Seminar on Coarse Geometry

December 19, 2014

Schedule of the talks

- **Part I: Fundamentals**

  1. (Sylvie 20.10.) **The coarse perspective on metric and length spaces** [Ro, §1.2] and [NY, §1.1]

     – Metric space, length space, examples ([NY, p.3,4,5])
     geodesic, geodesic space, proper space.
     The Hopf–Rinow theorem (Gromov's formulation) [Ro, Theorem 1.5] *A length space is proper–i.e. its closed and bounded subsets are compact–if and only if it is complete and locally compact*;
     Lipschitz maps [Ro, §1.2]

     – Definition of proper maps, bornologous maps, quasi-isometries, coarse maps. Coarse equivalence [Ro, §1.3],[NY, §1.3]. Examples.

  2. (Matthias 27.10.) **Groups as metric spaces** [Ro, §1.4] and [NY, §1.2]
     The word metric on a group Γ;
     The *Švarc–Milnor theorem* (A length space is coarsely equivalent to a group which acts on it properly and compactly by isometries on it) [Ro, Theorem 1.18], [NY, Proposition 1.3.13] (this theorem implies in particular that the coarse geometry of the fundamental group of a Riemannian manifold is the same as the coarse geometry of its universal covering)

  3. (Andreas 3.11. and Oliver 10.11.) **Hyperbolic spaces**

     – The geometry of the hyperbolic space $\mathbb{H}^n$ [Ro, §1.6].
     – Hyperbolicity in the sense of Gromov [NY, §1.5]

     – *Morse lemma* $\Longrightarrow$ *quasi-isometry preserves hyperbolicity* [NY, Lemma 5.9, Theorem 5.10]

4. (Barbara 17.11. and Laurent 24.11.) **Graphs, trees and complexes** [Ro, §1.5]
   [Ro, Proposition 1.24] *A locally finite simplicial complex can be equipped with a length metric which makes its geometric realisation a complete proper geodesic space*
5. (Christian Becker 1.12 and 8.12) **The asymptotic dimension as a coarse invariant**
   - Topological dimension, asymptotic dimension [NY, §2.1, 2.2]; [NY, Theorem 2.25] *The asymptotic dimension is preserved under coarse equivalence*
   - The asymptotic dimension for trees (*The asymptotic dimension of a tree is* ≤ 1 [NY, Proposition 2.3.1]) versus the asymptotic dimension for hyperbolic groups [NY, §2.3]; [NY, Theorem 2.3.3] *The asymptotic dimension of a hyperbolic group is finite (Gromov)*

- **Part II: On the way to the Novikov conjecture:** It follows from a result of Guoliang Yu that the Novikov conjecture – that addresses the question of which characteristic classes are invariants of homotopy type– holds for manifolds whose fundamental group that can coarsely be embedded in a Hilbert space. This encompasses a large class of groups, including all linear groups, all hyperbolic groups and all amenable groups. Proving Guoliang Yu's result is out of the scope of the seminar in which we only introduce the tools of the proof, namely amenability and property A.

  1. (Sara Azzali 15.12) **Finiteness of the asymptotic dimension I** [NY, §2.4], [NY, §2.5]

     - Upper bounds for the asymptotic dimension [NY, Theorem 2.4.3] in [NY, §2.4]
     - Groups with infinite asymptotic dimension [NY, §2.6]
  2. (Sara Azzali 5.01) **Finiteness of the asymptotic dimension II**[NY, §2.4], [NY, §2.5]

     - *The asymptotic dimension of an almost connected–i.e. with finitely many connected components– Lie group is finite* [NY, Theorem 2.5.6] in [NY, §2.5]
  3. (Florian 12.01. and t.b.a 19.01.) **Amenability as a coarse property**

     - Definition of amenability for a group Γ by the Følner condition [NY, §3.1], main examples [NY, pp.49-50]

- Properties of amenability; amenability of a subgroup H of a given group G (assuming G, H finitely generated); amenability is preserved by quasi-isometries (is a coarse property.)
- Characterisations of amenability [NY, Prop. 3.16, 3.17]
- *A finitely generated group* Γ *is amenable* $\iff l^\infty(\Gamma)$ *admits an invariant mean* [NY, Theorem 3.3.2] [Ro, Theorem 3.53]

4. (Andreas 26.01) **Property A** [NY, §4.1-4.3]

   - Finitely generated amenable groups and infinite trees have property A [NY, §4.1]
   - J. Roe's characterisation of property A ([NY, Theorem 4.2.1] assumed without proof)
   - **For uniformly discrete metric spaces, finite asymptotic dimension implies property A** [NY, Theorem 4.3.6]
   - Hyperbolic groups have property A [NY, Corollary 4.3.7].

5. (t.b.a 2.02) **Coarse embeddability**[NY, §5.1-5.2]

   - An isometric embedding is a coarse embedding; $\mathbb{Z}^n$ embeds coarsely in $\mathbb{R}^m$ for $m \geq n$ with the ℓ^1 norm, a tree embeds coarsely in the l^2 space of its vertices.[NY, §5.1,§5.2]
   - **A metric space with property A embeds coarsely into a Hilbert space** [NY, Theorem 5.2.4]
   - A uniformly discrete metric space with finite asymptotic dimension embeds coarsely into a Hilbert space [NY, Corollary5.2.6], see also [Ro, Example 11.5]
   - Characterisations of coarese embeddability [Ro, Theorem 11.16]

References

[NY] Piotr W. Nowak, Gouliang Yu, *Large scale geometry* EMS Textbooks in Mathematics 2012

[Ro] John Roe, *Lectures on coarse geometry* 2003 AMS University Lecture Series