Solving Matrix Nearness Problems using the Implicit Determinant Method

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Distance to instability - definition

- Stability of matrix $A \in \mathbb{C}^{n\times n}$: $\Lambda(A)$ in open left half plane
- Better measure of stability: distance of A to instability/stability radius

Define spectral abscissa

$$\eta(A) := \max\{\mathrm{Re}(\lambda) \,|\, \lambda \in \Lambda(A)\}$$

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Distance of a stable matrix A to instability

$$\beta(A) = \min\{\|E\| \mid \eta(A+E) = 0,\ E \in \mathbb{C}^{n\times n}\}$$

If $A + E$ has an eigenvalue on the imaginary axis, E is *destabilising perturbation.*

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If $A + E$ has an eigenvalue on the imaginary axis, E is *destabilising perturbation.*

$$(A + E - \omega i I)z = 0,$$

for some $\omega \in \mathbb{R}$ and $z \in \mathbb{C}^n$.

Distance to instability - known results

Distance to instability of a matrix (Van Loan 1984),

$$\beta(A) = \min_{\omega \in \mathbb{R}} \sigma_{\min}(A - \omega i I),$$

where $\sigma_{\min}(A - \omega i I)$ is the smallest singular value of $A - \omega i I$.

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Theorem (Byers 1988)

The $2n \times 2n$ Hamiltonian matrix

$$H(\alpha) = \begin{bmatrix} A & -\alpha I \\ \alpha I & -A^H \end{bmatrix}.$$

has an eigenvalue ωi on the imaginary axis if and only if $\alpha \geq \beta(A)$.

Results on $H(\alpha)$

$H(\alpha)$ has a pure imaginary eigenvalue ωi:

$$\underbrace{\begin{bmatrix} A & -\alpha I \\ \alpha I & -A^H \end{bmatrix}}_{H(\alpha)} \begin{bmatrix} v \\ u \end{bmatrix} = \omega i \begin{bmatrix} v \\ u \end{bmatrix}$$

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$\Rightarrow$

$$(A - \omega i I)v = \alpha u \quad \text{and} \quad (A - \omega i I)^H u = \alpha v.$$

If α^* is the minimum value of α at which $H(\alpha)$ has a pure imaginary eigenvalue $\omega^* i$ with corresponding $x^* = \begin{bmatrix} v^* \\ u^* \end{bmatrix}$ then $\alpha^* = \beta(A)$.
Assume $\alpha^* = \beta(A)$ is unique.

Existing numerical methods

- Bisection approach by Byers
  - choose lower and upper bound on α (0 and $\sigma_{\min}(A)$)
  - take mean value s and calculate all the eigenvalues of $H(s)$, update lower and upper bound according to pure imaginary eigenvalues of $H(s)$

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- Boyd/Balakrishnan method
  - given an upper bound $\alpha \geq \beta(A)$, compute all pure imaginary eigenvalues $iw_1, iw_2, \ldots, iw_l$ of $H(\alpha)$ ordered so that $w_1 \leq w_2 \leq \ldots \leq w_l$
  - set $s_k = \frac{w_k + w_{k+1}}{2}$, $k = 1, \ldots l-1$ and update $\alpha = \min_k \sigma_{\min}(A - s_k iI)$

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- He/Watson algorithm
  - find the minimum of $f(\omega) = \sigma_{\min}(A - \omega iI)$
  - uses inverse iteration algorithm to find a stationary ω
  - check on all the corresponding eigenvalues of $H(\alpha)$

Results on $H(\alpha)$

Assumption

$(\omega i, x)$ is a defective eigenpair of $H(\alpha) = \begin{bmatrix} A & -\alpha I \\ \alpha I & -A^H \end{bmatrix}$ of algebraic multiplicity 2.

