A new algorithm for calculating the distance to instability and the distance to a nearby defective matrix

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joint work with Alastair Spence

Part I

Distance to instability

Distance to instability

- Stability of matrix $A \in \mathbb{C}^{n \times n}$: $\Lambda(A)$ in open left half plane
- Measure of stability: distance of A to instability

Define spectral abscissa

$$\eta(A) := \max\{\mathrm{Re}(\lambda) \,|\, \lambda \in \Lambda(A)\}$$

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Distance to instability

Distance of a stable matrix A to instability

$$\beta(A) = \min\{\|E\| \,|\, \eta(A+E) = 0,\ E \in \mathbb{C}^{n\times n}\}$$

- If $\eta(A)$ is negative, A is stable.
- If $A+E$ has an eigenvalue on the imaginary axis, E is *destabilising perturbation*

Distance to instability - known results

- For a destabilising perturbation E

$$(A + E - \omega i I)z = 0,$$

for some $\omega \in \mathbb{R}$ and $z \in \mathbb{C}^n$.

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- Measure for distance to instability of a matrix (Van Loan 1984),

$$\beta(A) = \min_{\omega \in \mathbb{R}} \sigma_{\min}(A - \omega i I),$$

where $\sigma_{\min}(A - \omega i I)$ is the smallest singular value of $A - \omega i I$.

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Theorem (Byers 1988)

The $2n \times 2n$ Hamiltonian matrix

$$H(\alpha) = \begin{bmatrix} A & -\alpha I \\ \alpha I & -A^H \end{bmatrix}.$$

has an eigenvalue on the imaginary axis if and only if $\alpha \geq \beta(A)$.

Results on $H(\alpha)$

If $H(\alpha)$ has a pure imaginary eigenvalue ωi, then from

$$\underbrace{\begin{bmatrix} A & -\alpha I \\ \alpha I & -A^H \end{bmatrix}}_{H(\alpha)} \begin{bmatrix} v \\ u \end{bmatrix} = \omega i \begin{bmatrix} v \\ u \end{bmatrix}$$

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it follows that

$$(A - \omega i I)v = \alpha u \quad \text{and} \quad (A - \omega i I)^H u = \alpha v.$$

If α^* is the minimum value of α at which $H(\alpha)$ has a pure imaginary eigenvalue $\omega^* i$ with corresponding $x^* = \begin{bmatrix} v^* \\ u^* \end{bmatrix}$ then $\alpha^* = \beta(A)$.

Assume $\alpha^* = \beta(A)$ is unique.

Existing numerical methods

- Bisection approach by Byers
  - choose lower and upper bound on α (0 and $\sigma_{\min}(A)$)
  - take mean value s and calculate all the eigenvalues of $H(s)$, update lower and upper bound according to pure imaginary eigenvalues of $H(s)$

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- Boyd/Balakrishnan method
  - given an upper bound $\alpha \geq \beta(A)$, compute all pure imaginary eigenvalues $iw_1, iw_2, \ldots, iw_l$ of $H(\alpha)$ ordered so that $w_1 \leq w_2 \leq \ldots \leq w_l$
  - set $s_k = \frac{w_k + w_{k+1}}{2}$, $k = 1, \ldots l-1$ and update $\alpha = \min_k \sigma_{\min}(A - s_k iI)$

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  - set $s_k = \frac{w_k + w_{k+1}}{2}$, $k = 1, \ldots l-1$ and update $\alpha = \min_k \sigma_{\min}(A - s_k iI)$
- He/Watson algorithm
  - find the minimum of $f(\omega) = \sigma_{\min}(A - \omega iI)$
  - uses inverse iteration algorithm to find a stationary ω
  - check on all the corresponding eigenvalues of $H(\alpha)$

Results on $H(\alpha)$

Assumption

$(\omega i, x)$ is a defective eigenpair of $H(\alpha) = \begin{bmatrix} A & -\alpha I \\ \alpha I & -A^H \end{bmatrix}$ of algebraic multiplicity 2.

