On the characteristic initial value problem for nonlinear symmetric hyperbolic systems

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Plan of the presentation

1. Introduction (Prerequisites).
2. Quasi-linear first order hyperbolic system and the energy estimate
3. Iterative scheme and the continuity (Bootstrap) argument
4. Application to general semi-linear wave equations

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Cauchy Problems in normal form

- $u = (u^1, u^2, \dots, u^N)$ is a vector valued function of the $n+1$ variables $(t, x) = (x^0, x^1, \dots, x^n)$ of an open subset Ω of $\mathbb{R}^{1+n}$.
- Consider a system of N partial differential equations for the N unknown functions of order m of the form

$$\frac{\partial^m u}{\partial t^m} = F(t, x, u, \dots, u_\alpha, \dots) \tag{1}$$

where for a multi-index $\alpha = (\alpha_0, \dots, \alpha_n)$,

$$u_\alpha = \frac{\partial^{|\alpha|} u}{(\partial t)^{\alpha_0} (\partial x^1)^{\alpha_1} \dots (\partial x^n)^{\alpha_n}}$$

$$|\alpha| = \alpha_0 + \alpha_1 + \dots + \alpha_n \leq m,\ 0 \leq \alpha_0 < m\ ;$$

Definition

The initial value problem

$$\begin{cases} \dfrac{\partial^m u}{\partial t^m} = F(t, x, u, \ldots, u_\alpha, \ldots) \\ \left.\dfrac{\partial^\ell u}{\partial t^\ell}\right|_{t=t_0} = h_\ell, \quad 0 \le \ell \le m-1\,; \end{cases} \tag{2}$$

is said to be in its **normal form**.

Remark

Given a Cauchy problem in its normal form, one can compute all the (tangential and outwards) derivatives of the unknowns on the initial surface $\{t = t_0\}$. Thus the initial data together with the differential equations completely determined the Taylor series of u along the the initial surface $\{t = t_0\}$ provided that such a solution exists and is analytic which is the case when the data and the right hand side of the equations are analytic functions of all their arguments (Theorem of Cauchy-Kowalewski .

General Cauchy Problems

- In many situations, there is no preferred coordinate and the Cauchy data for a given (system of) partial differential equation(s) are prescribed on generic hypersurfaces (Σ).
- Consider a system of N PDEs in N unknown functions u in the general form:

$$F(x, u, \ldots, u_\alpha, \ldots) = 0, \quad |\alpha| \leq m \ , I = 1, \ldots, N \ , \tag{3}$$

- Suppose that the initial hypersurface (Σ) is given by

$$(\Sigma) : \phi(x) = 0 \ , \tag{4}$$

where the F's are smooth functions and ϕ is a smooth function with non vanishing gradient.

- Denote by ν the gradient of ϕ then, the normal derivatives of u along (Σ) are given by $(\phi_i = \frac{\partial \phi}{\partial x})$

$$\frac{\partial u}{\partial \nu} = \sum_{i=1}^{n} \phi_i \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial^\ell u}{(\partial \nu)^\ell} = \underbrace{\frac{\partial}{\partial \nu} \cdots \frac{\partial u}{\partial \nu}}_{\ell \ times}$$

- Then an initial value problem for the PDE (3) can be posed on (Σ) as:

$$\begin{cases} F(x, u, \ldots, u_\alpha, \ldots) = 0 \\ \left.\frac{\partial^\ell u}{(\partial \nu)^\ell}\right|_\Sigma = h_\ell, \end{cases} \quad |\alpha| \leq m \ , \quad 0 \leq \ell \leq m-1 \ ; \tag{5}$$

Transformation into normal form

Question

Can problem (5) be written in normal form?

To answer this question, we restrict ourself to the case where (3) is quasi-linear, i.e.

$$F(x, u, \dots, u_\beta, \dots) = \sum_{|\alpha|=m} A^\alpha u_\alpha + G \tag{6}$$

where

$$A^\alpha = A^\alpha(x, u, \dots, u_\beta, \dots), \quad |\beta| \leq m-1$$

is an $N \times N$ matrix valued function and

$$G = G(x, u, \dots, u_\beta, \dots), \quad |\alpha| \leq m-1$$

is a vector valued function.