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$$(H(\alpha) - \omega i I)x = 0, \quad x \neq 0, \quad \text{and} \quad \dim\ker(H(\alpha) - \omega i I) = 1,$$

$$y^H (H(\alpha) - \omega i I) = 0, \quad y \neq 0, \quad \text{and} \quad y^H x = 0,$$

$$y = Jx, \qquad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

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$$(H(\alpha) - \omega i I)\hat{x} = x, \quad \text{and} \quad y^H \hat{x} \neq 0,$$

Jordan block of dimension 2 at the critical value of α

Parameter dependent matrix eigenvalue problem $H(\omega, \alpha)$

Problem

How do we find a 2-dimensional Jordan block in $H(\alpha)$?

$$\underbrace{(H(\alpha) - \omega i I)}_{H(\omega,\alpha)} x = 0, \quad x \neq 0,$$

Bordered systems - a “new” method for finding eigenvalues

One-parameter problem $B(\lambda)x = 0$ or $y^H B(\lambda) = 0^H$ ($\det(B(\lambda)) = 0$)

Bordered system

$$\underbrace{\begin{bmatrix} B(\lambda) & b \\ c^H & 0 \end{bmatrix}}_{M(\lambda)} \begin{bmatrix} x(\lambda) \\ f(\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is nonsingular if $c^H x \neq 0$ and $y^H b \neq 0$ and $\operatorname{rank}(B(\lambda)) = n - 1$.

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$$B(\lambda)x(\lambda) = 0.$$

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Solve $f(\lambda) = 0$ using Newton's method $\lambda^+ = \lambda - \dfrac{f(\lambda)}{f_\lambda(\lambda)}$.

Parameter dependent matrix eigenvalue problem $H(\omega, \alpha)$

Problem

How do we find a 2-dimensional Jordan block in $H(\alpha)$?

$$\underbrace{(H(\alpha) - \omega i I)}_{H(\omega,\alpha)} x = 0, \quad x \neq 0,$$

The bordered matrix

Theorem (Two-parameter problem)

- *Let* (ω, α, x) *solve*
$$(H(\alpha) - \omega i I)x = 0, \quad x \neq 0,$$
- *Zero is a double eigenvalue of* $H(\alpha) - \omega i I$ *belonging to a 2-dimensional Jordan block with*
$$y^H x = 0, \quad \text{for} \quad y \in ker(H(\alpha) - \omega i I)^H \setminus \{0\}.$$

Then the bordered matrix

$$M(\omega, \alpha) = \begin{bmatrix} H(\alpha) - \omega i I & Jc \\ c^H & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

is nonsingular if $c^H x \neq 0$.

The implicit determinant method

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$$f(\omega,\alpha) = \frac{\det(H(\omega,\alpha))}{\det(M(\omega,\alpha))},$$

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Solve

$$f(\omega,\alpha) = 0 \quad \text{instead of} \quad \det(H(\omega,\alpha)) = 0,$$

where

$$f(\omega,\alpha) = x(\omega,\alpha)^H J (H(\alpha) - \omega i I) x(\omega,\alpha)$$

is real.

The implicit determinant method

Differentiate the linear system

Differentiate $\begin{bmatrix} H(\alpha) - \omega i I & Jc \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x(\omega,\alpha) \\ f(\omega,\alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with respect to ω:

$$\begin{bmatrix} H(\alpha) - \omega i I & Jc \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x_\omega(\omega,\alpha) \\ f_\omega(\omega,\alpha) \end{bmatrix} = \begin{bmatrix} ix(\omega,\alpha) \\ 0 \end{bmatrix}.$$

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First row

$$f_\omega(\omega,\alpha) = iy^H x$$

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$$f_\omega(\omega,\alpha) = i y^H x = 0,$$

because of Jordan block of dimension 2.

The implicit determinant method

Differentiate the linear system

Differentiate $\begin{bmatrix} H(\alpha) - \omega i I & Jc \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x(\omega, \alpha) \\ f(\omega, \alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with respect to ω:

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because of Jordan block of dimension 2. Solve

$$g(\omega, \alpha) = \begin{bmatrix} f(\omega, \alpha) \\ f_\omega(\omega, \alpha) \end{bmatrix} = 0.$$

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because of Jordan block of dimension 2. Solve

$$g(\omega,\alpha) = \begin{bmatrix} f(\omega,\alpha) \\ f_\omega(\omega,\alpha) \end{bmatrix} = 0.$$

Also,

$$(H(\alpha) - \omega i I)x_\omega(\omega,\alpha) = ix,$$

and $y^H x_\omega(\omega,\alpha) \neq 0$, hence $f_{\omega\omega}(\omega,\alpha) \neq 0$.