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$$y^H (H(\alpha) - \omega i I) = 0, \quad y \neq 0, \quad \text{and} \quad y^H x = 0,$$

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$$(H(\alpha) - \omega i I)\hat{x} = x, \quad \text{and} \quad y^H \hat{x} \neq 0,$$

Results on $H(\alpha)$

Problem

How do we find a 2-dimensional Jordan block in $H(\alpha)$?

$$\underbrace{(H(\alpha) - \omega i I)}_{H(\omega,\alpha)} x = 0, \quad x \neq 0,$$

Bordered systems

One-parameter problem $B(\lambda)x = 0$ or $y^H B(\lambda) = 0^H$ ($\det(B(\lambda)) = 0$)

Bordered system

$$\underbrace{\begin{bmatrix} B(\lambda) & b \\ c^H & 0 \end{bmatrix}}_{M(\lambda)} \begin{bmatrix} x(\lambda) \\ f(\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is nonsingular if $c^H x \neq 0$ and $y^H b \neq 0$ and $\operatorname{rank}(B(\lambda)) = n - 1$.

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Solve $f(\lambda) = 0$ using Newton's method $\lambda^+ = \lambda - \dfrac{f(\lambda)}{f_\lambda(\lambda)}$.

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$$\underbrace{(H(\alpha) - \omega i I)}_{H(\omega,\alpha)} x = 0, \quad x \neq 0,$$

The bordered matrix

Theorem (Two-parameter problem)

- *Let* (ω, α, x) *solve*

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- *For some* $c \in \mathbb{C}^n$ *assume*

$$c^H x \neq 0.$$

Then the bordered matrix

$$M(\omega, \alpha) = \begin{bmatrix} H(\alpha) - \omega i I & Jc \\ c^H & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

is nonsingular.

The implicit determinant method

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$$H(\omega, \alpha)x = 0 \quad \text{or} \quad \det(H(\omega, \alpha)) = 0$$

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$$H(\omega,\alpha)x = 0 \quad \text{or} \quad \det(H(\omega,\alpha)) = 0$$

Bordered system

$$\underbrace{\begin{bmatrix} H(\alpha)-\omega i I & Jc \\ c^H & 0 \end{bmatrix}}_{M(\omega,\alpha)} \begin{bmatrix} x(\omega,\alpha) \\ f(\omega,\alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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$$f(\omega,\alpha) = \frac{\det(H(\omega,\alpha))}{\det(M(\omega,\alpha))},$$

Solve

$$f(\omega,\alpha) = 0 \quad \text{instead of} \quad \det(H(\omega,\alpha)) = 0,$$

where

$$f(\omega,\alpha) = x(\omega,\alpha)^H J(H(\alpha) - \omega i I)x(\omega,\alpha)$$

is real.

The implicit determinant method

Differentiate the linear system

Differentiate $\begin{bmatrix} H(\alpha)-\omega i I & Jc \\ c^H & 0 \end{bmatrix}\begin{bmatrix} x(\omega,\alpha) \\ f(\omega,\alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with respect to ω:

$$\begin{bmatrix} H(\alpha)-\omega i I & Jc \\ c^H & 0 \end{bmatrix}\begin{bmatrix} x_\omega(\omega,\alpha) \\ f_\omega(\omega,\alpha) \end{bmatrix} = \begin{bmatrix} ix(\omega,\alpha) \\ 0 \end{bmatrix}.$$

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First row

$$f_\omega(\omega, \alpha) = iy^H x$$

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$$f_\omega(\omega,\alpha) = iy^H x = 0,$$

because of Jordan block of dimension 2.

The implicit determinant method

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$$f_\omega(\omega,\alpha) = iy^H x = 0,$$

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$$g(\omega,\alpha) = \begin{bmatrix} f(\omega,\alpha) \\ f_\omega(\omega,\alpha) \end{bmatrix} = 0.$$

The implicit determinant method

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$$g(\omega,\alpha) = \begin{bmatrix} f(\omega,\alpha) \\ f_\omega(\omega,\alpha) \end{bmatrix} = 0.$$

Also,

$$(H(\alpha)-\omega i I)x_\omega(\omega,\alpha) = ix,$$

and $y^H x_\omega(\omega,\alpha) \neq 0$, hence $f_{\omega\omega}(\omega,\alpha) \neq 0$.

