simply letting $\Sigma(\mathbb{R}^{2n+1}) = \Sigma(\mathbb{R}^{2n})$ and defining the action of e_{2n+1} to be multiplication by $\pm i$ on $\Sigma^{2n,+}$ and $\mp i$ on $\Sigma^{2n,-}$. For definiteness, we choose e_{2n+1} to act as $+i$ on $\Sigma^{2n,+}$. Thus, $\Sigma\left(\pi_M{}^{-1}(U_\alpha)\right) = \pi_M^*(\Sigma(U_\alpha))$ and $\Sigma\left(\pi^{-1}(V_\alpha)\right) = \pi^*(\Sigma(U_\alpha))$, and so

$$\begin{aligned} &\Sigma\left(\pi_M{}^{-1}(U_\alpha)\right) \otimes f_S^*\left(\Sigma\left(\pi^{-1}(V_\alpha)\right)\right)|_{\pi_M{}^{-1}(U_\alpha)} \\ &= \pi_M^*(\Sigma(U_\alpha)) \otimes \pi_M^* f^*(\Sigma(V_\alpha))|_{\pi_M{}^{-1}(U_\alpha)} \\ &\cong \pi_M^*\left(\Sigma(U_\alpha) \otimes f^*\Sigma(V_\alpha)|_{U_\alpha}\right). \end{aligned} \tag{5.6}$$

Consequently, we may take the bundle $E_{f_S} \to SM$ for the map $f_S : SM \to S^{2\mu+1}$ to be the pull back of $E_f \to M$ via π_M. Let

$$D^S : \Gamma(E_{f_S}) \to \Gamma(E_{f_S}), \tag{5.7}$$

be the corresponding operator which is locally the Dirac operator

$$D : \Gamma\left(\Sigma\left(\pi_M{}^{-1}(U_\alpha)\right) \otimes f_S^*\left(\Sigma\left(\pi^{-1}(V_\alpha)\right)\right)|_{\pi_M{}^{-1}(U_\alpha)}\right) \hookleftarrow. \tag{5.8}$$

LEMMA 5.2. *Let ω be the Kähler form of $\mathbb{P}^\mu(\mathbb{C})$. If $\psi \in \Gamma(E_M)$, then $\pi_M^*\psi \in \Gamma(E_{f_S})$ and*

$$D^S(\pi_M^*\psi) = \pi_M^*(D\psi) - \tfrac{1}{2}T_f \cdot (\pi_M^* f^*\omega) \cdot \pi_M^*\psi, \tag{5.9}$$

where "$\cdot$" is Clifford multiplication and T_f is the Killing field generating the S^1 action on SM.

The proof is contained in Lemma 4.3 and the proof of Theorem 4.1 in [**Am-Bä**]. Note that the "$d\omega$" in the proof of Theorem 4.1 in [**Am-Bä**] is $2\pi_M^* f^*\omega$, where ω denotes the Kähler form in the present paper.

LEMMA 5.3. *Assume that $f : M \to \mathbb{P}^\mu(\mathbb{C})$ is area-nonincreasing. If $\psi \in \Gamma(E_f)$ and $D\psi = 0$, then for $\nu := \mu - r = \frac{1}{2}\dim_{\mathbb{R}} M$, we have*

$$\left|D^S(\pi_M^*\psi)\right| \le \tfrac{1}{2}\nu\,|\psi| = \tfrac{1}{2}\nu\,|\pi_M^*\psi|. \tag{5.10}$$

PROOF. By (5.9), we have

$$D^S\left(\pi_M^*\psi\right) = -\tfrac{1}{2}\widetilde{e}_{2\nu+1}\cdot\pi_M^*\left(\omega^f\right)\cdot\pi_M^*\psi, \tag{5.11}$$

Since $\omega_x^f = f^*(\omega)_x$ is a 2-form (or equivalently an antisymmetric transformation) on T_xM of dimension 2ν, the orthonormal forms $\varphi_1,\cdots,\varphi_{2\nu}$ can be chosen such that there are nonnegative constants $\lambda_1,\cdots,\lambda_\nu$ for which

$$\omega_x^f = f^*(\omega)_x = \lambda_1\varphi_1\wedge\varphi_2+\cdots+\lambda_\nu\varphi_{2\nu-1}\wedge\varphi_{2\nu}. \tag{5.12}$$