Suppose that $y = (y^1, \dots, y^n)$ are independent coordinates on (Σ) and complete with $y^0 = \phi(x)$ to a system of coordinates (y^0, y). Then, computations show that the system of PDE (6) becomes

$$\sum_{|\alpha|=m} \frac{\partial\phi}{(\partial x^0)^{\alpha_0}} \cdots \frac{\partial\phi}{(\partial x^n)^{\alpha_n}} A^\alpha \frac{\partial^k \overline{u}}{(\partial y^0)^k} + g(y^0, y, \overline{u}, \dots, \overline{u}_\beta) = 0$$

$$|\beta| \leq m, \ \beta_0 < m$$

where $\overline{u}_\alpha$ are the derivatives of u with respect to the y's coordinates.

Thus the initial value problem,(5) can be recast into the normal form if and only if

$$\det\left(\sum_{|\alpha|=m}\frac{\partial\phi}{(\partial x^0)^{\alpha_0}}\cdots\frac{\partial\phi}{(\partial x^n)^{\alpha_n}}A^\alpha(x,u,\dots,u_\beta,\dots)\right)\Bigg|_{(\Sigma)}\neq 0\,.$$

In that case the Cauchy-Kowalewski Theorem applies to (5).

Definition

The hypersurface (Σ) is said to be characteristic with respect to the Cauchy problem (5) when

$$\det\left(\sum_{|\alpha|=m}\frac{\partial\phi}{(\partial x^0)^{\alpha_0}}\cdots\frac{\partial\phi}{(\partial x^n)^{\alpha_n}}A^\alpha(x,u,\dots,u_\beta,\dots)\right)\Bigg|_{(\Sigma)}=0\ .$$

In that case, we speak about "Characteristic Cauchy data".

Remark

1. Initial data on a characteristic surface cannot be prescribed freely: They must satisfy some compatibility conditions some times called the "transport Equations".
2. Correspondingly, the solution is not uniquely determined unless certain additional conditions are imposed on a hypersurface transverse to the initial surface.
3. Discontinuity (singularities) of a solution cannot occur except along characteristic surfaces
4. Characteristic surfaces are the only surfaces for which the same initial value problem may have several solutions

Example

As example, it is not difficult to see that in the case of semi-linear wave equation, the characteristic surfaces are null hypersurfaces

The Equation

- Y is a $(n-1)$-dimensional compact manifold without boundary.
- We are interested in quasi-linear first order symmetric hyperbolic systems of the form

$$Lf = G \;, \tag{7}$$

on subsets of

$$\widetilde{\mathscr{M}} := \{u \in [0,\infty), v \in [0,\infty), y \in Y\} \;.$$

where

  - f and G are sections of a real vector bundle E over $\widetilde{\mathscr{M}}$
  - E is equipped with a scalar product. We will use the same symbol ∇, respectively $\langle \cdot, \cdot \rangle$, to denote connections, respectively scalar products, on all relevant vector bundles.

- Both the scalar product and the connection coefficients are allowed to depend upon f, and we assume that ∇ is compatible with $\langle\cdot,\cdot\rangle$.
- $\widetilde{\mathscr{M}}$ will be assumed to be equipped with a measure $d\mu$, possibly dependent upon f.
- L is a first order operator of the form

$$L = A^\mu \nabla_\mu \ ,$$

where the A^μ's are self-adjoint (thus the system is symmetric), and are smooth functions of f and of the space-time coordinates.

The energy density

- Let q_r, $r = 1, \dots, m$, denote a collection of smooth vector fields on Y such that for each $y \in Y$ the vectors $q_r(y)$ span $T_y Y$; clearly $m \geq \dim Y$.
- For $k \in \mathbb{N}$, let $\mathscr{P}^k$ denote the collection of differential operators of the form

$$\mathring{\nabla}_{q_{r_1}} \dots \mathring{\nabla}_{q_{r_\ell}} , \quad 0 \leq \ell \leq k . \tag{8}$$

Here $\mathring{\nabla}$ is a fixed, arbitrarily chosen, smooth connection which is f, u, and v–independent. We number the operators (8) in an arbitrary way and call them P_r, thus

$$\mathscr{P}^k = \mathrm{Vect}\{P_r, r = 1, \dots, N(k)\} ,$$

for a certain $N(k)$.