Newton's method for *real* function g in two real variables

Solve

$$g(\omega,\alpha) = \begin{bmatrix} f(\omega,\alpha) \\ f_\omega(\omega,\alpha) \end{bmatrix} = 0,$$

using Newton's method:

$$G(\omega^{(i)},\alpha^{(i)}) \begin{bmatrix} \Delta\omega^{(i)} \\ \Delta\alpha^{(i)} \end{bmatrix} = -g(\omega^{(i)},\alpha^{(i)}),$$

$$\begin{bmatrix} \omega^{(i+1)} \\ \alpha^{(i+1)} \end{bmatrix} = \begin{bmatrix} \omega^{(i)} \\ \alpha^{(i)} \end{bmatrix} + \begin{bmatrix} \Delta\omega^{(i)} \\ \Delta\alpha^{(i)} \end{bmatrix}.$$

Jacobian for Newton's method

Jacobian

$$G(\omega^{(i)}, \alpha^{(i)}) = \begin{bmatrix} f_\omega(\omega^{(i)}, \alpha^{(i)}) & f_\alpha(\omega^{(i)}, \alpha^{(i)}) \\ f_{\omega\omega}(\omega^{(i)}, \alpha^{(i)}) & f_{\omega\alpha}(\omega^{(i)}, \alpha^{(i)}) \end{bmatrix}.$$

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and the Jacobian elements are evaluated by differentiating the system

$$\begin{bmatrix} H(\alpha)-\omega iI & Jc \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x(\omega,\alpha) \\ f(\omega,\alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

with respect to ω and α.

Implementation

- one (sparse) LU factorisation of

$$\begin{bmatrix} H(\alpha) - \omega i I & Jc \\ c^H & 0 \end{bmatrix}$$

- solve with bordered system matrix and 5 different right hand sides in order to obtain $f(\omega, \alpha)$ and entries for Jacobian

$$G(\omega, \alpha) = \begin{bmatrix} f_\omega(\omega, \alpha) & f_\alpha(\omega, \alpha) \\ f_{\omega\omega}(\omega, \alpha) & f_{\omega\alpha}(\omega, \alpha) \end{bmatrix}$$

- very fast quadratically convergent Newton method in 2 dimensions

Remarks

- full-rank Jacobian $G(\omega^*, \alpha^*) = \begin{bmatrix} 0 & f_\alpha(\omega^*, \alpha^*) \\ f_{\omega\omega}(\omega^*, \alpha^*) & f_{\omega\alpha}(\omega^*, \alpha^*) \end{bmatrix}$,

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- $f_{\omega\omega}(\omega^*,\alpha^*) < 0$ and $f_\alpha(\omega^*,\alpha^*) > 0$.

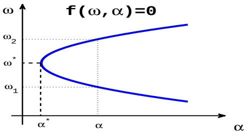

Figure: Curve $f(\omega,\alpha) = 0$ in the (ω,α)-plane for $f_{\omega\omega}(\omega^*,\alpha^*) < 0$

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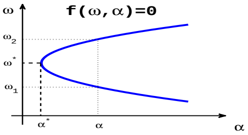

Figure: Curve $f(\omega, \alpha) = 0$ in the (ω, α)-plane for $f_{\omega\omega}(\omega^*, \alpha^*) < 0$

- Multiplication by $\begin{bmatrix} -J & 0 \\ 0^H & 1 \end{bmatrix}$ leads to the Hermitian system

$$\begin{bmatrix} -JH(\alpha) + \omega i J & c \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x(\omega, \alpha) \\ f(\omega, \alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

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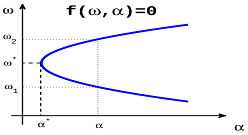

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- Test step.

Example 1

Consider

$$A = \begin{bmatrix} -0.4+6i & 1 & & \\ 1 & -0.1+i & 1 & \\ & 1 & -1-3i & 1 \\ & & 1 & -5+i \end{bmatrix}$$

which has eigenvalues (rounded to 3 significant digits)

$$\Lambda(A) = \{-0.41+5.80i, -0.04+0.95i, -0.92-2.62i, -5.13+0.87i\}$$

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Starting values:
$\alpha^{(0)} = 0$
$\omega^{(0)}$: imaginary part of the eigenvalue of A closest to the imaginary axis
$c = x^{(0)} = \begin{bmatrix} v(\omega^{(0)}, \alpha^{(0)}) \\ u(\omega^{(0)}, \alpha^{(0)}) \end{bmatrix}$, where $v(\omega^{(0)}, \alpha^{(0)})$ and $u(\omega^{(0)}, \alpha^{(0)})$ are right and left singular vectors of $A - \omega^{(0)} iI$

Example 1

Table: Results for Example 1.