By Lemma 3.1, we may assume that $g_N\left(f_*e_{2k-1},f_*e_{2k}\right)=0$, and since f is area-nonincreasing, $|f_*e_{2k-1}|\,|f_*e_{2k}|\le 1$. Note that

$$\begin{aligned}\lambda_k &= f^*(\omega)_x\left(e_{2k-1},e_{2k}\right) = \omega_{f(x)}\left(f_*e_{2k-1},f_*e_{2k}\right) = g_N\left(Jf_*e_{2k-1},f_*e_{2k}\right)\\ &\le |Jf_*e_{2k-1}|\,|f_*e_{2k}| = |f_*e_{2k-1}|\,|f_*e_{2k}|\le 1.\end{aligned} \tag{5.13}$$

Thus,

$$\begin{aligned}\left|D^S\left(\pi_M^*\psi\right)\right| &= \tfrac{1}{2}\left|T_f\cdot\pi_M^*\left(\omega^f\right)\cdot\pi_M^*\psi\right| = \tfrac{1}{2}\left|\pi_M^*\left(\omega^f\right)\cdot\pi_M^*\psi\right| = \tfrac{1}{2}\left|\omega^f\cdot\psi\right|\\ &= \tfrac{1}{2}\left|\left(\lambda_1\varphi_1\wedge\varphi_2+\cdots+\lambda_\nu\varphi_{2\nu-1}\wedge\varphi_{2\nu}\right)\cdot\psi\right|\\ &\le \tfrac{1}{2}\left|\left(\varphi_1\wedge\varphi_2\right)\cdot\psi\right|+\cdots+\tfrac{1}{2}\left|\left(\varphi_{2\nu-1}\wedge\varphi_{2\nu}\right)\cdot\psi\right|\\ &\le \tfrac{1}{2}\nu\,|\psi|.\end{aligned} \tag{5.14}$$

□

THEOREM 5.4. *Let $M = V^{\mu-r}(a_1,\cdots,a_r)$ be a complete intersection in $\mathbb{P}^\mu(\mathbb{C})$ defined by polynomials of degrees $a_1,\cdots,a_r$ and equipped with an arbitrary Riemannian metric g_M with scalar curvature κ_M. Let $f: M\to\mathbb{P}^\mu(\mathbb{C})$ be a smooth length-nonincreasing map homotopic to the inclusion $\iota: V^{\mu-r}(a_1,\cdots,a_r)\subseteq\mathbb{P}^\mu(\mathbb{C})$. Then for $\nu := \mu-r = \dim_{\mathbb{C}} V^{\mu-r}(a_1,\cdots,a_r)$*

$$\min\left(\kappa_M\right)\le\begin{cases}5\nu^2+4\nu & \text{for } \nu \text{ even}\\ 5\nu^2+12\nu & \text{for } \nu \text{ odd.}\end{cases} \tag{5.15}$$

Before the proof we state a special case.

COROLLARY 5.5. *Let $f: M = V^{\mu-r}(a_1,\cdots,a_r)\to\mathbb{P}^\mu(\mathbb{C})$ be an immersion homotopic through continuous maps to the inclusion $\iota: V^{\mu-r}(a_1,\cdots,a_r)\subseteq\mathbb{P}^\mu(\mathbb{C})$, and let κ_M be the scalar curvature of the metric on M induced by f. Then (5.15) holds.*

PROOF. (of Theorem 5.4) The scalar curvature of g_{SM} is given (see [**Be**], p. 253) by

$$\kappa_{SM} = \kappa_M - \tfrac{1}{2}\left|f^*d\theta\right|_{g_M}^2 = \kappa_M - 2\left|f^*\omega\right|_{g_M}^2 \ge \kappa_M - 2\nu, \tag{5.16}$$

