Solving eigenvalue problems and matrix nearness problems using the implicit determinant method

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Implicit Determinant Method

- Let $\boldsymbol{A}(\lambda,\mu) \in \mathbb{C}^{n\times n}$, $\lambda,\mu \in \mathbb{C}$ and $\boldsymbol{b},\boldsymbol{c} \in \mathbb{C}^n$. Consider the **bordered matrix**

$$\boldsymbol{M}(\lambda,\mu) = \begin{bmatrix} \boldsymbol{A}(\lambda,\mu) & \boldsymbol{b} \\ \boldsymbol{c}^H & 0 \end{bmatrix}.$$

- Assume rank$(\boldsymbol{A}(\lambda,\mu)) \geq n-1$ and $\boldsymbol{b},\boldsymbol{c}$ are chosen such that $\boldsymbol{M}(\lambda,\mu)$ is nonsingular.
- Consider

$$\begin{bmatrix} \boldsymbol{A}(\lambda,\mu) & \boldsymbol{b} \\ \boldsymbol{c}^H & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{f} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \boldsymbol{x} \in \mathbb{C}^n, \quad \boldsymbol{f} \in \mathbb{C}, \qquad (1)$$

then

(i) $\boldsymbol{f} = \boldsymbol{f}(\lambda,\mu)$ and $\boldsymbol{x} = \boldsymbol{x}(\lambda,\mu)$,
(ii) Given (λ,μ) then $\boldsymbol{f}(\lambda,\mu)$ and $\boldsymbol{x}(\lambda,\mu)$ can be found by solving (1),
(iii) Derivatives $\boldsymbol{f}_\lambda(\lambda,\mu)$, $\boldsymbol{f}_\mu(\lambda,\mu)$, $\boldsymbol{f}_{\lambda\lambda}(\lambda,\mu)$, etc. can be found easily by differentiation of (1) and then a solve with the same system matrix as in (1),
(iv) important equivalence:

$$\boldsymbol{f}(\lambda,\mu) = 0 \;\Leftrightarrow\; \det(\boldsymbol{A}(\lambda,\mu)) = 0$$

(v) If $\det(\boldsymbol{A}(\lambda^*,\mu^*)) = 0$ then $\boldsymbol{f}(\lambda^*,\mu^*) = 0$ and $\boldsymbol{x}(\lambda^*,\mu^*) \in \ker(\boldsymbol{A}(\lambda^*,\mu^*))$.

Applications: A (new) method for finding eigenvalues (for linear and nonlinear eigenvalue problems)

- $\boldsymbol{A}(\lambda,\mu) := \boldsymbol{A} - \lambda\boldsymbol{I}$

$$\boldsymbol{f}(\lambda) = 0 \;\Leftrightarrow\; \det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$$

- Newton's method for $\boldsymbol{f}(\lambda) = 0$:
  - Given λ_-, solve (1) with $\lambda = \lambda_-$ to get $\boldsymbol{f}(\lambda_-)$ and $\boldsymbol{x}(\lambda_-)$
  - Solve

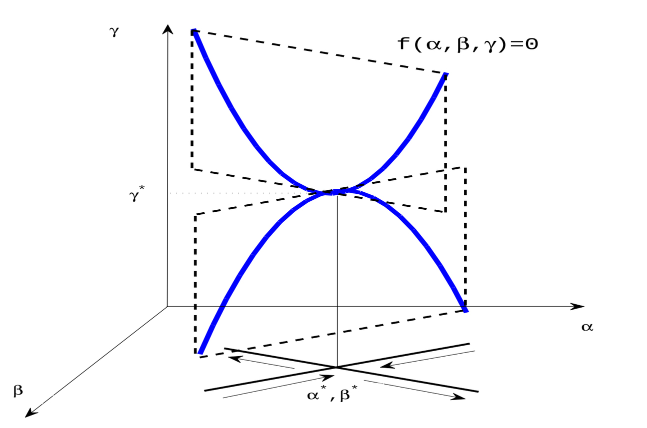
$$\begin{bmatrix} \boldsymbol{A} - \lambda_-\boldsymbol{I} & \boldsymbol{b} \\ \boldsymbol{c}^H & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_\lambda(\lambda_-) \\ \boldsymbol{f}_\lambda(\lambda_-) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}(\lambda_-) \\ 0 \end{bmatrix} \qquad (2)$$

  - $\lambda_+ = \lambda_- - \frac{f(\lambda_-)}{f_\lambda(\lambda_-)}$
- Close links with Inverse iteration
- Advantages if the matrix is defective (or nearly defective)
- Extends to general nonlinear analytic problem $\boldsymbol{A}(\lambda) \in \mathbb{C}^n$