	Newton method		
i	$\omega^{(i)}$	$\alpha^{(i)}$	$\|g(\omega^{(i)}, \alpha^{(i)})\|$
0	0.9530577740164838	0	-
1	0.953036248966048	0.031887014318100	1.5949900020014e-02
2	0.953014724735990	0.031887009443620	2.2577279982423e-04
3	0.953014724704841	0.031887014303200	2.4473093206567e-09
4	0.953014724704841	0.031887014303200	8.2762961087551e-16

Example 2

Brusselator matrix `bwm200.mtx`

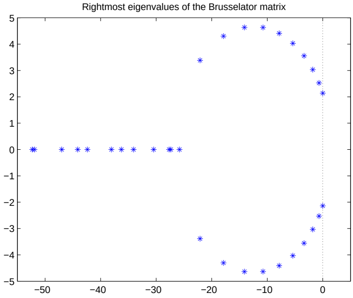

Figure: Rightmost eigenvalues of the Brusselator matrix in Example 2.

Example 2

Table: Results for Example 2.

	Newton method		
i	$\omega^{(i)}$	$\alpha^{(i)}$	$\lVert g(\omega^{(i)}, \alpha^{(i)})\rVert$
0	2.1394975220076343	0	-
1	2.1394975220045502	0.000008240971700	4.191183100020208e-06
2	2.139497522014727	0.000008240971687	3.828424651364583e-07
3	2.139497522014739	0.000008240971689	1.624908978281682e-10
4	2.139497522014746	0.000008240971691	8.163859421299612e-11

Example 2

Table: CPU times for Example 2.

Algorithm	"Inner" iterations		"Outer" iterations (Eigenvalue computation for Hamiltonian matrix)		Total CPU time
	quantity	CPU time	quantity	CPU time	
Boyd/Balakrishnan	5	1.38 s	5	2.30 s	3.68 s
He/Watson	90	1.68 s	1	0.44 s	2.12 s
Newton	4	0.84 s	1	0.45 s	1.29 s

Example 3

Orr-Sommerfeld operator

$$\frac{1}{\gamma R} L^2 v - i(UL - U'')v = \lambda L v, \quad \text{where} \quad L = \frac{d^2}{dx^2} - \gamma^2 \quad \text{and} \quad U = 1 - x^2.$$

Discretise the operator on $v \in [-1, 1]$ using finite differences with $\gamma = 1$, $R = 1000$ and $n = 1000$.

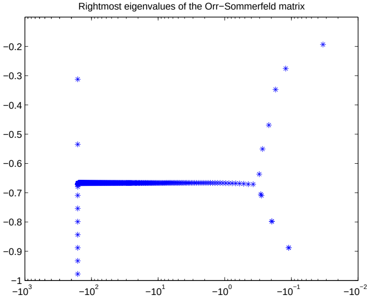

Figure: Eigenvalues of the Orr-Sommerfeld matrix in Example 3.

Example 3

Convergence to $\omega = 0.199755999447167$ and $\alpha = 0.001978172281960$ within 5 iterations.

Table: CPU times for Example 3.

Algorithm	"Inner" iterations		"Outer" iterations (Eigenvalue computation for Hamiltonian matrix)		Total CPU time
	quantity	CPU time	quantity	CPU time	
Boyd/Balakrishnan	6	3.49 s	6	63.28 s	66.77 s
He/Watson	1786	244.14 s	1	10.54 s	254.68 s
Newton	5	5.67 s	1	10.33 s	16.00 s

Example 4

Tolosa matrix `tols340.mtx`

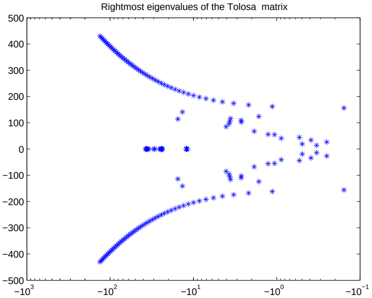

Figure: Eigenvalues of the Tolosa matrix in Example 4.