- Let w_r be any smooth functions on $\widetilde{\mathscr{M}}$, we set

$$X^\mu(k) := \sum_{r=1}^{N(k)} w_r \langle P_r f, A^\mu P_r f \rangle \ , \tag{9}$$

The splitting of f

We restrict our attention to f's which are of the form

$$f = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} , \tag{10}$$

with A^v and A^u satisfying

$$A^u = \begin{pmatrix} A^u_{\varphi\varphi} & 0 \\ 0 & 0 \end{pmatrix} , \; A^v = \begin{pmatrix} 0 & 0 \\ 0 & A^v_{\psi\psi} \end{pmatrix} , \tag{11}$$

and

$$A^v_{\psi\psi} > 0 \, , A^u_{\varphi\varphi} > 0$$

Characteristic hyperbolic system

From these hypotheses, we see that:

- the hypersurfaces $\{u = cst\}$ and $\{v = cst\}$ are characteristic hypersurfaces for the system (7)
- the first order system (7) is hyperbolic

We have

$$\nabla_\mu(X^\mu(k)) = \sum_r \Big\{ \underbrace{\langle P_r f, A^\mu P_r f\rangle \partial_\mu w_r}_{I_r} + w_r \Big(\underbrace{\langle P_r f, (\nabla_\mu A^\mu) P_r f\rangle}_{II_r} + \underbrace{2\langle P_r f, L P_r f\rangle}_{III_r} \Big) \Big\} ,$$

so that, for

$$\Omega_{a,b} = \underbrace{[0,a]}_{\ni u} \times \underbrace{[0,b]}_{\ni v} \times \underbrace{Y}_{\ni x^B} ,$$

and

$$d\mu = du\, dv\, d\mu_Y$$

any measure, absolutely continuous with respect to the coordinate Lebesgue measure, on Ω_{ab},

Stokes' theorem

from Stokes' theorem we have

$$\int_{\partial\Omega_{a,b}} X^\alpha(k)\,dS_\alpha = \int_{\Omega_{a,b}} \nabla_\mu(X^\mu(k))\,d\mu\ ,$$

which is,

$$\begin{aligned}\int_{u=a} X^\alpha(k)\,dS_\alpha + \int_{v=b} X^\alpha(k)\,dS_\alpha \\ = \int_{u=0} X^\alpha(k)\,dS_\alpha + \int_{v=0} X^\alpha(k)\,dS_\alpha \\ + \int_{\Omega_{a,b}} \nabla_\mu(X^\mu(k))\,d\mu\ .\end{aligned}$$

From the hypotheses on the matrices A^u and A^v, we see that for fields supported in a compact K we have:

$$\begin{aligned}
\int_{u=a} X^\alpha(k) dS_\alpha &\geq c(K) \sum_r \int_{u=a} w_r \langle P_r\varphi, P_r\varphi\rangle \, dv \, d\mu_Y \ , \\
&=: c(K)\, E_{k,\{w_r\}}[\varphi, a] \ , \qquad (12) \\
\int_{v=b} X^\alpha(k) dS_\alpha &\geq c(K) \sum_r \int_{v=b} w_r \langle P_r\psi, P_r\psi\rangle \, du \, d\mu_Y \\
&=: c(K)\, \mathscr{E}_{k,\{w_r\}}[\psi, b] \ . \qquad (13)
\end{aligned}$$

Thus,

$$E_{k,\{w_r\}}[\varphi, a] + \mathscr{E}_{k,\{w_r\}}[\psi, b] \leq C_1(K)\Big\{E_{k,\{w_r\}}[\varphi, 0] + \mathscr{E}_{k,\{w_r\}}[\psi, 0] + \int_{\Omega_{a,b}} \sum_r (I_r + w_r(II_r + III_r))\Big\}. \quad (14)$$

special weight

Let $\lambda \geq 0$, we choose the weight to be independent of r:

$$w_r = e^{-\lambda(u+v)} ,$$

and we will write $E_{k,\lambda}$ for $E_{k,\{w_r\}}$ with this choice of weight, similarly for $\mathscr{E}_{k,\lambda}$.

