Mathematical Beer Goggles or The Mathematics of Image Processing

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Outline

Outline

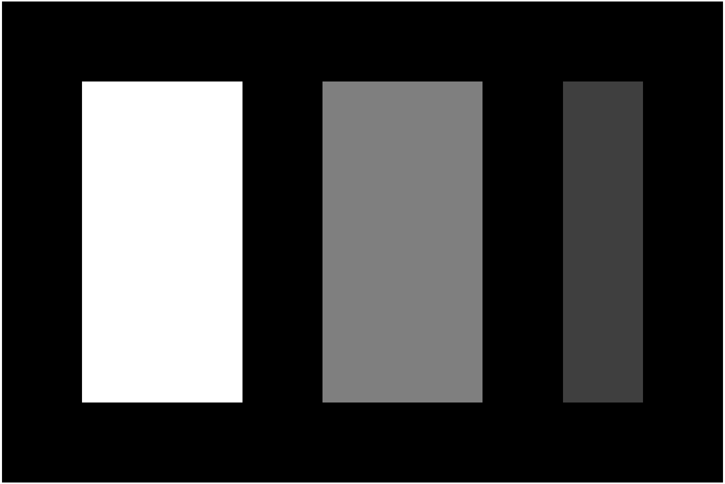
Grayscale images

One dimensional matrix

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 8 & 0 & 4 & 4 & 0 & 2 & 0 \\ 0 & 8 & 8 & 0 & 4 & 4 & 0 & 2 & 0 \\ 0 & 8 & 8 & 0 & 4 & 4 & 0 & 2 & 0 \\ 0 & 8 & 8 & 0 & 4 & 4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
imagesc(X), colormap(gray)
```

MATLAB image



RGB images

Three dimensional matrix

$$X(:,:,1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X(:,:,2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X(:,:,3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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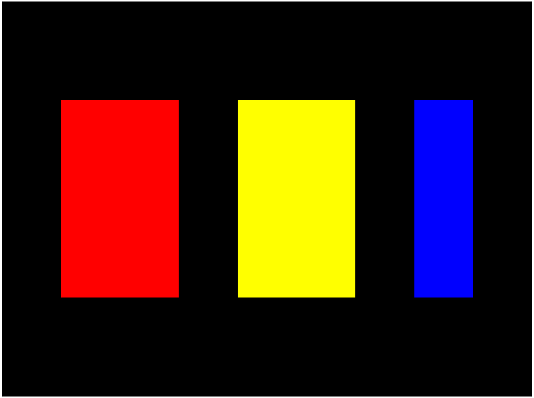
MATLAB image



```
X = imread('pic.jpg'), imwrite(X,'pic.jpg')
```

Outline

The singular value decomposition (SVD)

Existence and Uniqueness

Let $X \in \mathbb{C}^{m,n}$, $m \geq n$ Then

$$\begin{bmatrix} & X & \end{bmatrix} \left[v_1|v_2|\ldots|v_n\right] = \left[u_1|u_2|\ldots|u_m\right] \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \\ & & & 0 \end{bmatrix}$$

or

$$X = U\Sigma V^T,$$

where $U^TU = I$, with columns of U called left singular vectors and $V^TV = I$ with right singular vectors as columns of V and $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$ called singular values ordered such that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$.

Low-rank approximations

Theorem (The rank of a matrix)

The rank of X is r, the number of nonzero singular values in

$$X = U\Sigma V^T = U \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \cdots \\ & & & & 0 & \ddots \end{bmatrix} V^T.$$

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Theorem (Another representation)

X is the sum of r rank-one matrices

$$X = \sum_{j=1}^{r} \sigma_j u_j v_j^T.$$

Low-rank approximations

Theorem

For any ν with $0 \leq \nu \leq r$, define

$$X_\nu = \sum_{j=1}^{\nu} \sigma_j u_j v_j^T,$$

Then

$$\|X - X_\nu\|_2 = \inf_{B \in \mathbb{C}^{m,n},\, rank(B) \leq \nu} \|X - B\|_2 = \sigma_{\nu+1}.$$

Low-rank approximations

Proof ($m = n$)

$$\begin{aligned} \|X - X_\nu\|_2 &= \|\sum_{j=\nu+1}^{r} \sigma_j u_j v_j^T\|_2 = \|U \begin{bmatrix} 0 & & & \\ & \sigma_{\nu+1} & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} V^T\|_2 \\ &= \sigma_{\nu+1} \end{aligned}$$

Remains to show that there is no closer rank ν matrix to X.