Example 4

Convergence to $\omega = 1.559998439945282$ and $\alpha = 0.000019997968879$ within 4 iterations.

Table: CPU times for Example 4.

Algorithm	"Inner" iterations		"Outer" iterations (Eigenvalue computation for Hamiltonian matrix)		Total CPU time
	quantity	CPU time	quantity	CPU time	
Boyd/Balakrishnan	3	67.52 s	3	5.27 s	72.79 s
He/Watson	> 33000	> 2230 s	> 11	> 18 s	> 2248 s
Newton	4	2.01 s	1	1.69 s	3.7 s

Conclusions

- new algorithm for computing the distance to unstable matrix
- relies on finding a 2-dimensional Jordan block in 2-parameter matrix
- only one LU decomposition per Newton step of bordered matrix M necessary
- numerical results show that new method outperforms earlier algorithms

Part II

Distance to nearby defective matrix

Distance to nearest defective matrix

For a matrix $A \in \mathbb{C}^{n\times n}$ with n distinct eigenvalues

$$d(A) = \inf\{\|A - B\|, \quad B \text{ is a defective matrix}\}$$

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- condition number of a simple eigenvalue: $1/|y^H x|$, where x and y are normalised left and right eigenvectors

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Relation to pseudospectrum $\Lambda_\varepsilon(A) = \{\sigma_{\min}(A - zI) < \varepsilon\}$

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} \,|\, \det(A + E - zI) = 0, \text{ for some } E \in \mathbb{C}^{n\times n} \text{ with } \|E\| < \varepsilon\}.$$

- if $\Lambda_\varepsilon(A)$ has n components, then $A + E$ has n distinct eigenvalues for all perturbation matrices $E \in \mathbb{C}^{n\times n}$ with $\|E\| < \varepsilon$ and hence $A + E$ is not defective
- seek the smallest perturbation matrix E such that the pseudospectra of $A + E$ coalesce.

Distance to nearby defective - known results

Theorem (Alam,Bora 2005)

Let $A \in \mathbb{C}^{n\times n}$ and $z \in \mathbb{C} \setminus \Lambda(A)$, so that $A - zI$ has a simple smallest singular value $\varepsilon > 0$ with corresponding left and right singular vectors u and v such that

$$(A - zI)v = \varepsilon u.$$

Then z is an eigenvalue of $B = A - \varepsilon uv^H$ with geometric multiplicity 1 and corresponding left and right eigenvectors u and v respectively.

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Then z is an eigenvalue of $B = A - \varepsilon uv^H$ with geometric multiplicity 1 and corresponding left and right eigenvectors u and v respectively.

If $u^H v = 0$, then z (as an eigenvalue of B) has algebraic multiplicity greater than one, hence it is a nonderogatory defective eigenvalue of B and $\|A - B\| = \varepsilon$.

Problem

Problem

Find $z \in \mathbb{C}$, $u, v \in \mathbb{C}^n$ and $\varepsilon \in \mathbb{R}$ such that

$$(A - zI)v - \varepsilon u = 0$$
$$\varepsilon v - (A - zI)^H u = 0$$

and

$$u^H v = 0.$$

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Find $z \in \mathbb{C}$, $u, v \in \mathbb{C}^n$ and $\varepsilon \in \mathbb{R}$ such that

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$$u^H v = 0.$$

With $z = \alpha + i\beta$

$$\underbrace{\begin{bmatrix} -\varepsilon I & A - (\alpha + i\beta)I \\ (A - (\alpha + i\beta)I)^H & -\varepsilon I \end{bmatrix}}_{K(\alpha,\beta,\varepsilon)} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Problem

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With $z = \alpha + i\beta$

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- $x = \begin{bmatrix} u \\ v \end{bmatrix}$ is both a right and left null vector of $K(\alpha, \beta, \varepsilon)$,
- if $\varepsilon > 0$ is simple then $\dim \ker K(\alpha, \beta, \varepsilon) = 1$.