Newton's method for *real* function g in two real variables

Solve

$$g(\omega,\alpha) = \left[\begin{array}{c} f(\omega,\alpha) \\ f_\omega(\omega,\alpha) \end{array}\right] = 0,$$

using Newton's method:

$$G(\omega^{(i)},\alpha^{(i)}) \left[\begin{array}{c} \Delta\omega^{(i)} \\ \Delta\alpha^{(i)} \end{array}\right] = -g(\omega^{(i)},\alpha^{(i)}),$$

$$\left[\begin{array}{c} \omega^{(i+1)} \\ \alpha^{(i+1)} \end{array}\right] = \left[\begin{array}{c} \omega^{(i)} \\ \alpha^{(i)} \end{array}\right] + \left[\begin{array}{c} \Delta\omega^{(i)} \\ \Delta\alpha^{(i)} \end{array}\right].$$

Jacobian for Newton's method

Jacobian

$$G(\omega^{(i)}, \alpha^{(i)}) = \left[\begin{array}{cc} f_{\omega}(\omega^{(i)}, \alpha^{(i)}) & f_{\alpha}(\omega^{(i)}, \alpha^{(i)}) \\ f_{\omega\omega}(\omega^{(i)}, \alpha^{(i)}) & f_{\omega\alpha}(\omega^{(i)}, \alpha^{(i)}) \end{array} \right].$$

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and the Jacobian elements are evaluated by differentiating the system

$$\begin{bmatrix} H(\alpha) - \omega i I & Jc \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x(\omega, \alpha) \\ f(\omega, \alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

with respect to ω and α.

Implementation

- one (sparse) LU factorisation of

$$\begin{bmatrix} H(\alpha) - \omega i I & Jc \\ c^H & 0 \end{bmatrix}$$

- solve with bordered system matrix and 5 different right hand sides in order to obtain $f(\omega, \alpha)$ and entries for Jacobian

$$G(\omega, \alpha) = \begin{bmatrix} f_\omega(\omega, \alpha) & f_\alpha(\omega, \alpha) \\ f_{\omega\omega}(\omega, \alpha) & f_{\omega\alpha}(\omega, \alpha) \end{bmatrix}$$

- very fast quadratically convergent Newton method in 2 dimensions

Remarks

- full-rank Jacobian $G(\omega^*, \alpha^*) = \begin{bmatrix} 0 & f_\alpha(\omega^*, \alpha^*) \\ f_{\omega\omega}(\omega^*, \alpha^*) & f_{\omega\alpha}(\omega^*, \alpha^*) \end{bmatrix}$,

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- $f_{\omega\omega}(\omega^*, \alpha^*) < 0$ and $f_\alpha(\omega^*, \alpha^*) > 0$.

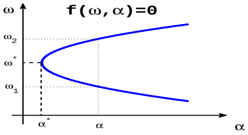

Figure: Curve $f(\omega, \alpha) = 0$ in the (ω, α)-plane for $f_{\omega\omega}(\omega^*, \alpha^*) < 0$

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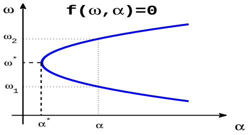

Figure: Curve $f(\omega, \alpha) = 0$ in the (ω, α)-plane for $f_{\omega\omega}(\omega^*, \alpha^*) < 0$

- Multiplication by $\begin{bmatrix} -J & 0 \\ 0^H & 1 \end{bmatrix}$ leads to the Hermitian system

$$\begin{bmatrix} -JH(\alpha) + \omega iJ & c \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x(\omega, \alpha) \\ f(\omega, \alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

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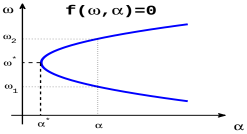

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- Test step.

Example 1

Consider

$$A = \begin{bmatrix} -0.4+6i & 1 & & \\ 1 & -0.1+i & 1 & \\ & 1 & -1-3i & 1 \\ & & 1 & -5+i \end{bmatrix}$$

which has eigenvalues (rounded to 3 significant digits)

$$\Lambda(A) = \{-0.41+5.80i, -0.04+0.95i, -0.92-2.62i, -5.13+0.87i\}$$

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Starting values:
$\alpha^{(0)} = 0$
$\omega^{(0)}$: imaginary part of the eigenvalue of A closest to the imaginary axis
$c = x^{(0)} = \begin{bmatrix} v(\omega^{(0)}, \alpha^{(0)}) \\ u(\omega^{(0)}, \alpha^{(0)}) \end{bmatrix}$, where $v(\omega^{(0)}, \alpha^{(0)})$ and $u(\omega^{(0)}, \alpha^{(0)})$ are right and left singular vectors of $A - \omega^{(0)} iI$

Example 1

Table: Results for Example 1.