where we have used (5.12). Note that $\kappa_M\le\kappa_{SM}+2\nu$, and in the special case $M=\mathbb{P}^\mu(\mathbb{C})$, $SM = S^{2\mu+1}$ and f is the identity, we obtain the known result $\kappa_{\mathbb{P}^\mu(\mathbb{C})} = (2\mu+1)\,2\mu+2\mu = 4\mu(\mu+1)$. We will assume until further notice that $a_1+\cdots+a_r$ is even so that f is a spin mapping. Suppose that there exists a nonzero $\psi\in\ker\left(D:\Gamma(E_f)\to\Gamma(E_f)\right)$. We apply Proposition 2.1, with $f_S: SM\to S^{2\mu+1}$ playing the role of $f: M\to N$ and $\pi_M^*\psi$ playing the role of ϕ. By virtue of Lemma 5.3 we may take $\eta=\nu^2$. Moreover, by the computation (3.11), we may choose ρ_{f_S} to be $2\nu(2\nu+1)$, since $\kappa_{S^{2\mu+1}}/\left((2\mu+1)(2\mu+2)\right)=1$. Thus, by (5.16) and an application of Proposition 2.1, we obtain

$$\min\left(\kappa_M\right)\le\min\left(\kappa_{SM}\right)+2\nu\le\left(\nu^2+2\nu(2\nu+1)\right)+2\nu = 5\nu^2+4\nu. \tag{5.17}$$

Thus, it suffices to show that $\ker(D : \Gamma(E_f) \to \Gamma(E_f)) \neq 0$. In fact, we will show that *index* $\left(D^+ : \Gamma\left(E_f^+\right) \to \Gamma\left(E_f^-\right)\right) > 0$. For this, we may assume that f is the inclusion $\iota : V^{\mu-r}(a_1, \cdots, a_r) \to \mathbb{P}^\mu(\mathbb{C})$, since the index is invariant under homotopy. Again, let $\mathbf{x} = \mathbf{c}_1(\xi^*)$ be the standard generator of $H^2(\mathbb{P}^\mu(\mathbb{C}), \mathbb{Z})$. The $\widehat{\mathbf{A}}$-class of a complete intersection is easily computed to be

$$\widehat{\mathbf{A}}(TV) = \frac{\widehat{\mathbf{A}}(\xi^*|_V)^{\mu+1}}{\widehat{\mathbf{A}}\left(\bigoplus_{i=1}^r \otimes^{a_i} \xi^*|_V\right)} = \left(\frac{\mathbf{x}/2}{\sinh(\mathbf{x}/2)}\right)^{\mu+1} \prod_{i=1}^r \frac{\sinh(a_i\mathbf{x}/2)}{a_i\mathbf{x}/2}. \tag{5.18}$$

Moreover,

$$\frac{\iota^*\left(\widehat{\mathbf{L}}(\mathbb{P}^\mu(\mathbb{C}))\right)}{\iota^*\left(\widehat{\mathbf{A}}(\mathbb{P}^\mu(\mathbb{C}))\right)} = \frac{\left(\frac{\mathbf{x}/2}{\tanh(\mathbf{x}/2)}\right)^{\mu+1}}{\left(\frac{\mathbf{x}/2}{\sinh(\mathbf{x}/2)}\right)^{\mu+1}} = \cosh^{\mu+1}(\mathbf{x}/2). \tag{5.19}$$

Thus, using the fact that $\mathbf{x}^{\mu-r}[V^{\mu-r}(a_1, \cdots, a_r)] = a_1 \cdots a_r$, we have

$$\begin{aligned}
I(\mu; a_1, \cdots, a_r) &:= \operatorname{index}(D^+) = 2^\mu \left(\iota^*\left(\widehat{\mathbf{L}}(\mathbb{P}^\mu(\mathbb{C}))\right) \frac{\widehat{\mathbf{A}}(V)}{\iota^*\left(\widehat{\mathbf{A}}(\mathbb{P}^\mu(\mathbb{C}))\right)}\right)[V] \\
&= 2^\mu \left(\tfrac{1}{2}\mathbf{x}\coth(\mathbf{x}/2)\right)^{\mu+1} \prod_{i=1}^r \frac{\sinh(a_i\mathbf{x}/2)}{a_i\mathbf{x}/2}[V] \\
&= a_1 \cdots a_r \cdot \text{coefficient of } \mathbf{x}^{\mu-r} \text{ in } 2^\mu \left(\tfrac{1}{2}\mathbf{x}\coth(\mathbf{x}/2)\right)^{\mu+1} \prod_{i=1}^r \frac{\sinh(a_i\mathbf{x}/2)}{a_i\mathbf{x}/2} \\
&= a_1 \cdots a_r \cdot res_{x=0}\left(x^{-(\mu-r)-1} 2^\mu \left(\tfrac{1}{2}x\coth(x/2)\right)^{\mu+1} \prod_{i=1}^r \frac{\sinh(a_ix/2)}{a_ix/2}\right) \\
&= res_{x=0}\left(\tfrac{1}{2}x^{-(\mu-r)-1}x^{\mu+1}\coth^{\mu+1}(x/2) \prod_{i=1}^r \frac{\sinh(a_ix/2)}{x/2}\right) \\
&= res_{x=0}\left(\tfrac{1}{2}\coth^{\mu+1}(x/2) \prod_{i=1}^r 2\sinh(a_ix/2)\right).
\end{aligned} \tag{5.20}$$