The bordered matrix

Theorem

- *Let* $(\alpha, \beta, \varepsilon, x)$ *solve*

$$K(\alpha, \beta, \varepsilon)x = 0, \quad x \neq 0,$$

so that $\dim \ker K(\alpha, \beta, \varepsilon) = 1$ *and* $x \in \ker(K(\alpha, \beta, \varepsilon)) \setminus \{0\}$.

- *For some* $c \in \mathbb{C}^{2n}$ *assume*

$$c^H x \neq 0.$$

- *Then the bordered Hermitian matrix*

$$M(\alpha, \beta, \varepsilon) = \begin{bmatrix} K(\alpha, \beta, \varepsilon) & c \\ c^H & 0 \end{bmatrix}$$

is nonsingular.

The implicit determinant method

Parameter dependent problem (three parameters!)

$$K(\alpha, \beta, \varepsilon)x = 0 \quad \text{or} \quad \det(K(\alpha, \beta, \varepsilon)) = 0$$

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Bordered system

$$\begin{bmatrix} K(\alpha,\beta,\varepsilon) & c \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x(\alpha,\beta,\varepsilon) \\ f(\alpha,\beta,\varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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Cramer's rule

$$f(\alpha,\beta,\varepsilon) = \frac{\det\left(K(\alpha,\beta,\varepsilon)\right)}{\det\left(M(\alpha,\beta,\varepsilon)\right)},$$

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Cramer's rule

$$f(\alpha,\beta,\varepsilon) = \frac{\det\left(K(\alpha,\beta,\varepsilon)\right)}{\det\left(M(\alpha,\beta,\varepsilon)\right)},$$

Solve

$$f(\alpha,\beta,\varepsilon) = 0.$$

instead of $\det(K(\alpha,\beta,\varepsilon)) = 0$, where $f(\alpha,\beta,\varepsilon)$ is real.

The implicit determinant method

Differentiating the bordered system with respect to α and β

Differentiate the bordered system

$$\begin{bmatrix} K(\alpha,\beta,\varepsilon) & c \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x(\alpha,\beta,\varepsilon) \\ f(\alpha,\beta,\varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with respect to α.

The implicit determinant method

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with respect to α.

$$\begin{bmatrix} K(\alpha,\beta,\varepsilon) & c \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x_\alpha(\alpha,\beta,\varepsilon) \\ f_\alpha(\alpha,\beta,\varepsilon) \end{bmatrix} = \begin{bmatrix} v(\alpha,\beta,\varepsilon) \\ u(\alpha,\beta,\varepsilon) \\ 0 \end{bmatrix},$$

The implicit determinant method

Differentiating the bordered system with respect to α and β

Differentiate the bordered system

$$\begin{bmatrix} K(\alpha,\beta,\varepsilon) & c \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x(\alpha,\beta,\varepsilon) \\ f(\alpha,\beta,\varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with respect to α.

$$\begin{bmatrix} K(\alpha,\beta,\varepsilon) & c \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x_\alpha(\alpha,\beta,\varepsilon) \\ f_\alpha(\alpha,\beta,\varepsilon) \end{bmatrix} = \begin{bmatrix} v(\alpha,\beta,\varepsilon) \\ u(\alpha,\beta,\varepsilon) \\ 0 \end{bmatrix},$$

multiply first row by $x^H = \begin{bmatrix} u^H & v^H \end{bmatrix}$ from the left

$$f_\alpha(\alpha,\beta,\varepsilon) = \begin{bmatrix} u^H & v^H \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = u^H v + v^H u = 2\mathrm{Re}(u^H v).$$

The implicit determinant method

Differentiating the bordered system with respect to α and β

Differentiate the bordered system

$$\begin{bmatrix} K(\alpha,\beta,\varepsilon) & c \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x(\alpha,\beta,\varepsilon) \\ f(\alpha,\beta,\varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with respect to β.