	Newton method		
i	$\omega^{(i)}$	$\alpha^{(i)}$	$\|g(\omega^{(i)}, \alpha^{(i)})\|$
0	0.953057740164838	0	-
1	0.953036248966048	0.031887014318100	1.5949900020014e-02
2	0.953014724735990	0.031887009443620	2.2577279982423e-04
3	0.953014724704841	0.031887014303200	2.4473093206567e-09
4	0.953014724704841	0.031887014303200	8.2762961087551e-16

Example 2

Orr-Sommerfeld operator

$$\frac{1}{\gamma R}L^2 v - i(UL - U'')v = \lambda L v, \quad \text{where} \quad L = \frac{d^2}{dx^2} - \gamma^2 \quad \text{and} \quad U = 1 - x^2.$$

Discretise the operator on $v \in [-1, 1]$ using finite differences with $\gamma = 1$, $R = 1000$ and $n = 1000$.

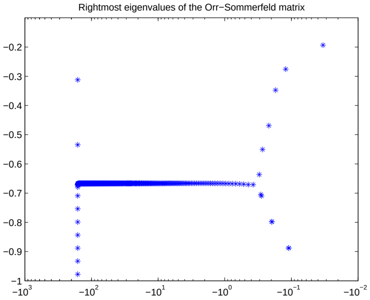

Figure: Eigenvalues of the Orr-Sommerfeld matrix in Example 3.

Example 2

Convergence to $\omega = 0.199755999447167$ and $\alpha = 0.001978172281960$ within 5 iterations.

Table: CPU times for Example 3.

Algorithm	"Inner" iterations		"Outer" iterations (Eigenvalue computation for Hamiltonian matrix)		Total CPU time
	quantity	CPU time	quantity	CPU time	
Boyd/Balakrishnan	6	3.49 s	6	63.28 s	66.77 s
He/Watson	1786	244.14 s	1	10.54 s	254.68 s
Newton	5	5.67 s	1	10.33 s	16.00 s

Example 3

Tolosa matrix `tols340.mtx`

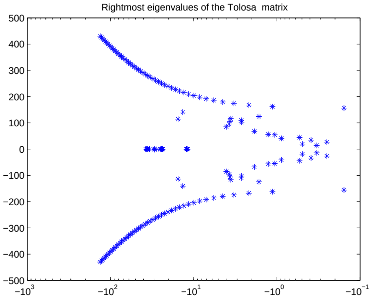

Figure: Eigenvalues of the Tolosa matrix in Example 4.

Example 3

Convergence to $\omega = 1.559998439945282$ and $\alpha = 0.000019997968879$ within 4 iterations.

Table: CPU times for Example 4.

Algorithm	"Inner" iterations		"Outer" iterations (Eigenvalue computation for Hamiltonian matrix)		Total CPU time
	quantity	CPU time	quantity	CPU time	
Boyd/Balakrishnan	3	67.52 s	3	5.27 s	72.79 s
He/Watson	> 33000	> 2230 s	> 11	> 18 s	> 2248 s
Newton	4	2.01 s	1	1.69 s	3.7 s

Real versus complex stability radius

- Complex stability radius

$$\beta_{\mathbb{C}}(A) = \min_{\omega \in \mathbb{R}} \sigma_{\min}(A - \omega i I),$$

where $\sigma_{\min}(A - \omega i I)$ is the smallest singular value of $A - \omega i I$.

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- Real stability radius (Qiu et al 1995)

$$\beta_{\mathbb{R}}(A) = \min_{\omega \in \mathbb{R}} \max_{\gamma \in (0,1]} \sigma_{2n-1}\left(\begin{bmatrix} A & -\omega\gamma I \\ \frac{\omega}{\gamma} I & A \end{bmatrix}\right),$$

where $\sigma_{2n-1}(\cdot)$ is the second smallest singular value of the matrix $(\cdot)$. The function to be maximised with respect to γ is unimodal in $(0, 1]$ and hence the local maximiser will be a global maximiser.

