Fast computation of the stability radius of a matrix

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Outline

Introduction

Background

Implicit Determinant Method

Examples

A simple example

- Consider the 10×10 matrix

$$A = \begin{bmatrix} J_9(-0.1) & 0 \\ 0 & -0.001 \end{bmatrix}$$

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$$\lambda_1 = -0.1 \quad \text{and} \quad \lambda_2 = -0.001$$

The matrix is stable.

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Observation

Matrix A is stable, with all the eigenvalues well away from the imaginary axis $\lambda_2 = -0.001$. But $A + E$ is unstable, where perturbation is only $\|E\| = 10^{-9}$!

Distance to instability - definition

- Stability of matrix $A \in \mathbb{C}^{n\times n}$: $\Lambda(A)$ in open left half plane

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Distance of a stable matrix A to instability

$$\beta(A) = \min\{\|E\| \mid \eta(A+E) = 0,\ E \in \mathbb{C}^{n\times n}\}$$

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- If $A+E$ has an eigenvalue on the imaginary axis, E is *destabilising perturbation*

Distance to instability - known results

- For a destabilising perturbation E

$$(A + E - \omega i I)z = 0,$$

for some $\omega \in \mathbb{R}$ and $z \in \mathbb{C}^n$.

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$$A - \omega i I = U \Sigma V^H.$$

  The minimising destabilising perturbation is given by $E_{\min} = -\sigma_{\min} u_n v_n^H$, where $\sigma_{\min}$ is the minimum singular value of $A - \omega i I$.

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- Measure for distance to instability of a matrix (Van Loan 1984),

$$\beta(A) = \min_{\omega \in \mathbb{R}} \sigma_{\min}(A - \omega i I),$$

where $\sigma_{\min}(A - \omega i I)$ is the smallest singular value of $A - \omega i I$.

Distance to instability - known results

Consider the singular values of $A - \omega i I$:

$$(A - \omega i I)v = \alpha u \quad \text{and} \quad (A - \omega i I)^H u = \alpha v.$$

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$$\underbrace{\begin{bmatrix} A & -\alpha I \\ \alpha I & -A^H \end{bmatrix}}_{H(\alpha)} \begin{bmatrix} v \\ u \end{bmatrix} = \omega i \begin{bmatrix} v \\ u \end{bmatrix}$$

$H(\alpha)$ has a pure imaginary eigenvalue ωi

Results on $H(\alpha)$

Theorem (Byers 1988)

The $2n \times 2n$ Hamiltonian matrix

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has an eigenvalue on the imaginary axis if and only if $\alpha \geq \beta(A)$.

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If α^* is the minimum value of α at which $H(\alpha)$ has a pure imaginary eigenvalue $\omega^* i$ with corresponding $x^* = \begin{bmatrix} v^* \\ u^* \end{bmatrix}$ then $\alpha^* = \beta(A)$.

Outline

Existing numerical methods

- Bisection approach by Byers
  - choose lower and upper bound on α (0 and $\sigma_{\min}(A)$)
  - take mean value s and calculate all the eigenvalues of $H(s)$, update lower and upper bound according to pure imaginary eigenvalues of $H(s)$

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- Boyd/Balakrishnan method
  - given an upper bound $\alpha \geq \beta(A)$, compute all pure imaginary eigenvalues $iw_1, iw_2, \ldots, iw_l$ of $H(\alpha)$ ordered so that $w_1 \leq w_2 \leq \ldots \leq w_l$
  - set $s_k = \frac{w_k + w_{k+1}}{2}$, $k = 1, \ldots l-1$ and update $\alpha = \min_k \sigma_{\min}(A - s_k iI)$

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- He/Watson algorithm
  - find the minimum of $f(\omega) = \sigma_{\min}(A - \omega iI)$
  - uses inverse iteration algorithm to find a stationary ω
  - check on all the corresponding eigenvalues of $H(\alpha)$

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$(\omega i, x)$ is a defective eigenpair of $H(\alpha) = \begin{bmatrix} A & -\alpha I \\ \alpha I & -A^H \end{bmatrix}$ of algebraic multiplicity 2.

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$$(H(\alpha) - \omega i I)x = 0, \quad x \neq 0, \quad \text{and} \quad \dim\ker(H(\alpha) - \omega i I) = 1,$$

$$y^H(H(\alpha) - \omega i I) = 0, \quad y \neq 0, \quad \text{and} \quad y^H x = 0,$$

$$y = Jx, \qquad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

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$$(H(\alpha) - \omega i I)\hat{x} = x, \quad \text{and} \quad y^H \hat{x} \neq 0,$$

Jordan block of dimension 2 at the critical value of α

Parameter dependent matrix eigenvalue problem $H(\omega, \alpha)$

Problem

How do we find a 2-dimensional Jordan block in $H(\alpha)$?

$$\underbrace{(H(\alpha) - \omega i I)}_{H(\omega,\alpha)} x = 0, \quad x \neq 0,$$

Outline

Bordered systems - a "new" method for finding eigenvalues

One-parameter problem $B(\lambda)x = 0$ or $y^H B(\lambda) = 0^H$ ($\det(B(\lambda)) = 0$)

Bordered system for $\text{rank}(B(\lambda)) = n - 1$

$$\underbrace{\begin{bmatrix} B(\lambda) & b \\ c^H & 0 \end{bmatrix}}_{M(\lambda)} \begin{bmatrix} x(\lambda) \\ f(\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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Solve $f(\lambda) = 0$ using Newton's method $\lambda^+ = \lambda - \dfrac{f(\lambda)}{f_\lambda(\lambda)}$.