Low-rank approximations

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Remains to show that there is no closer rank ν matrix to X.

- Let B have rank ν, null space of (B) has dimension $n - \nu$
- $\{v_1, \ldots, v_{\nu+1}\}$ has dimension $\nu + 1$
- Let h be a unit vector in their intersection:

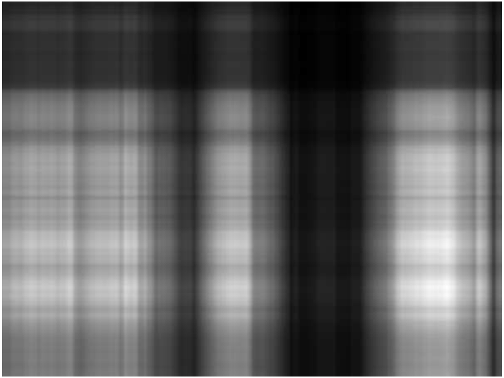
$$\begin{aligned} \|X - B\|^2 &\geq \|(X - B)h\|^2 = \|Xh\|^2 = \|U\Sigma V^T h\|^2 \\ &= \|\Sigma(V^T h)\|^2 \geq \sigma_{\nu+1}^2 \|V^T h\|^2 \geq \sigma_{\nu+1}^2. \end{aligned}$$

Example $m = 604$, $n = 453$, $m * n = 273612$

Compression ratio

$$c := \frac{(m+n)\nu}{mn}$$

Rank-1 approximation $m = 604$, $n = 453$, $m + n = 1057$



Compression ratio

$$c := 3.8631e-03$$

Rank-2 approximation

Compression ratio

$$c := 7.7263e - 03$$

Rank-3 approximation

Compression ratio

$$c := 1.1589e-02$$

Rank-4 approximation

Compression ratio

$$c := 1.5453e-02$$

Rank-5 approximation

Compression ratio

$$c := 1.9316e - 02$$

Rank-10 approximation

Compression ratio

$$c := 3.8631e - 02$$

Rank-20 approximation

Compression ratio

$$c := 7.7263e - 02$$

Rank-30 approximation

Compression ratio

$$c := 0.11589$$

Rank-40 approximation

Compression ratio

$$c := 0.15453$$

Rank-60 approximation

Compression ratio

$$c := 0.23179$$

Rank-80 approximation

Compression ratio

$$c := 0.30905$$

Outline

Blurred and exact images

- Let X be the exact image
- Let B be the blurred image

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- If the blurring of the columns is independent of the blurring in the rows then

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First attempt at deblurring

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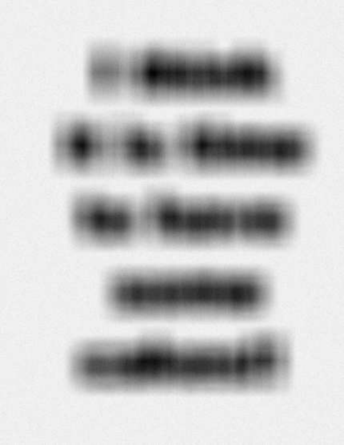

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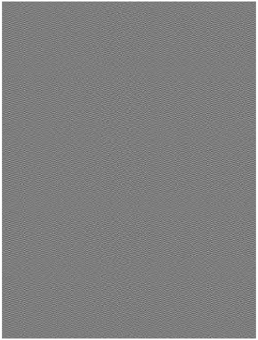

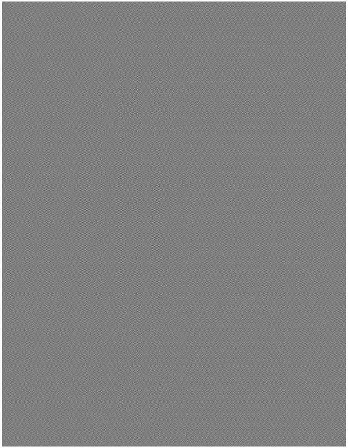

What is the problem?