From (12) we find

$$E_{k,\lambda}[\varphi, a] = \sum_{0\le j\le k} \int_{[0,b]\times Y} |\mathring{\nabla}_{q_{r_1}} \dots \mathring{\nabla}_{q_{r_j}} \varphi(a, v, \cdot)|^2 e^{-\lambda(a+v)} dv d\mu_Y =: \int_0^b e^{-\lambda(a+v)} \|\varphi(a, v)\|^2_{H^k(Y)} dv \,, \tag{15}$$

where one recognises the usual Sobolev norms $H^k(Y)$ on Y. One similarly has

$$\mathscr{E}_{k,\lambda}[\psi, b] = \sum_{0\le j\le k} \int_{[0,a]\times Y} |\mathring{\nabla}_{q_{r_1}} \dots \mathring{\nabla}_{q_{r_j}} \psi(\cdot, b, \cdot)|^2 e^{-\lambda(u+b)} du d\mu_Y =: \int_0^a e^{-\lambda(u+b)} \|\psi(u, b)\|^2_{H^k(Y)} du \,. \tag{16}$$

Writing LP_rf as $P_rLf + [L, P_r]f$, and assuming

$$\langle \varphi, A^u_{\varphi\varphi}\varphi\rangle \geq c|\varphi|^2 \,, \qquad \langle \psi, A^v_{\psi\psi}\psi\rangle \geq c|\psi|^2 \,, \tag{17}$$

with $c > 0$, one obtains for $k > \frac{n-1}{2}$

$$\begin{aligned} & E_{k,\lambda}[\varphi, a] + \mathscr{E}_{k,\lambda}[\psi, b] \\ \leq \quad & C_1(K)\Big\{ E_{k,\lambda}[\varphi, 0] + \mathscr{E}_{k,\lambda}[\psi, 0] \\ & + \int_{\mathscr{U}} e^{-\lambda(u+v)}\Big\{ \left(\|\nabla_\mu A^\mu\|_{L^\infty(Y)} - c\lambda\right) \|f\|^2_{H^k(Y)} \\ & + C(Y, k)\|f\|_{H^k(Y)}\|G\|_{H^k(Y)} \\ & + 2\int_{\mathscr{U}\times Y} \langle P_r f, [L, P_r]f\rangle e^{-\lambda(u+v)}\, d\mu \Big\} \,. \end{aligned} \tag{18}$$

Moser type Inequalities

We recall here the following Moser inequalities which will be used repeatedly:

- Moser product inequality:

$$\|fg\|_{H^k(Y)} \leq C_M(Y,k)\Big(\|f\|_{L^\infty(Y)}\|g\|_{H^k(Y)} + \|f\|_{H^k(Y)}\|g\|_{L^\infty(Y)}\Big) .$$

- Moser commutation inequality, for $0 \leq r \leq k$:

$$\|P_r(fg) - P_r(f)g\|_{L^2(Y)} \leq C_M(Y,k)\Big(\|f\|_{L^\infty(Y)}\|g\|_{H^k(Y)} + \|f\|_{H^{k-1}(Y)}\|g\|_{W^{1,\infty}(Y)}\Big) .$$

and

The Moser composition inequality:

$$\|F(f,\cdot)\|_{H^k(Y)} \leq \hat{C}_M\Big(Y,k,F,\|f\|_{L^\infty(Y)}\Big)\Big(\|F(f=0,\cdot)\|_{H^k(Y)}+\|f\|_{H^k(Y)}\Big) .$$

Proposition: The main estimate

After analyzing the commutator terms in (18) using the Moser inequalities, we obtain the following:

Proposition

There exists a constant

$$\hat{C}_1 = C\Big(Y, k, \|f\|_{W^{1,\infty}(Y)}, \|A\|_{W^{1,\infty}(Y)}, \|\tilde{A}\|_{L^\infty(Y)}, \|\gamma\|_{W^{1,\infty}}, \|\Gamma\|_{W^{1,\infty}}, \|G\|_{W^{1,\infty}} + \|\tilde{G}\|_{L^\infty}\Big) > 0 \qquad (19)$$

such that:

$$\begin{aligned}
&E_{k,\lambda}[\varphi, a] + \mathscr{E}_{k,\lambda}[\psi, b] \\
&\leq \quad C_1(K)\Big\{ E_{k,\lambda}[\varphi, 0] + \mathscr{E}_{k,\lambda}[\psi, 0] \\
&\quad + \int_{\mathscr{U}} e^{-\lambda(u+v)}\Big\{ \left(\|\nabla_\mu A^\mu\|_{L^\infty(Y)} - c\lambda\right) \|f\|^2_{H^k(Y)} \\
&\quad + \hat{C}_1 \|f\|_{H^k(Y)} \times \Big(\|f\|_{H^k(Y)} + \|A\|_{H^k(Y)} + \|\tilde{A}\|_{H^{k-1}(Y)} \\
&\quad + \|\Gamma\|_{H^k(Y)} + \|\gamma\|_{H^k(Y)} + \|G\|_{H^k(Y)} + \|\tilde{G}\|_{H^{k-1}(Y)} \Big) \Big\} .
\end{aligned}$$

