Transcritical flow modelling with the Box Scheme

UNIVERSITY OF BATH

Melina Freitag
Supervisor: Bill Morton

28th July 2003

The Saint Venant Equations

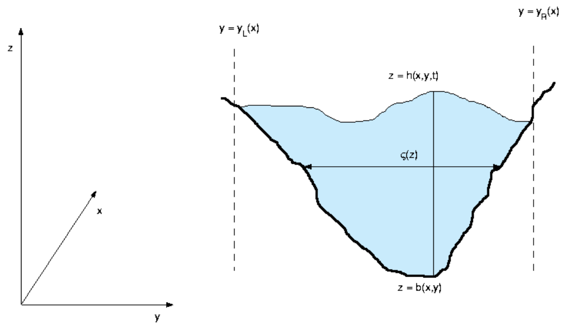


Figure 1: Channel cross section

- Mass Conservation

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

- Momentum Conservation

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\beta Q^2}{A} + gI_1\right) = gI_2 + gA(S_0 - S_f)$$

The Box Scheme

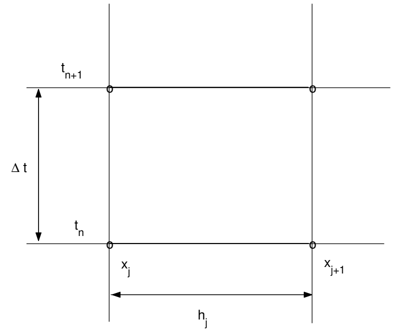


Figure 2: The Preissmann Box Scheme Stencil

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{s}(\mathbf{u}, x), \quad \mathbf{u} = [A, Q]^T$$

$$\frac{\partial \mathbf{u}}{\partial t} \approx \frac{\mathbf{u}_{j+1}^{n+1} - \mathbf{u}_{j+1}^{n} + \mathbf{u}_{j}^{n+1} - \mathbf{u}_{j}^{n}}{2\Delta t},$$

$$\frac{\partial \mathbf{f}}{\partial x} \approx \frac{\theta(\mathbf{f}_{j+1}^{n+1} - \mathbf{f}_{j}^{n+1}) + (1-\theta)(\mathbf{f}_{j+1}^{n} - \mathbf{f}_{j}^{n})}{h_j},$$

Implementation and Solution Procedure

- $2N+2$ unknowns, $2N$ equations
- need nonlinear iteration technique to solve $\mathbf{R}_{j+\frac{1}{2}} = 0 \quad \forall\, j = 0, \ldots, N-1$
- Newton system

$$\begin{bmatrix} x & x & \cdots & \cdots & \cdots & \cdots & \cdots \\ x & x & x & x & \cdots & \cdots & \cdots \\ x & x & x & x & \cdots & \cdots & \cdots \\ \cdots & \cdots & x & x & x & x & \cdots \\ \cdots & \cdots & x & x & x & x & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \Delta \mathbf{u} = \mathbf{R}$$

- Thomas Algorithm

Transcritical Flow

- subcritical and supercritical regions
- *Froude Number:*
$$F = \frac{|v|}{c}$$
- subcritical $F < 1$, supercritical $F > 1$
- steep slopes or rapidly changing channel widths
- counting problem, stability problems