Letting $z = x/2$, we have

$$\tfrac{1}{2}\coth^{\mu+1}(x/2) \prod_{i=1}^r 2\sinh(a_ix/2)\, dx = \coth^{\mu+1}(z) \prod_{i=1}^r 2\sinh(a_iz)\, dz. \tag{5.21}$$

Thus,

$$I(\mu; a_1, \cdots, a_r) = res_{z=0}\left(\coth^{\mu+1}(z) \prod_{i=1}^r 2\sinh(a_iz)\right). \tag{5.22}$$

Since $\coth z$ has a pole of order one at $z = 0$, $\coth^{\mu+1}(z)$ has a pole of order $\mu+1$. As a consequence of Lemma 5.6 below, the singular part of the Laurent expansion of $\coth^{\mu+1}(z)$ is of the form

$$\sum_{k=0}^{\left[\frac{\mu}{2}\right]} \frac{c_{2k}(\mu)}{z^{\mu+1-2k}} = \frac{c_0(\mu)}{z^{\mu+1}} + \frac{c_2(\mu)}{z^{\mu-1}} + \cdots + \begin{cases} \frac{c_{\mu-1}(\mu)}{z^2} & \mu \text{ odd} \\ \frac{c_\mu(\mu)}{z} & \mu \text{ even} \end{cases}, \tag{5.23}$$

where $c_0(\mu) = 1$ and $c_{2k}(\mu) > 0$ (strict!) for all integers k with $0 \le k \le \left[\frac{\mu}{2}\right]$. We have

$$\prod_{i=1}^{r} 2\sinh(a_i z) = \prod_{i=1}^{r} 2a_i z \frac{\sinh(a_i z)}{a_i z}$$

$$= 2^r a_1 \cdots a_r z^r \left(\sum_{j=0}^{\infty} g_{2j}(a_1, \cdots, a_r) z^{2j} \right) \tag{5.24}$$

for some constants $g_{2k}(a_1, \cdots, a_r)$. The $g_{2j}(a_1, \cdots, a_r)$ are strictly positive, since $\sinh(a_i z) / (a_i z)$ has strictly positive Taylor coefficients multiplying the even powers of z. Hence, if we assume that $\nu := \mu - r = \dim_{\mathbb{C}} V^{\mu-r}(a_1, \cdots, a_r)$ is even,

$$I(\mu; a_1, \cdots, a_r)$$

$$= res_{z=0}\left(\coth^{\mu+1}(z) \prod_{i=1}^{r} 2\sinh(a_i z) \right)$$

$$= res_{z=0}\left(\sum_{k=0}^{\left[\frac{\mu}{2}\right]} \frac{c_{2k}(\mu)}{z^{\mu+1-2k}} \cdot 2^r a_1 \cdots a_r z^r \left(\sum_{j=0}^{\infty} g_{2j}(a_1, \cdots, a_r) z^{2j} \right) \right)$$

$$= 2^r a_1 \cdots a_r \, res_{z=0}\left(\sum_{k=0}^{\left[\frac{\mu}{2}\right]} \sum_{j=0}^{\infty} c_{2k}(\mu) g_{2j}(a_1, \cdots, a_r) z^{r+2j+2k-\mu-1} \right)$$

$$= 2^r a_1 \cdots a_r \sum_{k=0}^{(\mu-r)/2} c_{2k}(\mu) g_{\mu-r-2k}(a_1, \cdots, a_r) > 0. \tag{5.25}$$