Iterative scheme and the argument

- For the purpose of the arguments, we let

$$\mathscr{N}^- := \{u = 0\ , v \in [0, b_0]\} \times Y\ , \quad \mathscr{N}^+ := \{u \in [0, a_0], v = 0\} \times Y$$

- The initial data $\overline{f} \equiv f|_{\mathscr{N}}$ are prescribed on

$$\mathscr{N} := \mathscr{N}^- \cup \mathscr{N}^+\ ,$$

- Note that we are free to prescribe $\overline{\varphi}(v) \equiv \varphi(0, v)$ on $\mathscr{N}^-$ and $\overline{\psi}(u) \equiv \psi(u, 0)$ on $\mathscr{N}^+$, and then the fields $\psi(0, v)$ on $\mathscr{N}^-$ and $\varphi(u, 0)$ on $\mathscr{N}^+$ can be calculated by solving transport equations; it is part of our hypothesis that these equations have global solutions on $\mathscr{N}^{\pm}$.

smooth approximation

- Choose a sequence $\overline{f}_i$ of smooth initial data approaching $\overline{f}$
- Let f_0 be any smooth extension of $\overline{f}_0$ to Ω_{a_0,b_0}.
- For given f_i, the field f_{i+1} is defined as the solution of the linear system

$$L_i f_{i+1} = G_i \ , \tag{20}$$

where

$$L_i = A^\mu(f_i,\cdot)\nabla(i)_\mu \ , \quad G_i = G(f_i,\cdot) \ , \tag{21}$$

and where we have used the symbol $\nabla(i)$ to denote ∇, as determined by f_i.

Solution of linear Problem

For smooth initial data and f_i, (20) always has a global smooth solution on Ω_{a_0,b_0} by A. Rendall, 1990.

The argument

- Define

$$C_0 = 1 + \sup_{i\in\mathbb{N},(u,v)\in([0,a_0]\times\{0\})\cup(\{0\}\times[0,b_0])} \|\overline{f}_i(u,v)\|_{W^{1,\infty}(Y)} .$$

- $$C_{\text{div}} := \sup|\nabla_\mu A^\mu| + 1 . \qquad (22)$$

In the second constant, the supremum is taken over all points in $\mathscr{N}^+ \cup \mathscr{N}^-$ and over all $(\varphi, \psi, \nabla\varphi, \nabla\psi)$ satisfying

$$(\varphi,\psi) \in \mathscr{K} \ , |\mathring{\nabla}_B f(u,v)| \le 2C_0 \ , |\partial_u \psi| \le 2\sup_i \|\overline{\frac{\partial \psi_i}{\partial u}}\|_{L^\infty(\mathscr{N}^+\cup\mathscr{N}^-)} + 1 ,$$
$$|\partial_v \varphi| \le 2\sup_i \|\overline{\frac{\partial \varphi_i}{\partial v}}\|_{L^\infty(\mathscr{N}^+\cup\mathscr{N}^-)} + 1 .$$

Let a_i be the largest number in $(0, a_0]$ such that

$$\begin{aligned} \|(\nabla_\mu A^\mu)_i\|_{L^\infty(\Omega_{a_i,b_0})} &\leq C_{\text{div}} \ , \\ \sup_{(u,v)\in[0,a_i]\times[0,b_0]} \|f_i(u,v)\|_{W^{1,\infty}(Y)}, &\leq 4C_0 \end{aligned} \tag{24}$$

By continuity,

$$a_i > 0$$

Theorem

Now using the energy estimate of the previous section, we prove

Theorem

There exists $a_* > 0$ such that $\forall i \in \mathbb{N},\ a_i \geq a_*$, so that there is a common domain

$$\Omega_* := \{u \in [0, a_*]\,, v \in [0, b_0]\} \times Y$$

on which inequalities (24) are satisfied by all the f_i's.
The sequence (f_i) converges to a solution of the original problem on Ω_*.