Calculating a Jordan block

- $\boldsymbol{A}(\lambda,\mu) := \boldsymbol{A}(\mu) - \lambda\boldsymbol{I}$
- Assume $\boldsymbol{A}(\mu^*)$ has a Jordan block at eigenvalue λ^*
- At the Jordan block

$$\boldsymbol{f}(\lambda^*,\mu^*) = 0 \;\Leftrightarrow\; \det(\boldsymbol{A}(\lambda^*,\mu^*)) = 0$$
$$\boldsymbol{f}_\lambda(\lambda^*,\mu^*) = 0 \;\Leftrightarrow\; \frac{\partial}{\partial\lambda}\det(\boldsymbol{A}(\lambda,\mu))\,|_{\lambda^*,\mu^*} = 0$$

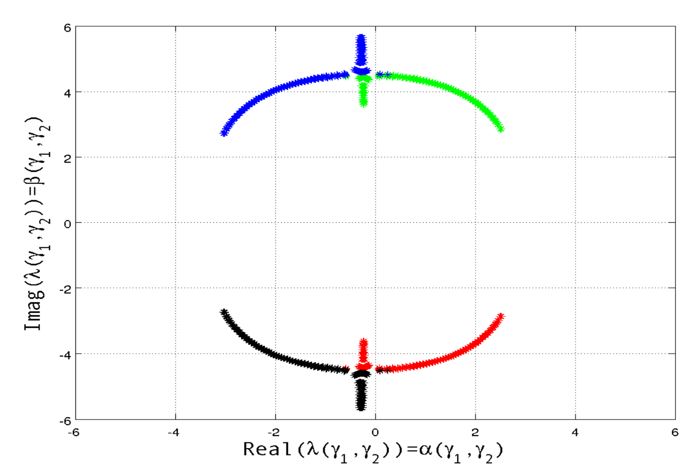


(a) Strong coupling of eigenvalues at a saddle point of $\boldsymbol{f}$

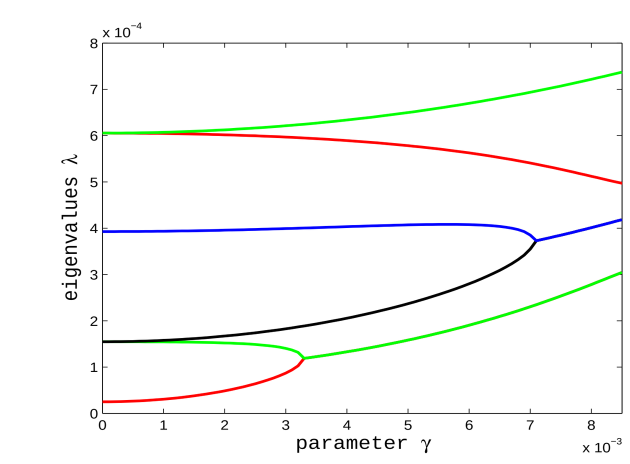
- Numerical method: Newton's method applied to

$$\boldsymbol{f}(\lambda,\mu) = 0$$
$$\boldsymbol{f}_\lambda(\lambda,\mu) = 0$$

- $\boldsymbol{f}_\lambda(\lambda,\mu)$ is calculated as in (2) and elements of the Jacobian $\boldsymbol{f}_\mu(\lambda,\mu)$, $\boldsymbol{f}_{\lambda\lambda}(\lambda,\mu)$ are calculated similarly



(b) Coalescence of complex eigenvalues bus problem



(c) Coalescence of smallest eigenvalues for panel flutter problem

- Applications in aerodynamical stability, stability of electrical power systems, quantum mechanics etc.

Distance to instability

- Distance to instability $\beta(\boldsymbol{A}) = \min\{\|\boldsymbol{E}\| \mid \eta(\boldsymbol{A}+\boldsymbol{E}) = 0,\ \boldsymbol{E} \in \mathbb{C}^{n\times n}\}$, where $\eta(\boldsymbol{A}) := \max\{\mathrm{Re}(\lambda) \mid \lambda \in \Lambda(\boldsymbol{A})\}$
- $\boldsymbol{E}$ is *destabilising perturbation*: $(\boldsymbol{A}+\boldsymbol{E}-\omega i\boldsymbol{I})\boldsymbol{z} = 0$, $\omega \in \mathbb{R}$, $\boldsymbol{z} \in \mathbb{C}^n$.
- Measure for distance to instability of a matrix (Van Loan 1984):

$$\beta(\boldsymbol{A}) = \min_{\omega\in\mathbb{R}} \sigma_{\min}(\boldsymbol{A} - \omega i\boldsymbol{I}),$$

where $\sigma_{\min}(\boldsymbol{A} - \omega i\boldsymbol{I})$ is the smallest singular value of $\boldsymbol{A} - \omega i\boldsymbol{I}$