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The implicit determinant method

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$$f(\omega,\alpha) = 0 \quad \text{instead of} \quad \det(H(\omega,\alpha)) = 0,$$

where

$$f(\omega,\alpha) = x(\omega,\alpha)^H J(H(\alpha) - \omega i I)x(\omega,\alpha)$$

is real.

The implicit determinant method

Differentiate the linear system

Differentiate $\begin{bmatrix} H(\alpha) - \omega i I & Jc \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x(\omega,\alpha) \\ f(\omega,\alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with respect to ω:

$$\begin{bmatrix} H(\alpha) - \omega i I & Jc \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x_\omega(\omega,\alpha) \\ f_\omega(\omega,\alpha) \end{bmatrix} = \begin{bmatrix} i x(\omega,\alpha) \\ 0 \end{bmatrix}.$$

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$$f_\omega(\omega, \alpha) = i y^H x$$

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$$f_\omega(\omega, \alpha) = iy^H x = 0,$$

because of Jordan block of dimension 2.

The implicit determinant method

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Also,

$$(H(\alpha) - \omega i I) x_\omega(\omega,\alpha) = ix,$$

and $y^H x_\omega(\omega,\alpha) \neq 0$, hence $f_{\omega\omega}(\omega,\alpha) \neq 0$.

Newton's method for *real* function g in two real variables

Solve

$$g(\omega, \alpha) = \begin{bmatrix} f(\omega, \alpha) \\ f_\omega(\omega, \alpha) \end{bmatrix} = 0,$$

using Newton's method:

$$G(\omega^{(i)}, \alpha^{(i)}) \begin{bmatrix} \Delta\omega^{(i)} \\ \Delta\alpha^{(i)} \end{bmatrix} = -g(\omega^{(i)}, \alpha^{(i)}),$$

$$\begin{bmatrix} \omega^{(i+1)} \\ \alpha^{(i+1)} \end{bmatrix} = \begin{bmatrix} \omega^{(i)} \\ \alpha^{(i)} \end{bmatrix} + \begin{bmatrix} \Delta\omega^{(i)} \\ \Delta\alpha^{(i)} \end{bmatrix}.$$

Jacobian for Newton's method

Jacobian

$$G(\omega^{(i)}, \alpha^{(i)}) = \begin{bmatrix} f_\omega(\omega^{(i)}, \alpha^{(i)}) & f_\alpha(\omega^{(i)}, \alpha^{(i)}) \\ f_{\omega\omega}(\omega^{(i)}, \alpha^{(i)}) & f_{\omega\alpha}(\omega^{(i)}, \alpha^{(i)}) \end{bmatrix}.$$

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and the Jacobian elements are evaluated by differentiating the system

$$\left[\begin{array}{cc} H(\alpha) - \omega i I & Jc \\ c^H & 0 \end{array} \right] \left[\begin{array}{c} x(\omega, \alpha) \\ f(\omega, \alpha) \end{array} \right] = \left[\begin{array}{c} 0 \\ 1 \end{array} \right],$$

with respect to ω and α.

Implementation

- one (sparse) LU factorisation of

$$\begin{bmatrix} H(\alpha) - \omega i I & Jc \\ c^H & 0 \end{bmatrix}$$

- solve with bordered system matrix and 5 different right hand sides in order to obtain $f(\omega, \alpha)$ and entries for Jacobian

$$G(\omega, \alpha) = \begin{bmatrix} f_\omega(\omega, \alpha) & f_\alpha(\omega, \alpha) \\ f_{\omega\omega}(\omega, \alpha) & f_{\omega\alpha}(\omega, \alpha) \end{bmatrix}$$

- very fast quadratically convergent Newton method in 2 dimensions

Remarks

- full-rank Jacobian $G(\omega^*, \alpha^*) = \begin{bmatrix} 0 & f_\alpha(\omega^*, \alpha^*) \\ f_{\omega\omega}(\omega^*, \alpha^*) & f_{\omega\alpha}(\omega^*, \alpha^*) \end{bmatrix}$,

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- $f_{\omega\omega}(\omega^*, \alpha^*) < 0$ and $f_\alpha(\omega^*, \alpha^*) > 0$.