Model Problem 1

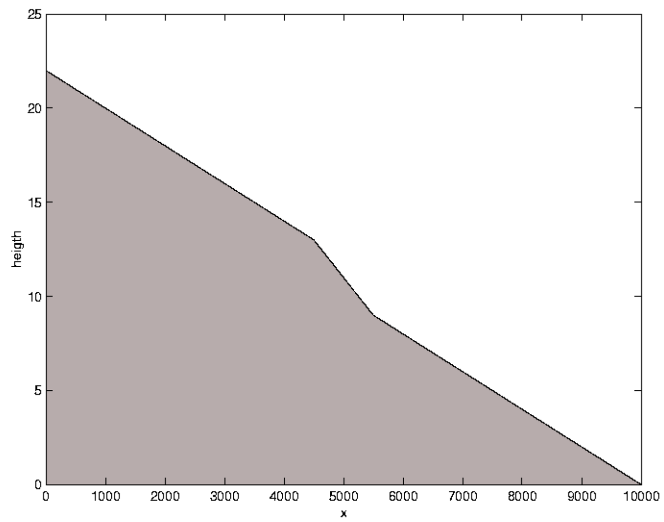


Figure 3: Channel bed

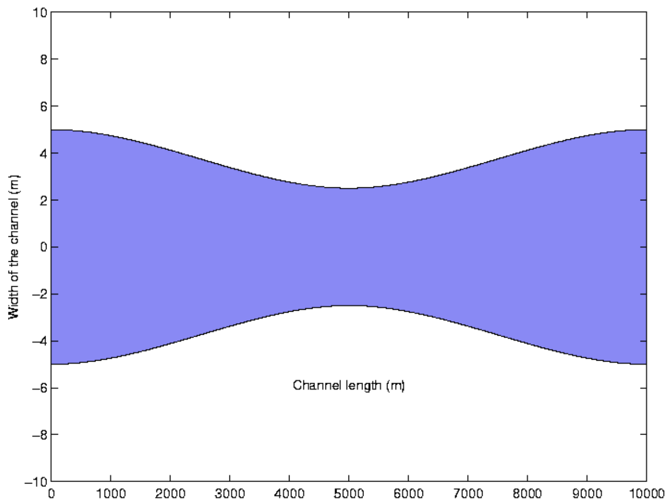
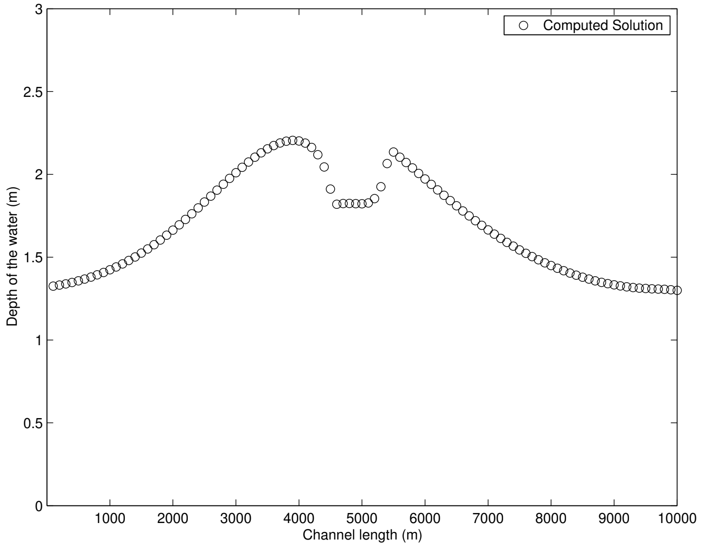


Figure 4: Channel width

- upstream boundary condition $Q = 20m^3/s$ at inflow,
- downstream boundary condtion $h = 1.3m$ at outflow

Model Problem 1 - Results



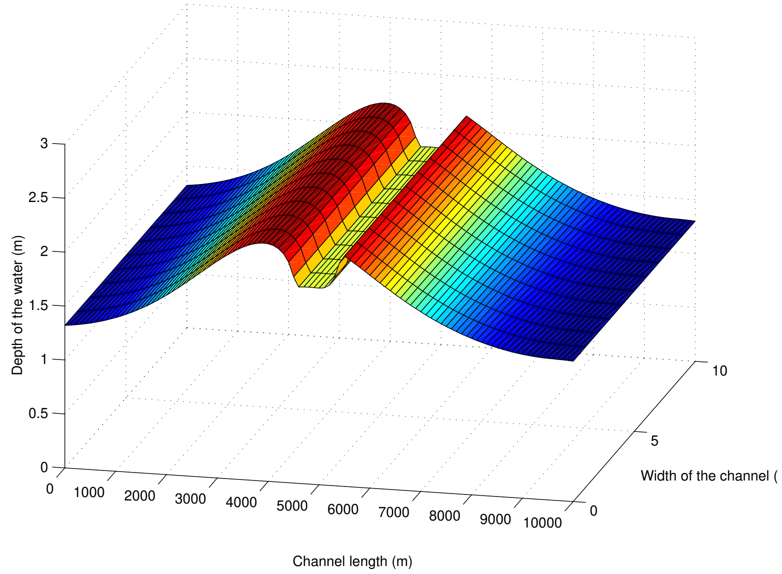


Figure 5: Subcritical Problem

- $\Delta x = 100$, $\Delta t = 100$, $\theta = \frac{2}{3}$
- $3 - 6$ iterations, Froude Number: $F = 0.52$