- Byers (1988): The $2n \times 2n$ Hamiltonian matrix

$$\boldsymbol{H}(\alpha) = \begin{bmatrix} \boldsymbol{A} & -\alpha\boldsymbol{I} \\ \alpha\boldsymbol{I} & -\boldsymbol{A}^H \end{bmatrix}.$$

has pure imaginary eigenvalue if and only if $\alpha \geq \beta(\boldsymbol{A})$:

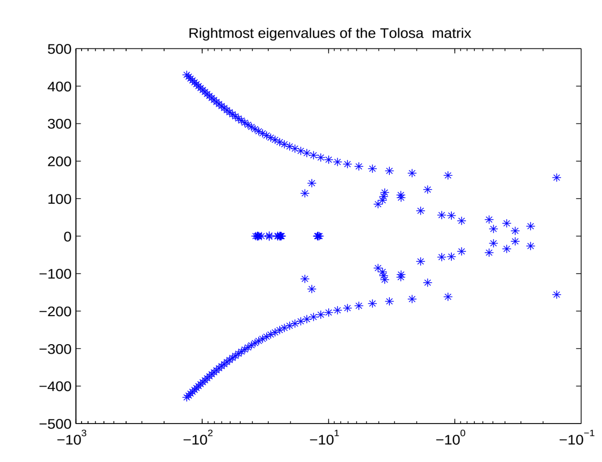
$$\begin{bmatrix} \boldsymbol{A} & -\alpha\boldsymbol{I} \\ \alpha\boldsymbol{I} & -\boldsymbol{A}^H \end{bmatrix}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix} = \omega i \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix} \qquad \omega i \text{ is a defective eigenvalue.}$$

- Method: Use implicit determinant method to find smallest α such that $\det(\boldsymbol{H}(\alpha) - i\omega\boldsymbol{I}) = 0$
- Set up

$$\begin{bmatrix} \boldsymbol{H}(\alpha) - i\omega\boldsymbol{I} & \boldsymbol{J}\boldsymbol{c} \\ \boldsymbol{c}^H & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{x}(\alpha,\omega) \\ \boldsymbol{f}(\alpha,\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Solve for

$$\boldsymbol{f}(\alpha,\omega) = 0$$
$$\boldsymbol{f}_\omega(\alpha,\omega) = 0$$



(d) Eigenvalues of the Tolosa matrix

Table: Results for Tolosa matrix

	Newton method			
i	$\omega^{(i)}$	$\alpha^{(i)}$	$\|\boldsymbol{g}(\omega^{(i)},\alpha^{(i)})\|$	$\boldsymbol{f}_{\omega\omega}(\omega^{(i)},\alpha^{(i)})$
0	155.99992199998	0	-	-
1	155.99988299728	0.0019997968878	9.99903e-04	-8.217446e-02
2	155.99984399453	0.0019997968253	1.60251e-06	-4.108720e-02
3	155.99984399452	0.0019997968879	3.12541e-11	-4.108718e-02
4	155.99984399452	0.0019997968878	3.78571e-16	-4.108718e-02

Table: CPU times

Algorithm	"Inner" iterations		"Outer" iterations (Eigenvalue computation for Hamiltonian matrix)		Total CPU time
	quantity	CPU time	quantity	CPU time	
Boyd/Balakrishnan	**3**	**67.52** s	**3**	**5.27** s	**72.79** s
He/Watson	$>$ **33000**	$>$ **2230** s	$>$ **11**	$>$ **18** s	$>$ **2248** s
Newton	**4**	**2.01** s	**1**	**1.69** s	**3.7** s

Summary of applications

- Computations of paths $\det\boldsymbol{A}(\lambda,\mu) = 0$ [S./Poulton, 2005]
- Computation of Jordan blocks [Akinola/F./S., 2013]
- Distance to nearby defective matrix [Akinola/F./S., 2013]
- Distance to instability [F./S., 2011]
- Computing the real stability radius [F./S., 2013]
- Calculating the $\boldsymbol{H}_\infty$-norm [F./S./Van Dooren, 2014]

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