Relation to Hamiltonian matrix problem

Real stability radius

$$\beta_{\mathbb{R}}(A) = \min_{\omega\in\mathbb{R}} \max_{\gamma\in(0,1]} \sigma_{2n-1} \underbrace{\left(\begin{bmatrix} A & -\omega\gamma I \\ \frac{\omega}{\gamma} I & A \end{bmatrix} \right)}_{B(\omega,\gamma)} = \min_{\omega\in\mathbb{R}} \max_{\gamma\in(0,1]} \sigma_{2n-1} B(\omega,\gamma),$$

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Theorem (Boyd, Balakrishnan, Kabamba 1989)

Let A be a stable matrix and $\gamma \in (0,1]$. Then, for all $\omega \in \mathbb{R}$, $\alpha > 0$ is a singular value of $B(\omega,\gamma)$ if and only if $i\omega$ is an eigenvalue of $H(\alpha,\gamma)$ given by

$$H(\alpha,\gamma) := \begin{bmatrix} \hat{A} & -\alpha T_\gamma T_\gamma^T \\ \alpha F T_\gamma^T T_\gamma F & -\hat{A}^T \end{bmatrix} \in \mathbb{R}^{4n\times 4n}$$

where

$$\hat{A} := \begin{bmatrix} -A & 0 \\ 0 & A \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \quad F = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}, \quad T_\gamma = \frac{1}{\sqrt{2}} \begin{bmatrix} I & \gamma I \\ \frac{1}{\gamma} I & -I \end{bmatrix} \in \mathbb{R}^{2n\times 2n}.$$

Results on $H(\alpha, \gamma)$

Generic case

For fixed γ, if α is a simple singular value, then $(\omega i, x)$ is a defective eigenpair of $H(\alpha, \gamma)$ of algebraic multiplicity 2 if α is a simple .

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The bordered matrix

$$M(\omega, \alpha, \gamma) = \begin{bmatrix} H(\alpha, \gamma) - i\omega I & Jc \\ c^H & 0 \end{bmatrix}$$

is nonsingular if $c^H x \neq 0$.

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$$\begin{bmatrix} H(\alpha, \gamma) - \omega i I & Jc \\ c^H & 0 \end{bmatrix} \begin{bmatrix} x(\omega, \alpha, \gamma) \\ f(\omega, \alpha, \gamma) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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which now has an extra parameter γ. Using similar theory as for the complex stability radius case we obtain

$$g(\omega, \alpha, \gamma) = \begin{bmatrix} f(\omega, \alpha, \gamma) \\ f_\omega(\omega, \alpha, \gamma) \\ f_\gamma(\omega, \alpha, \gamma) \end{bmatrix} = 0.$$

Characteristics of the $(1, 1)$ block of the system are inherited to the scalar function f

Newton's method

Solve the sequence of linear systems

$$G(\omega^{(i)}, \alpha^{(i)}, \gamma^{(i)}) \begin{bmatrix} \omega^{(i+1)} - \omega^{(i)} \\ \gamma^{(i+1)} - \gamma^{(i)} \\ \alpha^{(i+1)} - \alpha^{(i)} \end{bmatrix} = -g(\omega^{(i)}, \alpha^{(i)}, \gamma^{(i)}), \quad , i = 1, 2, \ldots$$

with starting value $(\omega^{(0)}, \alpha^{(0)}, \gamma^{(0)})$.
Jacobian

$$G(\omega^{(i)}, \alpha^{(i)}, \gamma^{(i)}) = \begin{bmatrix} f_\omega(\omega^{(i)}, \alpha^{(i)}, \gamma^{(i)}) & f_\gamma(\omega^{(i)}, \alpha^{(i)}, \gamma^{(i)}) & f_\alpha(\omega^{(i)}, \alpha^{(i)}, \gamma^{(i)}) \\ f_{\omega\omega}(\omega^{(i)}, \alpha^{(i)}, \gamma^{(i)}) & f_{\omega\gamma}(\omega^{(i)}, \alpha^{(i)}, \gamma^{(i)}) & f_{\omega\alpha}(\omega^{(i)}, \alpha^{(i)}, \gamma^{(i)}) \\ f_{\omega\gamma}(\omega^{(i)}, \alpha^{(i)}, \gamma^{(i)}) & f_{\gamma\gamma}(\omega^{(i)}, \alpha^{(i)}, \gamma^{(i)}) & f_{\gamma\alpha}(\omega^{(i)}, \alpha^{(i)}, \gamma^{(i)}) \end{bmatrix}$$