Model Problem 2 - Results

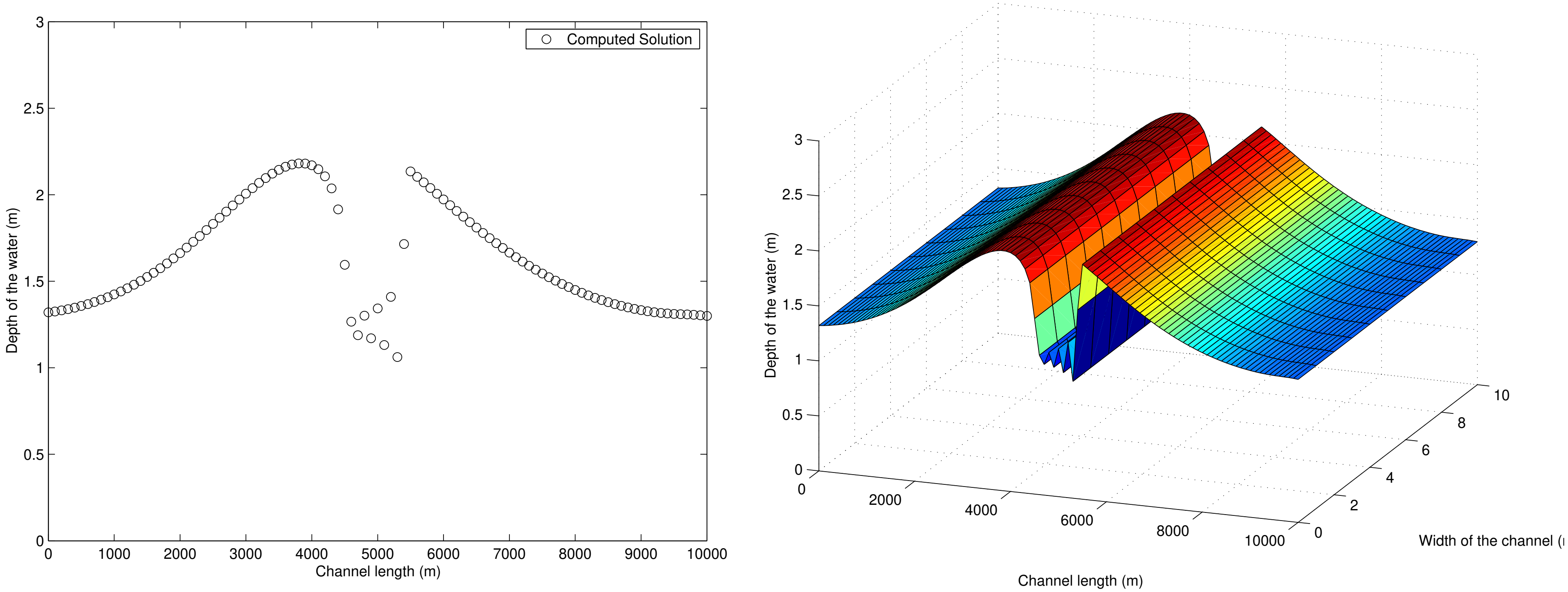


Figure 6: Transcritical Problem

- $\Delta x = 100$, $\Delta t = 100$, $\theta = \frac{2}{3}$, Froude Number: $F = 1.1545$

Model Problem 3 - Results

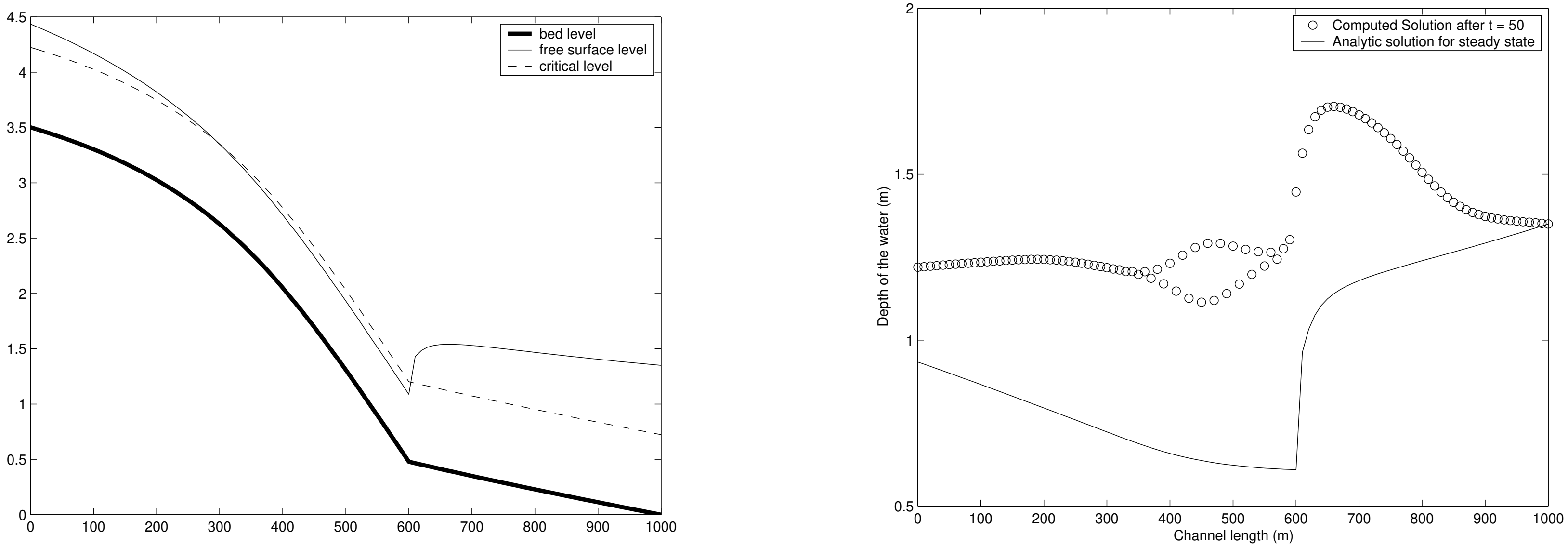


Figure 7: Transcritical Problem

- $\Delta x = 10$, $\Delta t = 5$, $\theta = \frac{2}{3}$,
- Froude Number: $F = 1.7606$, scheme breaks down

Further Work

- Why does it break down?
- Find a technique to implement the Box Scheme for transcritical flow.