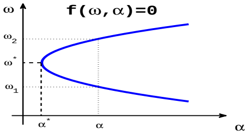

Figure: Curve $f(\omega, \alpha) = 0$ in the (ω, α)-plane for $f_{\omega\omega}(\omega^*, \alpha^*) < 0$

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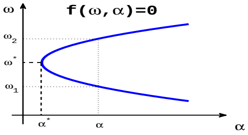

Figure: Curve $f(\omega, \alpha) = 0$ in the (ω, α)-plane for $f_{\omega\omega}(\omega^*, \alpha^*) < 0$

- Multiplication by $\begin{bmatrix} -J & 0 \\ 0^H & 1 \end{bmatrix}$ leads to the Hermitian system

$$\begin{bmatrix} -JH(\alpha) + \omega i J & c \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x(\omega, \alpha) \\ f(\omega, \alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Outline

Example 1

Consider

$$A = \begin{bmatrix} -0.4+6i & 1 & & \\ 1 & -0.1+i & 1 & \\ & 1 & -1-3i & 1 \\ & & 1 & -5+i \end{bmatrix}$$

which has eigenvalues (rounded to 3 significant digits)

$$\Lambda(A) = \{-0.41+5.80i, -0.04+0.95i, -0.92-2.62i, -5.13+0.87i\}$$

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Starting values:

$\alpha^{(0)} = 0$

$\omega^{(0)}$: imaginary part of the eigenvalue of A closest to the imaginary axis

$c = x^{(0)} = \begin{bmatrix} v(\omega^{(0)}, \alpha^{(0)}) \\ u(\omega^{(0)}, \alpha^{(0)}) \end{bmatrix}$, where $v(\omega^{(0)}, \alpha^{(0)})$ and $u(\omega^{(0)}, \alpha^{(0)})$ are right and left singular vectors of $A - \omega^{(0)} iI$

Example 1

Table: Results for Example 1.

	Newton method		
i	$\omega^{(i)}$	$\alpha^{(i)}$	$\lVert g(\omega^{(i)}, \alpha^{(i)}) \rVert$
0	0.953057740164838	0	-
1	0.953036248966048	0.031887014318100	1.5949900020014e-02
2	0.953014724735990	0.031887009443620	2.2577279982423e-04
3	0.953014724704841	0.031887014303200	2.4473093206567e-09
4	0.953014724704841	0.031887014303200	8.2762961087551e-16

Example 2

Orr-Sommerfeld operator

$$\frac{1}{\gamma R}L^2 v - i(UL - U'')v = \lambda L v, \quad \text{where} \quad L = \frac{d^2}{dx^2} - \gamma^2 \quad \text{and} \quad U = 1 - x^2.$$

Discretise the operator on $v \in [-1, 1]$ using finite differences with $\gamma = 1$, $R = 1000$ and $n = 1000$.

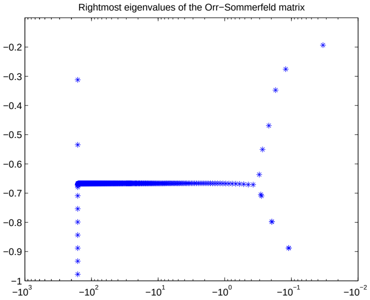

Figure: Eigenvalues of the Orr-Sommerfeld matrix in Example 2.

Example 2

Convergence to $\omega = 0.199755999447167$ and $\alpha = 0.001978172281960$ within 5 iterations.

Table: CPU times for Example 2.

Algorithm	"Inner" iterations		"Outer" iterations (Eigenvalue computation for Hamiltonian matrix)		Total CPU time
	quantity	CPU time	quantity	CPU time	
Boyd/Balakrishnan	6	3.49 s	6	63.28 s	66.77 s
He/Watson	1786	244.14 s	1	10.54 s	254.68 s
Newton	5	5.67 s	1	10.33 s	16.00 s

Example 3

Tolosa matrix `tols340.mtx`

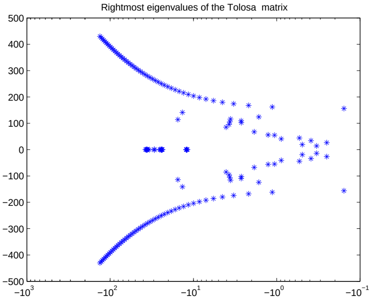

Figure: Eigenvalues of the Tolosa matrix in Example 3.

Example 3

Convergence to $\omega = 1.559998439945282$ and $\alpha = 0.000019997968879$ within 4 iterations.

Table: CPU times for Example 3.

Algorithm	"Inner" iterations		"Outer" iterations (Eigenvalue computation for Hamiltonian matrix)		Total CPU time
	quantity	CPU time	quantity	CPU time	
Boyd/Balakrishnan	3	67.52 s	3	5.27 s	72.79 s
He/Watson	> 33000	> 2230 s	> 11	> 18 s	> 2248 s
Newton	4	2.01 s	1	1.69 s	3.7 s

Final remarks

Conclusions

- new algorithm for computing the distance to unstable matrix
- relies on finding a 2-dimensional Jordan block in 2-parameter matrix
- only one LU decomposition per Newton step of bordered matrix M necessary
- numerical results show that new method outperforms earlier algorithms

Final remarks

Conclusions

- new algorithm for computing the distance to unstable matrix
- relies on finding a 2-dimensional Jordan block in 2-parameter matrix
- only one LU decomposition per Newton step of bordered matrix M necessary
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Extensions

- structured stability radius
- discrete distance to instabilty (Gürbüzbalaban et al)
- H_∞-norm

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Thank you.