$$\begin{bmatrix} K(\alpha,\beta,\varepsilon) & c \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x_\beta(\alpha,\beta,\varepsilon) \\ f_\beta(\alpha,\beta,\varepsilon) \end{bmatrix} = i \begin{bmatrix} v(\alpha,\beta,\varepsilon) \\ -u(\alpha,\beta,\varepsilon) \\ 0 \end{bmatrix}.$$

multiply first row by $x^H = \begin{bmatrix} u^H & v^H \end{bmatrix}$ from the left

$$f_\beta(\alpha,\beta,\varepsilon) = i(u^H v - v^H u) = -2\mathrm{Im}(u^H v).$$

The implicit determinant method

Summary

Solution to $\det K(\alpha, \beta, \varepsilon) = 0$ with $u^H v = 0$

$$\Updownarrow$$

The implicit determinant method

Summary

Solution to $\det K(\alpha, \beta, \varepsilon) = 0$ with $u^H v = 0$

$$\Updownarrow$$

Solution to $g(\alpha, \beta, \varepsilon) = 0$, where

$$g(\alpha, \beta, \varepsilon) = \begin{bmatrix} f(\alpha, \beta, \varepsilon) \\ f_\alpha(\alpha, \beta, \varepsilon) \\ f_\beta(\alpha, \beta, \varepsilon) \end{bmatrix}.$$

Newton's method for *real* function g in three real variables

Newton's method:

$$G(\alpha^{(i)}, \beta^{(i)}, \varepsilon^{(i)}) \begin{bmatrix} \Delta\alpha^{(i)} \\ \Delta\beta^{(i)} \\ \Delta\varepsilon^{(i)} \end{bmatrix} = -g(\alpha^{(i)}, \beta^{(i)}, \varepsilon^{(i)}),$$

with Jacobian

$$G(\alpha^{(i)}, \beta^{(i)}, \varepsilon^{(i)}) = \begin{bmatrix} f_\alpha^{(i)} & f_\beta^{(i)} & f_\varepsilon^{(i)} \\ f_{\alpha\alpha}^{(i)} & f_{\alpha\beta}^{(i)} & f_{\alpha\varepsilon}^{(i)} \\ f_{\beta\alpha}^{(i)} & f_{\beta\beta}^{(i)} & f_{\beta\varepsilon}^{(i)} \end{bmatrix},$$

where the Jacobian elements are evaluated by differentiating the system

$$\begin{bmatrix} K(\alpha, \beta, \varepsilon) & c \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x(\alpha, \beta, \varepsilon) \\ f(\alpha, \beta, \varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with respect to α, β and ε.

Implementation

- one LU factorisation of

$$\begin{bmatrix} K(\alpha,\beta,\varepsilon) & c \\ c^H & 0 \end{bmatrix}$$

  per Newton step
- very fast quadratically convergent Newton method in 3 dimensions
- Jacobian

$$G(\alpha^*,\beta^*,\varepsilon^*) = \begin{bmatrix} 0 & 0 & f_\varepsilon^* \\ f_{\alpha\alpha}^* & f_{\alpha\beta}^* & f_{\alpha\varepsilon}^* \\ f_{\beta\alpha}^* & f_{\beta\beta}^* & f_{\beta\varepsilon}^* \end{bmatrix},$$

  is nonsingular.

Example 1

Kahan matrix

$$A = \begin{bmatrix} 1 & -c & -c & -c & -c \\ & s & -sc & -sc & -sc \\ & & s^2 & -s^2c & -s^2c \\ & & & \ddots & \vdots \\ & & & & s^{n-1} \end{bmatrix},$$

where $s^{n-1} = 0.1$ and $s^2 + c^2 = 1$. We consider this matrix for $n = 6$.

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$$A = \begin{bmatrix} 1 & -c & -c & -c & -c \\ & s & -sc & -sc & -sc \\ & & s^2 & -s^2c & -s^2c \\ & & & \ddots & \vdots \\ & & & & s^{n-1} \end{bmatrix},$$

where $s^{n-1} = 0.1$ and $s^2 + c^2 = 1$. We consider this matrix for $n = 6$.