A noisy blurred image

$$B = B_{\text{exact}} + E = A_c X A_r^T + E$$

and therefore

$$X_{\text{Naive}} = X + A_c^{-1} E A_r^{-T}.$$

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Error

The naive solution satisfies

$$\frac{\|X_{\text{Naive}} - X\|_F}{\|X\|_F} \leq \text{cond}(A_c)\text{cond}(A_r)\frac{\|E\|_F}{\|B\|_F}.$$

Deblurring using a general model

The blurring process as a linear model

We assume the blurring process is linear, i.e.

$$x = \mathrm{vec}(X) = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^N, \quad b = \mathrm{vec}(B) = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix} \in \mathbb{R}^N$$

$N = m * n$ are related by the linear model

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$N = m * n$ are related by the linear model

$$Ax = b$$

$b = b_{exact} + e$

$$x_{\text{Naive}} = A^{-1}b = A^{-1}b_{\text{exact}} + A^{-1}e = x + A^{-1}e$$

Separable two dimensional blurs

The Kronecker product

If horizontal and vertical flow can be separated then

$$Ax = b \quad \Leftrightarrow \quad A\mathrm{vec}(X) = \mathrm{vec}(B) = \mathrm{vec}(A_c X A_r^T)$$

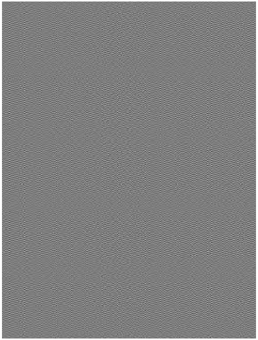
$$(A_r \otimes A_c)\mathrm{vec}(X) = \mathrm{vec}(A_c X A_r^T),$$

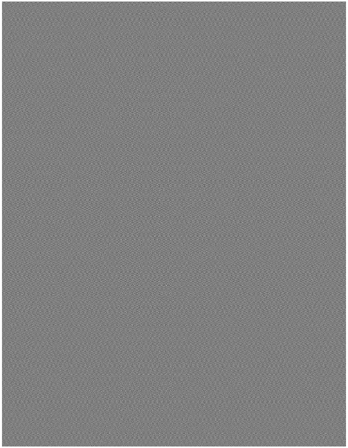
where

$$A = A_r \otimes A_c = \begin{bmatrix} a_{11}^r A_c & \dots & a_{1n}^r A_c \\ \vdots & \vdots & \vdots \\ a_{n1}^r A_c & \dots & a_{nn}^r A_c \end{bmatrix},$$

$$(U_r \Sigma_r V_r^T) \otimes (U_c \Sigma_c V_c^T) = (U_r \otimes U_c)(\Sigma_r \otimes \Sigma_c)(V_r \otimes V_c)^T.$$

$x_{\text{Naive}} = x + A^{-1}e$





Outline

Taking bad pictures

Sources of bad pictures

- defocus the camera lens (limitations in the optical system)
- motion blur
- air turbulence
- atmospheric blurring

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Noise E

- background photons from both natural or artificial sources
- signal represented by finite number of bits (quantisation error)

Modelling the blurring matrix A

Single bright pixel

$$x = e_i \Rightarrow Ae_i = \text{ column } i \text{ of } A$$

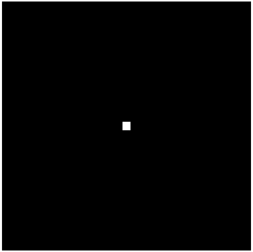

Figure: Point source (single bright pixel)