We will consider the case ν odd momentarily, but let us first address the case that $a_1 + \cdots + a_r$ is odd. We proceed as follows. Regard $\mathbb{P}^{\mu}(\mathbb{C})$ as the hypersurface $z_{\mu+2} = 0$ in $\mathbb{P}^{\mu+1}(\mathbb{C})$ with homogeneous coordinates $z_1, \cdots, z_{\mu+2}$. Then $V^{\mu-r}(a_1, \cdots, a_r)$ may be regarded as $V^{(\mu+1)-(r+1)}(a_1, \cdots, a_r, a_{r+1}) \subseteq \mathbb{P}^{\mu+1}(\mathbb{C})$, where $a_{r+1} = 1$ is the degree of the polynomial $z_{\mu+2}$ which is adjoined to the polynomials of degree $a_1, \cdots, a_r$ in $z_1, \cdots, z_{\mu+1}$ which originally defined $V^{\mu-r}(a_1, \cdots, a_r)$. Since $a_1 + \cdots + a_r + a_{r+1}$ is now even and $\mu + 1 \equiv r + 1 \bmod 2$, our considerations thus far may be applied to $V^{(\mu+1)-(r+1)}(a_1, \cdots, a_r, a_{r+1})$. The inequality (5.17) only involves ν (i.e., it is independent of μ), and so it still holds even though we have replaced μ by $\mu + 1$.

If $\mu - r$ is odd, then (without loss of generality) we assume that $a_1 + \cdots + a_r$ is even, but in place of $D : \Gamma(E_f) \to \Gamma(E_f)$, we consider

$$D : \Gamma(f^*(\xi^*) \otimes E_f) \to \Gamma(f^*(\xi^*) \otimes E_f), \tag{5.26}$$

where $\xi^* \to \mathbb{P}^{\mu}(\mathbb{C})$ is the dual of the tautological line bundle $\xi \to \mathbb{P}^{\mu}(\mathbb{C})$. Note that ξ is the associated bundle $S^{2\mu+1} \times_{Id} \mathbb{C}$, where $Id : U(1) \to End(\mathbb{C})$ is the identity. We need to choose ξ^* instead of ξ in order that we have $e^{\mathbf{x}}$ instead of $e^{-\mathbf{x}}$ in (5.28) below. We now apply Proposition 2.2 with $f_S : SM \to S^{2\mu+1}$ playing the role of $f : M \to N$, with $L = \pi_M^* f^*(\xi^*)$, and with $\pi_M^* \psi \in \Gamma(\pi_M^*(f^*(\xi^*) \otimes E_f))$ playing the role of ϕ. Note that the curvature F^L is $-2i\pi_M^* f^* \omega$. If we choose the basis $e_1, \cdots, e_{2\nu}$ so as to put $(f^*\omega)_x$ in canonical form (see (5.12)), we have

$$\rho_L(x) \leq 2 \sum_{h,k=1}^{2\nu} \left|-2i\left(\pi_M^* f^*\omega\right)(\widetilde{e}_h, \widetilde{e}_k)\right| \leq 8\nu,$$

where we used the fact that f is area-nonincreasing in order to obtain the last inequality. Thus, assuming that there is a nonzero harmonic section $\psi \in \Gamma(f^*(\xi^*) \otimes E_f)$, by Proposition 2.2 and Theorem 4.1 we get (as in (5.17) but with the extra term 8ν) the result

$$\min(\kappa_M) \leq \min(\kappa_{SM}) + 2\nu \leq \left(\nu^2 + 2\nu(2\nu+1) + 8\nu\right) + 2\nu = 5\nu^2 + 12\nu, \tag{5.27}$$

when ν is odd. To produce a nonzero harmonic section ϕ it suffices to prove that *index* $\left(D^{f^*(\xi^*)+} : \Gamma\left(f^*(\xi^*) \otimes E_f^+\right) \hookleftarrow \Gamma\left(f^*(\xi^*) \otimes E_f^-\right)\right) > 0$. We have

$$\begin{aligned}\operatorname{index}\left(D^{f^*(\xi^*)+}\right) &= 2^\mu \left(\mathbf{ch}(f^*(\xi^*))\, f^*\left(\widehat{\mathbf{L}}(\mathbb{P}^\mu(\mathbb{C}))\right) \frac{\widehat{\mathbf{A}}(M)}{f^*\left(\widehat{\mathbf{A}}(\mathbb{P}^\mu(\mathbb{C}))\right)}\right)[M] \\ &= 2^\mu \left(e^{\mathbf{x}} \tfrac{1}{2}\mathbf{x} \coth(\mathbf{x}/2)\right)^{\mu+1} \prod_{i=1}^{r} \frac{\sinh(a_i \mathbf{x}/2)}{a_i \mathbf{x}/2}[V].\end{aligned} \tag{5.28}$$