Starting values:

$\alpha^{(0)} = \beta^{(0)} = 0$

$\varepsilon^{(0)} = \sigma_{\min}$, $u^{(0)} = u_{\min}$ and $v^{(0)} = v_{\min}$, where $\sigma_{\min}$ is the minimum singular value of A with corresponding left and right singular vectors $u_{\min}$ and $v_{\min}$

$c = x^{(0)}$

$$\|g(\alpha^{(i)}, \beta^{(i)}, \varepsilon^{(i)})\| < \tau, \quad \text{where} \quad \tau = 10^{-14}.$$

Example 1

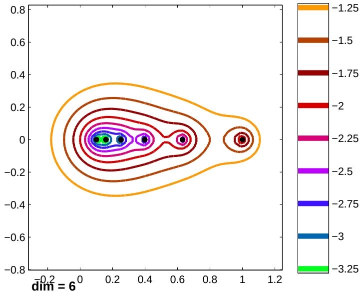

Figure: Pseudospectra plot for Kahan matrix for $n = 6$.

Example 1

Table: Results for Example 1

i	$\alpha^{(i)}$	$\beta^{(i)}$	$\varepsilon^{(i)}$	$\|g(\alpha^{(i)}, \beta^{(i)}, \varepsilon^{(i)})\|$	$F_{\alpha\beta}^{(i)}$
0	0	0	9.9694e-03	0	0
1	1.3643e-01	0	1.2145e-02	8.1049e-02	3.9318e-01
2	1.3319e-01	0	7.1339e-04	3.9165e-02	-1.0032e+00
3	1.2767e-01	0	4.9351e-04	4.3976e-03	-4.5529e-01
4	1.2763e-01	0	4.7049e-04	8.2870e-05	-4.3191e-01
5	1.2763e-01	0	4.7049e-04	4.7344e-08	-4.3136e-01
6	1.2763e-01	0	4.7049e-04	5.3655e-15	-4.3136e-01

Eigenvalues 1.5849×10^{-1} and 10^{-1} coalesce at 1.2763×10^{-1} for a value of $\varepsilon = 4.7049 \times 10^{-4}$.

Example 2

$A \in \mathbb{C}^{n\times n}$ taken from Matlab `A = gallery('grcar',n)`

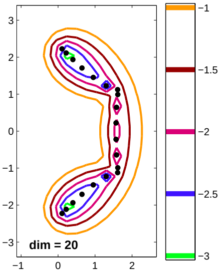

Figure: Pseudospectra plot for grcar matrix for $n = 20$.

Example 2

Table: Results for Example 2.

i	$\alpha^{(i)}$	$\beta^{(i)}$	$\varepsilon^{(i)}$	$\|g(\alpha^{(i)}, \beta^{(i)}, \varepsilon^{(i)})\|$	$F_{\alpha\beta}^{(i)}$
0	0	-2.5000e+00	0	0	0
1	9.5854e-02	-2.3299e+00	1.7989e-02	1.3806e-01	9.9103e-01
2	1.3904e-01	-2.2465e+00	1.3564e-03	3.2308e-02	-2.3623e-01
3	1.6141e-01	-2.2042e+00	7.2914e-04	1.1930e-02	-1.5963e-01
4	1.5554e-01	-2.1818e+00	4.5435e-04	3.4851e-03	-2.7982e-02
5	1.5338e-01	-2.1815e+00	4.9060e-04	3.4265e-04	-2.4693e-02
6	1.5331e-01	-2.1817e+00	4.9141e-04	2.3240e-05	-2.3956e-02
7	1.5331e-01	-2.1817e+00	4.9141e-04	1.6942e-08	-2.4012e-02
8	1.5331e-01	-2.1817e+00	4.9141e-04	4.6672e-14	-2.4012e-02
9	1.5331e-01	-2.1817e+00	4.9141e-04	4.5263e-17	-2.4012e-02

Eigenvalue pairs $1.0802 \times 10^{-1} \pm 2.2253i$ and $2.1882 \times 10^{-1} \pm 2.1132i$ coalesce at $1.5331 \times 10^{-1} \pm 2.1817i$ for a value of $\varepsilon = 4.9141 \times 10^{-4}$

Conclusions

- new algorithm for computing a nearby defective matrix (in our examples: nearest defective matrix)
- only one LU decomposition per Newton step of bordered matrix M necessary

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Thank you.

Thank you.

More on this:
Talks by M. Overton, M. Gurbuzbalaban and A. Spence tomorrow.