The precise statement of the Theorem

The main Theorem

Let Y be a $(n-1)-$dimensional compact manifold without boundary, let a_0 and b_0 two positive real numbers and set

$$\Omega_0 = [0, a_0] \times [0, b_0] \times Y$$

Consider the symmetric hyperbolic system (7) on Ω_0 with the splitting (10) and assume that (11) holds. Let $\overline{\varphi}$ and $\overline{\psi}$ be defined respectively on $\mathscr{N}^-$ and $\mathscr{N}^+$, providing Cauchy data for (7):

$$\varphi = \overline{\varphi} \quad \text{on} \quad \mathscr{N}^- \qquad \text{and} \qquad \psi = \overline{\psi} \quad \text{on} \quad \mathscr{N}^+ . \tag{25}$$

Let $\ell \in \mathbb{N}$, $\ell > \frac{n+9}{2}$ and suppose that

$$\overline{\varphi} \in \cap_{0 \le j \le \ell} C^j([0, b_0]; H^{\ell-j}(Y)) \text{ and } \overline{\psi} \in \cap_{0 \le j \le \ell} C^j([0, a_0]; H^{\ell-j}(Y)) .$$

Assume that the *transport equations*

$$A^{\mu}_{\varphi\varphi}|_{v=0}\partial_{\mu}\varphi|_{v=0} = \big(- A^{\mu}_{\varphi\psi}\partial_{\mu}\psi + G_{\varphi}\big)\Big|_{v=0} \,, \qquad (26)$$

$$A^{\mu}_{\psi\psi}|_{u=0}\partial_{\mu}\psi|_{u=0} = \big(- A^{\mu}_{\psi\varphi}\partial_{\mu}\varphi + G_{\psi}\big)\Big|_{u=0} \,, \qquad (27)$$

with initial data

$$\varphi|_{u=v=0} = \overline{\varphi}|_{v=0} \text{ and } \psi|_{u=v=0} = \overline{\psi}|_{u=0},$$

have a global solution on $([0, a_0] \times Y) \cup ([0, b_0] \times Y)$. Then there exists an ℓ-independent constant $a_* \in (0, a_0]$ such that the Cauchy problem (7), (25) has a unique solution f defined on $[0, a_*] \times [0, b_0] \times Y$ satisfying

$$f \in \cap_{0\le j\le \ell-2} C^j([0, a_*]\times[0, b_0]; H^{\ell-j-2}(Y)) \subset C^1([0, a]\times[0, b_0]\times Y) \,.$$

The solution f is smooth if $\overline{\varphi}$ and $\overline{\psi}$ are.

Double-null coordinates system

- $(\mathscr{M}, g)$ is a smooth $(n+1)$-dimensional space-time
- $\widehat{\mathscr{N}}^{\pm}$ are two null hypersurfaces in $\mathscr{M}$ emanating from a spacelike manifold Y of codimension two.
- Denote by $\mathscr{N}^{\pm}$ the intersection of $\widehat{\mathscr{N}}^{\pm}$ with the causal future of Y.

In order to apply our results above to semi-linear wave equations with initial data on $\mathscr{N}^{\pm}$ we need to construct local coordinate systems (u, v, x^A), where the x^A's are local coordinates on Y, near

$$\mathscr{N} := \mathscr{N}^{+} \cup \mathscr{N}^{-}$$

so that

$$\mathscr{N}^{-} := \{u = 0\}\ , \quad \mathscr{N}^{+} := \{v = 0\}\ . \tag{28}$$

and

$$g(\nabla u, \nabla u) = 0 = g(\nabla v, \nabla v)\ , \tag{29}$$

wherever defined.

Construction of the (u, v, x^A) coordinates

- Let ℓ_Y and ω_Y be any smooth null future pointing vector fields defined along Y and normal to Y such that ℓ_Y is tangent to $\mathscr{N}^+$ and ω_Y is tangent to $\mathscr{N}^-$.
- Both $\widehat{\mathscr{N}}^+$ and $\mathscr{N}^+$ are threaded by the null geodesics issued from Y with initial tangent ℓ_Y at Y. These geodesics will be referred to as the *generators* of $\widehat{\mathscr{N}}^+$, respectively of $\mathscr{N}^+$. The associated field of tangents will be denoted by ℓ^+.
- Let r_+ denote the corresponding parameter along the integral curves of ℓ^+, with $r_+ = 0$ at Y.
- Similarly $\widehat{\mathscr{N}}^-$ and $\mathscr{N}^-$ are threaded by their null geodesic generators issued from Y, tangent to ω_Y at Y, with field of tangents ω^- and parameter r_-.