Hence, the same computation that led to (5.22) yields

$$\operatorname{index}\left(D^{f^*(\xi^*)+}\right) = res_{z=0}\left(e^{z/2} \coth^{\mu+1}(z) \prod_{i=1}^{r} 2\sinh(a_i z)\right) \tag{5.29}$$

However, the coefficients of the power series of $e^{z/2}$ are all positive and the singular part of the $\coth^{\mu+1}(z)\prod_{i=1}^r 2\sinh(a_i z)$ has positive coefficients for the even negative powers in the Laurent expansion (and coefficients 0 on the odd powers). Note that the singular part of $\coth^{\mu+1}(z)\prod_{i=1}^r 2\sinh(a_i z)$ is nontrivial since $r \leq \mu$. Thus, $\operatorname{index}\left(D^{f^*(\xi^*)+}\right) > 0$. □

LEMMA 5.6. *For $\mu \geq 1$, let*

$$\coth^\mu(z) = \sum_{k=-\infty}^{\infty} d_k(\mu)\, z^k, \tag{5.30}$$

where $d_k(\mu) = 0$ for $k < -\mu$. Then $d_k(\mu) > 0$ for $-\mu \leq k \leq -1$ and $k \equiv \mu \bmod 2$.

PROOF. We have

$$\begin{aligned}\tfrac{d}{dz}\left(\coth^{\mu+1}(z)\right) &= -(\mu+1)\coth^\mu z\left(\coth^2 z - 1\right) \\ &= -(\mu+1)\left(\coth^{\mu+2} z - \coth^\mu z\right),\end{aligned} \tag{5.31}$$

or

(5.32)
$$\sum_{k=-\infty}^{\infty} k d_k(\mu+1)\, z^{k-1} = -(\mu+1)\left(\sum_{k=-\infty}^{\infty} d_k(\mu+2)\, z^k - \sum_{k=-\infty}^{\infty} d_k(\mu)\, z^k\right).$$

Thus,

$$\sum_{k=-\infty}^{\infty} (k+1)\, d_{k+1}(\mu+1)\, z^k = -(\mu+1)\left(\sum_{k=-\infty}^{\infty} d_k(\mu+2)\, z^k - \sum_{k=-\infty}^{\infty} d_k(\mu)\, z^k\right),$$

$$(k+1)\, d_{k+1}(\mu+1) = -(\mu+1)\, d_k(\mu+2) + (\mu+1)\, d_k(\mu), \text{ or}$$

$$d_k(\mu+2) = d_k(\mu) - \tfrac{k+1}{\mu+1} d_{k+1}(\mu+1). \tag{5.33}$$

We wish to prove that $d_k(\mu) > 0$ for $-\mu \leq k \leq -1$ and $k \equiv \mu \bmod 2$. We see that this is true for $\mu = 1$ and $\mu = 2$, since

$$\coth z = z^{-1} + \tfrac{1}{3}z - \tfrac{1}{45}z^3 + \cdots \text{ and } \coth^2 z = z^{-2} + \tfrac{2}{3} + \tfrac{1}{15}z^2 + \cdots. \tag{5.34}$$