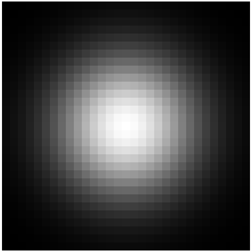

Figure: Point spread function (PSF)

Modelling the blurring matrix A

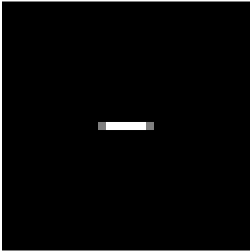

Figure: Motion blur

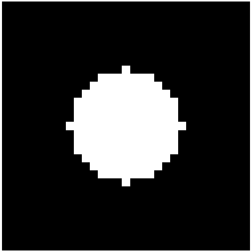

Figure: Out-of-focus blur

$$p_{ij} = \begin{cases} 1/(\pi r)^2 & \text{if} \quad (i-k)^2 + (j-l)^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases}$$

Modelling the blurring matrix A

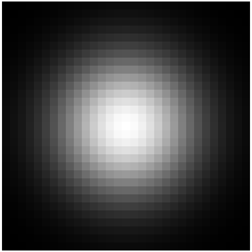

Figure: Atmospheric turbulence blur

$$p_{ij} = \exp\left(-\frac{1}{2}\begin{bmatrix} i-k \\ j-l \end{bmatrix}^T \begin{bmatrix} s_1^2 & \rho^2 \\ \rho^2 & s_2^2 \end{bmatrix}^{-1} \begin{bmatrix} i-k \\ j-l \end{bmatrix}^T\right)$$

$$p_{ij} = \left(1 + \begin{bmatrix} i-k \\ j-l \end{bmatrix}^T \begin{bmatrix} s_1^2 & \rho^2 \\ \rho^2 & s_2^2 \end{bmatrix}^{-1} \begin{bmatrix} i-k \\ j-l \end{bmatrix}^T\right)^{-\beta}$$

Boundary conditions and structured matrix computations

Boundary conditions

- Zero boundary conditions
- Periodic boundary conditions
- Reflexive boundary conditions

The matrix A which is obtained from P by convolution becomes

- Block Toeplitz matrix
- Block Circulant matrix
- Sum of Block Toeplitz and Block Hankel and Block Toeplitz plus Hankel matrices

Spectral filtering

The SVD again

With

$$A = U\Sigma V^T = \left[u_1 | u_2 | \dots | u_N\right] \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_N \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_N^T \end{bmatrix}$$

we have

$$x_{\text{Naive}} = A^{-1}b = V\Sigma^{-1}U^T b = \sum_{i=1}^{N} \frac{u_i^T b}{\sigma_i} v_i$$

$$X_{\text{Naive}} = \sum_{i=1}^{N} \frac{u_i^T b}{\sigma_i} V_i = \sum_{i=1}^{N} \frac{u_i^T b_{\text{exact}}}{\sigma_i} V_i + \sum_{i=1}^{N} \frac{u_i^T e}{\sigma_i} V_i$$

Spectral filtering

Behaviour of singular values

- $\sigma_i \to 0$ as i grows
- the more "blurry" the function, the faster the decay rate
- $\mathrm{cond}(A) = \sigma_1 / \sigma_N$

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The regularised solution

Introduce filter factors Φ_i

$$x_{\text{Naive}} = \sum_{i=1}^{N} \Phi_i \frac{u_i^T b}{\sigma_i} v_i$$

