Numerical approaches to singular value asymptotics for specific integral operators

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Problem formulation

- Given: $B = M \circ J$ as Fredholm integral equation

$$[Bx](s) = \int_0^1 k(s,t)x(t)dt, \quad (0 \leq s \leq 1)$$

with

$$k(s,t) = \begin{cases} m(s), & (0 \leq t \leq s \leq 1) \\ 0, & (0 \leq s \leq t \leq 1). \end{cases}$$

where $m(s)$ has got a zero, for example $m(s) = s^{\alpha}$ or $m(s) = e^{-\frac{1}{s^{\alpha}}}$

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- well-known: singular values of integral operator J:

$$\sigma_n(J) = \frac{2}{\pi(2n-1)}$$

- analytical results for the singular value decomposition (Chang (1952), Reade (1984), minimax principle):

$$\sigma_n(B) = \mathcal{O}(n^{-1})$$

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Formulation as a Sturm-Liouville problem

Formulate

$$[Bx](s) = \int_0^s m(s)x(t)dt, \quad (0 \leq s \leq 1),$$

as a boundary value problem, using the eigenvalue equation $B^*Bu = \sigma^2 u$:

$$\sigma^2 \left(\frac{u'(\tau)}{m^2(\tau)} \right)' = -u(\tau), \quad m(\tau) \neq 0, u \in C^2[0,1]$$

or

$$-(a(\tau)u'(\tau))' = \lambda u(\tau), \quad u(1) = u'(0) = 0,$$

where $\lambda = \dfrac{1}{\sigma^2}$ and $a(\tau) = \dfrac{1}{m^2(\tau)}$.

Finite difference methods for the Sturm-Liouville problem

- Boundary value problem:

$$-(a(\tau)u'(\tau))' = \lambda u(\tau)$$

$$u(1) = 0 \quad \text{and} \quad \lim_{\tau \to 0} a(\tau)u'(\tau) = 0.$$

- apply classical finite difference method:

$$\left(\frac{a_{i+1} - a_{i-1}}{4h^2} - \frac{a_i}{h^2}\right) u_{i-1} + \frac{2a_i}{h^2} u_i + \left(\frac{a_{i-1} - a_{i+1}}{4h^2} - \frac{a_i}{h^2}\right) u_{i+1} = \lambda u_i$$

- right hand side $\tau_0 := \varepsilon$.

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Results for $m(s) = s^\alpha$

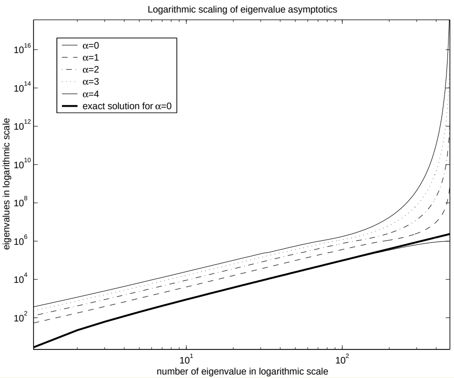

Figure: Computed eigenvalues of Sturm-Liouville problem $-(au')' = \lambda u$ for $n = 500$ and different values for α and exact eigenvalues for $\alpha = 0$ in logarithmic scales

Results for $m(s) = e^{-\frac{1}{s^{\alpha}}}$

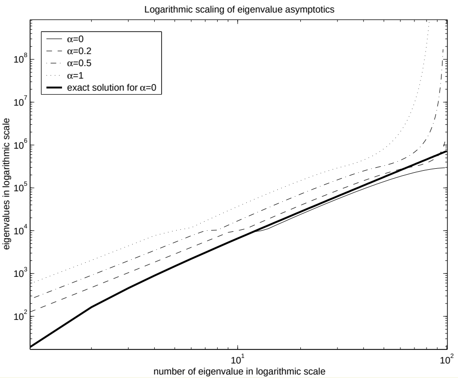

Figure: Computed eigenvalues of Sturm-Liouville problem $-(au')' = \lambda u$ for $n = 100$ and different values for α and exact eigenvalues for $\alpha = 0$ in logarithmic scales

Galerkin method for integral equations

$$[B(x)](s) = \int_0^1 k(s,t)x(t)dt, \quad 0 \le s \le 1, \quad 0 \le t \le 1$$

- singular value expansion for square integrable kernels:

$$k(s,t) = \sum_{j=1}^{\infty} \sigma_j u_j(t) v_j(s), \; 0 \le s \le 1, \; 0 \le t \le 1.$$

- algebraic singular value decomposition of $A \in \mathbb{R}^{n,n}$:

$$A = U\Sigma V^T = \sum_{j=1}^{n} s_j \mathbf{u}_j \mathbf{v}_j^T,$$

$$\|B\|_{HS}^2 := \sum_{j=1}^{\infty} \sigma_j^2 < \infty \quad \|A\|_F^2 := \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 = \sum_{j=1}^{n} s_j^2.$$

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Algorithm for Galerkin's method