Assume that we have the result for μ and $\mu+1$. We need to show that

$$d_k(\mu+2) = d_k(\mu) - \tfrac{k+1}{\mu+1} d_{k+1}(\mu+1) > 0, \tag{5.35}$$

for $-(\mu+2) \leq k \leq -1$ and $k \equiv (\mu+2) \bmod 2$. If $k = -1$ and $k \equiv (\mu+2) \bmod 2$ (i.e., μ odd), then $d_k(\mu+2) = d_k(\mu) > 0$ (in fact $d_{-1}(\mu) = d_{-1}(1) = 1$ for all odd μ). If $-(\mu+2) \leq k \leq -2$ and $k \equiv (\mu+2) \bmod 2$, then $-(\mu+1) \leq k+1 \leq -1$ and $k+1 \equiv \mu + 1 \bmod 2$, in which case $-\frac{k+1}{\mu+1} d_{k+1}(\mu+1) > 0$ and $d_k(\mu) \geq 0$ (Indeed $d_k(\mu) > 0$ for $-\mu \leq k \leq -1$ and $d_k(\mu) = 0$ for $k = -(\mu+1)$ or $-(\mu+2)$). Thus, $d_k(\mu+2) = d_k(\mu) - \frac{k+1}{\mu+1} d_{k+1}(\mu+1) > 0$ for $-\mu \leq k \leq -1$ and $k \equiv (\mu+2) \bmod 2$. □

6. Concluding remarks

We close with some useful observations with respect to the scalar curvature of certain metrics on complete intersections. For a Kähler manifold (M, g, J) of complex dimension ν with Kähler form ω, there is the following formula which has been attributed to Chern (see [**Gro**], p. 58)

$$\int_M \kappa_g\, dM = \frac{4\pi}{(\nu-1)!}\left(c_1[M] \smile [\omega]^{\nu-1}\right)[M]. \tag{6.1}$$

For $\nu = 1$, this is the Gauss-Bonnet theorem. We give a short proof. It is well-known (see [**Be**], p. 79) that $c_1[M]$ is represented by the 2-form $\frac{1}{2\pi}\rho$ where ρ is the Ricci form associated with the Ricci tensor via $\rho(X,Y) := \mathit{Ricci}(JX, Y)$. Thus (where $\left\{\widehat{e_i}, \widehat{Je_i}\right\}$ is the basis dual to $\{e_i, Je_i\}$),

$$\begin{aligned}
&\frac{4\pi}{(\nu-1)!}\left(c_1[M]\smile[\omega]^{\nu-1}\right)[M]\\
&=\frac{4\pi}{(\nu-1)!}\int_M \frac{1}{2\pi}\rho\wedge\omega^{\nu-1}=\frac{2}{(\nu-1)!}\int_M \sum_i \rho(e_i,Je_i)\,\widehat{e_i}\wedge\widehat{Je_i}\wedge\omega^{\nu-1}\\
&=\frac{2}{(\nu-1)!}\int_M\left(\sum_i\rho(e_i,Je_i)\right)\frac{1}{\nu}\omega^\nu=\int_M\left(2\sum_i\rho(e_i,Je_i)\right)\frac{\omega^\nu}{\nu!}\\
&=\int_M\left(2\sum_i \mathit{Ricci}(Je_i,Je_i)\right)dM\\
&=\int_M\left(\sum_i \mathit{Ricci}(e_i,e_i)+\mathit{Ricci}(Je_i,Je_i)\right)dM\\
&=\int_M\kappa_g\,dM. \qquad (6.2)
\end{aligned}$$

This was generalized in [**Bl**] to the case of quasi-Kähler manifolds (where J is not necessarily parallel, but still with $\omega(X,Y)=g(JX,Y)$ and ω closed), where it is shown that

$$\int_M\left(\kappa_g+\tfrac{1}{4}|\nabla J|^2\right)dM=\frac{4\pi}{(\nu-1)!}\left(c_1[M]\smile[\omega]^{\nu-1}\right)[M]. \qquad (6.3)$$

Specializing to the case M is a complete intersection $V^\nu(a_1,\cdots,a_r)$ with the induced Kähler metric g_0 from $\mathbb{P}^{\nu+r}(\mathbb{C})$ with the usual Fubini-Study metric, we compute

$$\begin{aligned}
\int_M\kappa_{g_0}\,dM &= \frac{4\pi}{(\nu-1)!}\left(c_1[M]\smile[\omega]^{\nu-1}\right)[M]\\
&= \frac{4\pi}{(\nu-1)!}\left((\mu+1-|a|)\,\mathbf{x}\smile[\omega]^{\nu-1}\right)[M]\\
&= \frac{4\pi}{(\nu-1)!}\left((\mu+1-|a|)\left[\frac{\omega}{\pi}\right]\smile[\omega]^{\nu-1}\right)[M]\\
&= \frac{4(\mu+1-|a|)}{(\nu-1)!}[\omega]^\nu[M]=\frac{4(\mu+1-|a|)}{(\nu-1)!}\pi^\nu\mathbf{x}^\nu[M]\\
&= 4\pi^\nu\frac{(\mu+1-|a|)}{(\nu-1)!}a_1\cdots a_r. \qquad (6.4)
\end{aligned}$$