Two methods

TSVD

$$\Phi_i = \begin{cases} 1 & i = 1, \ldots, k \\ 0 & i = k+1, \ldots, N \end{cases}$$

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Tikhonov regularisation

$$\Phi_i = \frac{\sigma_i^2}{\sigma_i^2 + \alpha^2}$$

where $\alpha > 0$ is a regularisation parameter, This choice of filter factor yields solution to the minimisation problem

$$\min_x \{\|b - Ax\|_2^2 + \alpha^2 \|x\|_2^2\}.$$

Regularisation and perturbation errors

Regularised solution

$$\begin{aligned} x_{\text{filt}} &= V\Phi\Sigma^{-1}U^T b \\ &= V\Phi\Sigma^{-1}U^T A x_{\text{exact}} + V\Phi\Sigma^{-1}U^T e \\ &= V\Phi V^T x_{\text{exact}} + V\Phi\Sigma^{-1}U^T e \end{aligned}$$

$$x_{\text{exact}} - x_{\text{filt}} = \underbrace{(I - V\Phi V^T)x_{\text{exact}}}_{\text{Regularisation error}} - \underbrace{V\Phi\Sigma^{-1}U^T e}_{\text{Perturbation error}}$$

Regularisation and perturbation errors

Regularised solution

$$
\begin{aligned}
x_{\text{filt}} &= V\Phi\Sigma^{-1}U^T b \\
&= V\Phi\Sigma^{-1}U^T A x_{\text{exact}} + V\Phi\Sigma^{-1}U^T e \\
&= V\Phi V^T x_{\text{exact}} + V\Phi\Sigma^{-1}U^T e
\end{aligned}
$$

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$$

Oversmoothing and undersmoothing

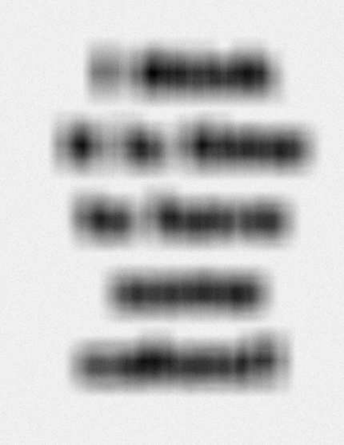
- small regularisation error, large perturbation error leads to undersmoothed solution
- large regularisation error, small perturbation error leads to oversmoothed solution

Regularisation and perturbation errors

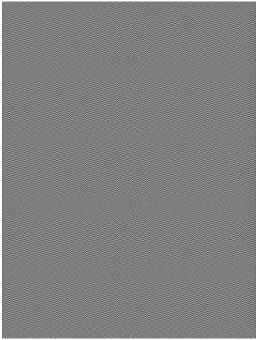
Parameter choice methods

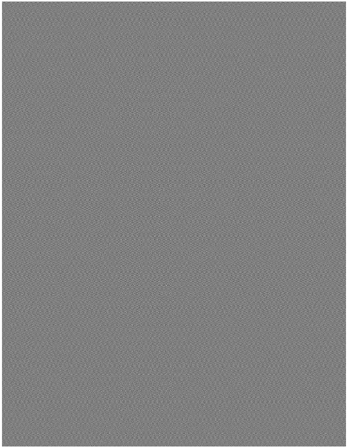
- Discrepancy Principle
- Generalised Cross Validation
- L-Curve Criterion

A second attempt at deblurring



$X_{\text{Naive}} = A_c^{-1} B A_r^{-T}$





Filtered solution using TSVD

Figure: $k = 4801$, $N = 83000$

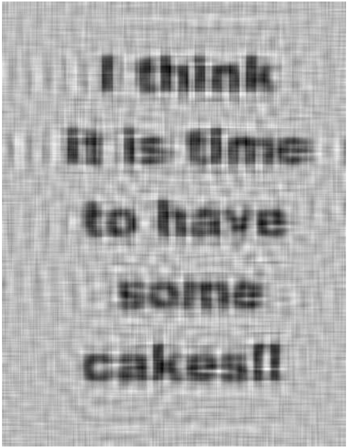


Figure: $k = 6630$, $N = 238650s$

Tikhonov regularisation

Figure: $\alpha = 0.0276$

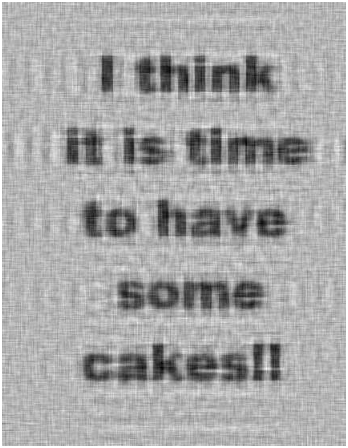


Figure: $\alpha = 0.0137$

P. C. Hansen, J. G. Nagy, and D. P. O'Leary, *Deblurring Images - Matrices, Spectra and Filtering*, SIAM, Philadelphia, 1st ed., 2006.