- Let x^A_Y be any local coordinates on an open subset $\mathscr{O}$ of Y, we propagate them to functions $x^A_\pm$ on $\mathscr{N}^\pm$ by requiring the $x^A_\pm$'s to be equal to x^A_Y along the corresponding null geodesic generators of $\mathscr{N}^\pm$. Then $(r_\pm, x^A_\pm)$ define local coordinates on $\mathscr{N}^\pm$ near each of the relevant generators.
- On $\widehat{\mathscr{N}^+}$ we let ω^+ be any smooth field of null vectors transverse to $\widehat{\mathscr{N}^+}$ and normal to the level-sets of r_+ such that $\omega^+|_Y = \omega_Y$. The function u is defined by the requirement that u is constant along the null geodesics issued from $\widehat{\mathscr{N}^+}$ with initial tangent ω^+, equal to r_+ at $\widehat{\mathscr{N}^+}$. We denote by ω the field of tangents to those geodesics, normalised in any suitable way. Thus

$$\omega(u) = 0\,, \quad u|_{\mathscr{N}^-} = 0\,. \tag{30}$$

- On can prove that the level sets of u, say $\mathscr{N}^-_u$, are null hypersurfaces i.e. $g(\nabla u, \nabla u) = 0$.

Similarly

- on $\mathscr{N}^-$ we let ℓ^- be any smooth field of null vectors transverse to $\mathscr{N}^-$ and normal to the level-sets of r_- such that $\ell^-|_Y = \ell_Y$.
- The function v is defined by the requirement that v is constant along the null geodesics issued from $\mathscr{N}^-$ with initial tangent ℓ^-, and with initial value r_- at $\mathscr{N}^-$.
- Denote by ℓ the field of tangents to those geodesics.
- It holds that

$$\ell(v) = 0 \,, \quad v|_{\mathscr{N}^+} = 0 \,, \quad g(\nabla v, \nabla v) = 0 \,. \qquad (31)$$

Note that by construction we have

$$\ell|_{\mathscr{N}^\pm} = \ell^\pm \,, \qquad \omega|_{\mathscr{N}^\pm} = \omega^\pm \,. \tag{32}$$

Finally,

- The functions x^A are defined through the requirement that the x^A's be constant along the null geodesics starting from $\mathscr{N}^-$ with initial tangent ℓ^-, and taking the values x^A_- at the intersection point.

Remark

This construction breaks down when the geodesics start intersecting. However, it always provides the desired coordinates in a neighborhood of $\mathscr{N}$. In particular, given two generators of $\mathscr{N}^{\pm}$ emanating from the same point on Y, there exists a neighborhood of those generators on which (u, v, x^A) form a coordinate system.

Note that

$$g(\omega,\omega) = g(\ell,\ell) = 0 \,, \tag{33}$$

and that we also have

$$\ell^v = 0 = \ell^A \iff \ell = \ell^u \partial_u \,, \quad \omega^u = 0 \iff \omega = \omega^v \partial_v + \omega^A \partial_A \,. \tag{34}$$

Finally, once the coordinates u and v have been constructed, we rescale ℓ, or ω, or both, so that

$$g(\omega,\ell) = -\frac{1}{2}\ . \tag{35}$$

and, by a re-parametrization, we can assume that the functions u and v run from zero to infinity on all generators of $\mathscr{N}^+$ and $\mathscr{N}^-$.

The wave-equation in double-null coordinates

- Let W be a vector bundle over $\mathscr{M}$.
- We will be seeking a section h of W, defined on a neighborhood of $\mathscr{N}^-$ and of differentiability class at least C^2 there, such that the following hold:

$$\begin{aligned} \Box_g h &= H(h, \nabla h, \cdot) \quad \text{on } I^+(\mathscr{N}^+ \cup \mathscr{N}^-), \\ h &= h^+ \quad \text{on} \quad \mathscr{N}^+, \qquad (36) \\ h &= h^- \quad \text{on} \quad \mathscr{N}^- . \qquad (37) \end{aligned}$$

  for prescribed fields $h^{\pm}$, and for some map H, allowed to depend upon the coordinates.
- We assume H to be smooth in all its arguments. but the results here apply to maps of finite, sufficiently large, order of differentiability in h and ∇h, and of Sobolev differentiability in the coordinates.