We also have Wirtinger's formula (see [**Gra**], p. 125) for the volume

$$\int_M dM=\frac{1}{\nu!}\int_M\omega^\nu=\frac{1}{\nu!}\int_M\left(\pi\left(\frac{\omega}{\pi}\right)\right)^\nu=\frac{\pi^\nu}{\nu!}\mathbf{x}^\nu[M]=\frac{\pi^\nu}{\nu!}a_1\cdots a_r. \qquad (6.5)$$

Thus,

$$\mathit{ave}(\kappa_{g_0})=\frac{\int_M\kappa_{g_0}\,dM}{\int_M dM}=\frac{4\pi^\nu\frac{(\mu+1-|a|)}{(\nu-1)!}a_1\cdots a_r}{\frac{\pi^\nu}{\nu!}a_1\cdots a_r}=4\nu(\mu+1-|a|). \qquad (6.6)$$

In the case $|a|=\mu-\nu$ and $a_i=1$, we have M isometric to $\mathbb{P}^\nu(\mathbb{C})$ and the known result

$$\mathit{ave}(\kappa_{g_0})=4\nu(\mu+1-|a|)=4\nu(\mu+1-(\mu-\nu))=4\nu(\nu+1). \qquad (6.7)$$

We offer the

CONJECTURE 6.1. *Let g be any metric on the complete intersection $V^\nu(a_1, \cdots, a_r) \subseteq \mathbb{P}^\mu(\mathbb{C})$ that satisfies $g \geq g_0$, where g_0 is the metric induced by the Fubini-Study metric. Then*

$$\min(\kappa_g) \leq 4\nu(\nu+1).$$

Blair's result (6.3) verifies this in the case where g is associated with ω via an almost complex structure J. In fact, in this case $4\nu(\nu+1)$ can be replaced by $4\nu(\mu+1-|a|)$. However, $4\nu(\nu+1)$ cannot be replaced by $4\nu(\mu+1-|a|)$ in general, for the following reason. Consider the case of a nonsingular hypersurface $V^{\mu-1}(a_1) \subseteq \mathbb{P}^\mu(\mathbb{C})$ of degree a_1. If $\mu \geq 4$, then the Lefschetz Hyperplane Theorem (see **[Gri-Ha]**, pp. 156-159) implies not only that $V^{\mu-1}(a_1)$ is simply-connected, but also that $H^2(\mathbb{P}^\mu(\mathbb{C}), \mathbb{Z}) \overset{\iota^*}{\cong} H^2(V^{\mu-1}(a_1), \mathbb{Z})$, and hence $\mathbf{x}|_V$ generates $H^2(V^{\mu-1}(a_1), \mathbb{Z})$. Then by (5.2), $V^{\mu-1}(a_1)$ is not spin if μ and a_1 have the same parity. Hence, for $\mu \geq 4$ and $a_1 \equiv \mu \bmod 2$, a result in **[Gro-La]** implies that the simply-connected, nonspin $V^{\mu-1}(a_1)$ of dimension $2\mu-2 \geq 6$ admits a metric of positive scalar curvature, whereas $4\nu(\mu+1-a_1)$ is negative for a_1 sufficiently large. Of course, there is a sizable gap between the Conjecture and what may be concluded from Theorem 5.4, which begs to be closed if possible. We also mention that in his forthcoming thesis **[Kr]**, W. Kramer has verified the conjecture for $V^\nu(a_1, \cdots, a_r) = \mathbb{P}^\nu(\mathbb{C})$.

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Christian Bär
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg
Germany
E-mail address: baer@mathematik.uni-freiburg.de

David Bleecker
Department of Mathematics
University of Hawaii
Honolulu, HI 96822
USA
E-mail address: bleecker@math.hawaii.edu