- Choose $\{\Psi_j\}$ and $\{\Phi_j\}$ orthonormal sets of basis functions in $I_t = (0,1)$, $I_s = (0,1)$.
- Determine matrix $A \in \mathbb{R}^{n,n}$ with

$$a_{ij} = \langle B\Phi_j, \Psi_i \rangle_{L^2(0,1)} \quad i,j = 1,\dots,n.$$

- Compute the SVD of this matrix

$$A\mathbf{v} = s^{(n)}\mathbf{u}.$$

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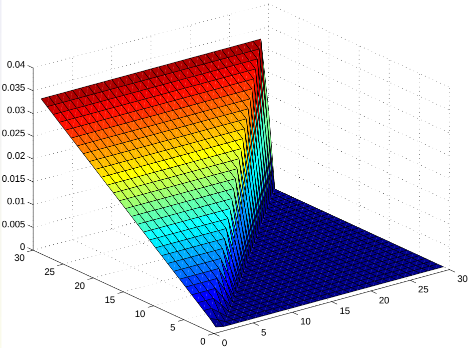
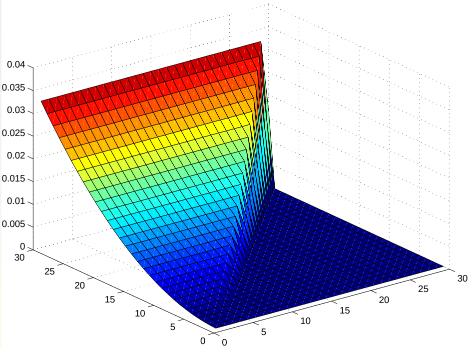
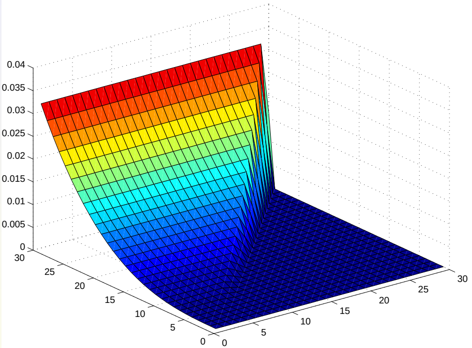
Matrix structure for $m(s) = s$



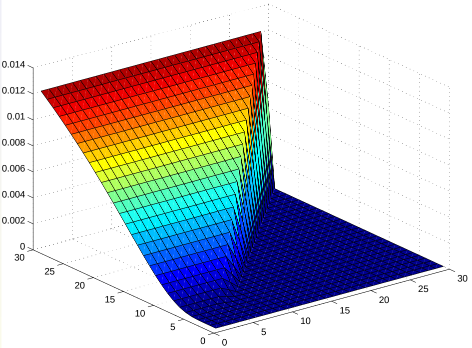
Matrix structure for $m(s) = s^2$



Matrix structure for $m(s) = s^3$



Matrix structure for $m(s) = e^{-\frac{1}{s}}$



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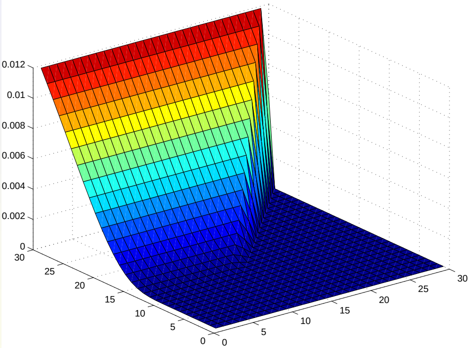


Approximation properties

Proposition Let

$$\|B\|^2 := \int_0^1 \int_0^1 |k(s,t)|^2 dtds = \sum_{i=1}^{\infty} \sigma_i^2.$$

Then

$$s_i^{(n)} \le s_i^{(n+1)} \le \sigma_i, \quad i = 1, \ldots, n.$$

Furthermore, the errors of the approximate singular values $s_i^{(n)}$ are bounded by

$$0 \le \sigma_i - s_i^{(n)} \le \delta_n, \quad i = 1, \ldots, n,$$

where

$$\delta_n^2 = \|B\|^2 - \|A\|_F^2.$$

Furthermore

$$s_i^{(n)} \le \sigma_i \le [(s_i^{(n)})^2 + \delta_n^2]^{\frac{1}{2}}, \quad i = 1, \ldots, n.$$

Results for $m(s) = s^{\alpha}$

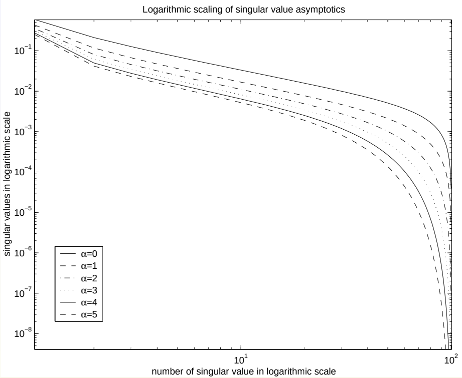
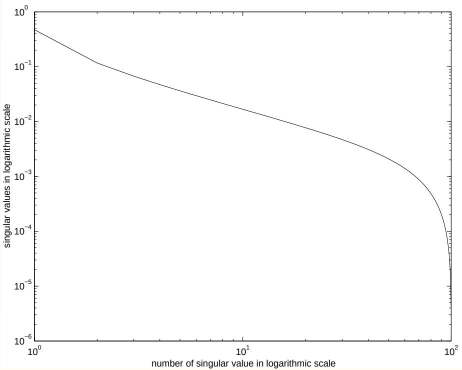