Let (u, v, x^A) be a coordinate system constructed in previous section, and let ω, and ℓ be the vector fields defined there, with

$$g(\omega, \omega) = g(\ell, \ell) = 0 \ , \quad g(\omega, \ell) = -\frac{1}{2} \ .$$

ON-basis

- Every vector orthogonal to ℓ is tangent to the level sets of v. Similarly, a vector orthogonal to ω is tangent to the level sets of u. Hence vectors orthogonal to both have no u- and v-components in the coordinate system above.
- We can thus write $(Vect\{\omega,\ell\})^{\perp} = Vect\{e_B;\ B=1,\dots,n-1\}$, where the e_B's form an ON-basis of TY. Thus

$$g(e_A, e_B) = \delta^A_B\ , \quad \text{and} \quad e_A = {e_A}^B \partial_B \iff {e_A}^u = 0 = {e_A}^v\ .$$

- The inverse metric in terms of this frame reads

$$g^\sharp = -\frac{1}{2}(\ell \otimes \omega + \omega \otimes \ell) + \sum_B e_B \otimes e_B \ ,$$

- Then, the wave operator takes the form

$$-\frac{1}{2}\nabla_\omega \nabla_\ell - \frac{1}{2}\nabla_\ell \nabla_\omega + \sum_C \nabla_{e_C} \nabla_{e_C} + \dots \ ,$$

  where ... denotes first- and zero-derivative terms arising from the precise nature of the field h.

- Setting

$$\varphi_0 = \psi_0 = h,\ \varphi_A = \psi_A = e_A(h),\ \varphi_+ = \omega(h),\ \psi_- = \ell(h) \ , \tag{38}$$

  leads to the following set of equations:

wave Equation as a symmetric system of first order PDE

$$\begin{aligned} \ell(\varphi_0) &= \psi_0 \,, \\ \ell(\varphi_+) - \sum_C e_C(\psi_C) &= H_{\varphi_+} \,, \\ \ell(\varphi_C) - e_C(\psi_-) &= H_{\varphi_C} \,, \end{aligned} \tag{39}$$

$$\begin{aligned} \omega(\psi_-) - \sum_C e_C(\varphi_C) &= H_{\psi_-} \,, \\ \omega(\psi_C) - e_C(\varphi_+) &= H_{\psi_C} \,, \\ \omega(\psi_0) &= \varphi_0 \,, \end{aligned} \tag{40}$$

where H_{φ_+}, etc., contains H and all remaining terms that do not involve second derivatives of h.

This system is a first-order system of PDEs in the unknown $f = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$, with $\varphi = \begin{pmatrix} \varphi_0 \\ \varphi_+ \\ \varphi_A \end{pmatrix}$ and $\psi = \begin{pmatrix} \psi_0 \\ \psi_- \\ \psi_A \end{pmatrix}$. System (39)-(40) can be written as

$$A^\mu \nabla_\mu f = G(f) \,,$$

or equivalently as

$$\begin{pmatrix} A^\mu_{\varphi\varphi} & A^\mu_{\varphi\psi} \\ A^\mu_{\psi\varphi} & A^\mu_{\psi\psi} \end{pmatrix} \nabla_\mu \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} G_\varphi \\ G_\psi \end{pmatrix} \,, \qquad (41)$$

with

$$A^u_{\varphi\varphi} = \ell^u \cdot Id \;, \quad A^u_{\varphi\psi} = A^u_{\psi\varphi} = A^u_{\psi\psi} = 0 \;,$$
$$A^v_{\psi\psi} = \omega^v \cdot Id \;, \quad A^v_{\varphi\psi} = A^v_{\psi\varphi} = A^v_{\varphi\varphi} = 0 \;,$$
$$A^B_{\varphi\psi} = A^B_{\psi\varphi} = \quad - \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \delta^B_1 & \dots & \delta^B_{n-1} \\ 0 & \delta^B_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \delta^B_{n-1} & 0 & \dots & 0 \end{pmatrix} \;,$$
$$A^B_{\varphi\varphi} = 0 \;, \quad A^B_{\psi\psi} = \omega^B \cdot Id \;,$$
$$G_\varphi(\varphi,\psi) = \begin{pmatrix} \psi_0 \\ H_{\varphi_+} \\ H_{\varphi_C} \end{pmatrix} \quad \text{and} \quad G_\psi(\varphi,\psi) = \begin{pmatrix} H_{\psi_-} \\ H_{\psi_C} \\ \varphi_0 \end{pmatrix} \;.$$