Figure: Computed singular values of integral equation $Bv = \sigma u$ for $n = 100$ and different values for α in logarithmic scales

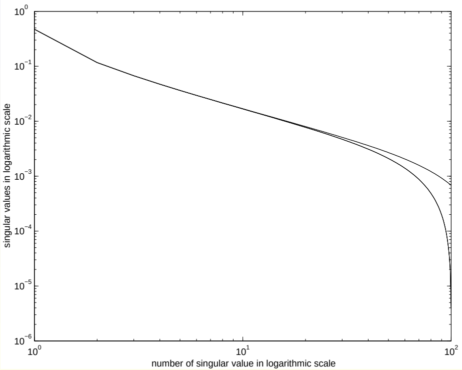
Discrete singular values for $n \longrightarrow \infty$

$$m(s) = s, \quad n = 100$$



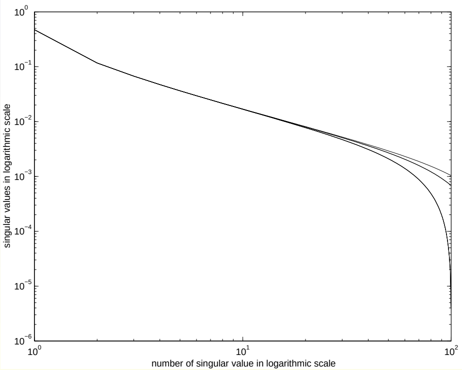
Discrete singular values for $n \longrightarrow \infty$

$$m(s) = s, \quad n = 150$$



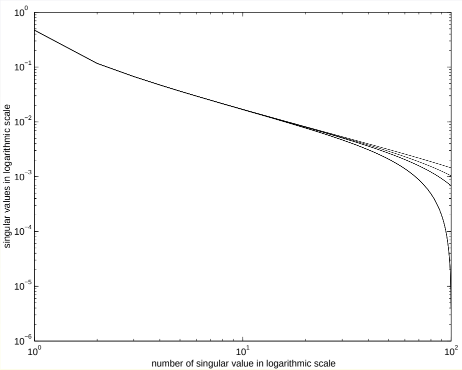
Discrete singular values for $n \to \infty$

$$m(s) = s, \quad n = 200$$



Discrete singular values for $n \longrightarrow \infty$

$$m(s) = s, \quad n = 400$$



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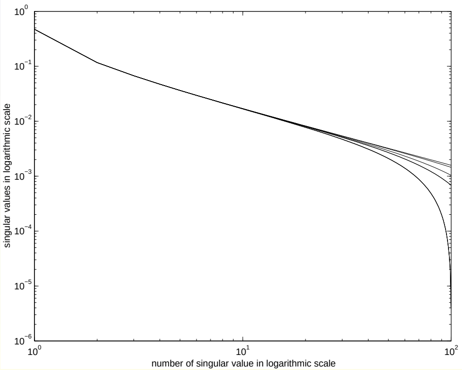

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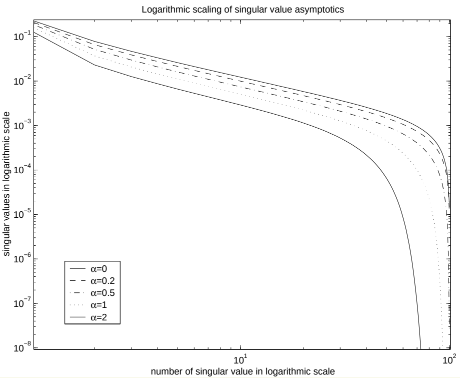

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Further numerical approaches and summary

- Rayleigh-Ritz method for symmetric kernel B^*B
- Orthonormal/Non-orthonormal basis functions
- Generalized singular value problem/Generalized eigenproblem
- Comparison yields

$$\sigma_n(B) = \int_0^1 m(s)ds \cdot \sigma_n(J) = \int_0^1 m(s)ds \cdot \frac{2}{\pi(2n-1)},$$

for the integral operator $B = M \circ J$.

- Combination of analytical and numerical results:

$$\underline{C} n^{-1} = \sigma_n^{\text{approx}}(B) \le \sigma_n^{\text{exact}}(B) \le \overline{C} n^{-1}, \quad n \to \infty$$

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Conclusions

- numerical computation of the SVD of the discretized problem through various methods
- error estimates and approximation properties
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