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with starting value $(\omega^{(0)}, \alpha^{(0)}, \gamma^{(0)})$
In the limit the Jacobian is

$$G(\omega^*, \alpha^*, \gamma^*) = \begin{bmatrix} 0 & 0 & f^*_{\alpha} \\ f^*_{\omega\omega} & f^*_{\omega\gamma} & f^*_{\omega\alpha} \\ f^*_{\omega\gamma} & f^*_{\gamma\gamma} & f^*_{\gamma\alpha} \end{bmatrix}.$$

Newton's method

Solve the sequence of linear systems

$$G(\omega^{(i)}, \alpha^{(i)}, \gamma^{(i)}) \begin{bmatrix} \omega^{(i+1)} - \omega^{(i)} \\ \gamma^{(i+1)} - \gamma^{(i)} \\ \alpha^{(i+1)} - \alpha^{(i)} \end{bmatrix} = -g(\omega^{(i)}, \alpha^{(i)}, \gamma^{(i)}), \quad , i = 1, 2, \ldots$$

with starting value $(\omega^{(0)}, \alpha^{(0)}, \gamma^{(0)})$
In the limit the Jacobian is

$$G(\omega^*, \alpha^*, \gamma^*) = \begin{bmatrix} 0 & 0 & f^*_\alpha \\ f^*_{\omega\omega} & f^*_{\omega\gamma} & f^*_{\omega\alpha} \\ f^*_{\omega\gamma} & f^*_{\gamma\gamma} & f^*_{\gamma\alpha} \end{bmatrix}.$$

- Can show $f^*_\alpha > 0$.
- $f^*_{\omega\omega} f^*_{\gamma\gamma} - {f^*_{\omega\gamma}}^2 < 0$ as $\alpha(\omega^*, \gamma^*)$ is a saddle point of $\alpha(\omega, \gamma)$.

Example 4

Consider the matrix

$$A = \begin{bmatrix} -0.4 & 7 & 0 & 0 & 0 & 0 \\ -5 & -0.4 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 0 \\ 0 & 0 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix},$$

which has the eigenvalues

$$\Lambda(A) = \{-0.3823 \pm 5.8081i, -0.9360 \pm 2.8210i, -5.1633, -5.0000\},$$

all in the left half plane with the leftmost eigenvalue $-0.3823 \pm 5.8081i$,

Example 4 - Surface of $\sigma_{2n-1}(\omega, \gamma)$

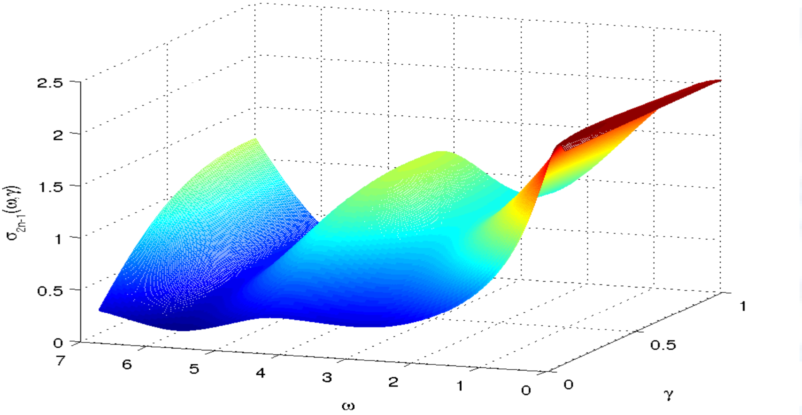

Figure: Surface $\sigma_{2n-1}(\omega, \gamma)$ of matrix A in Example 1.

Example 4 - Results

Table: Results for Example 4.

i	$\alpha^{(i)}$	$\omega^{(i)}$	$\gamma^{(i)}$	$\|g(\alpha^{(i)},\omega^{(i)},\gamma^{(i)})\|$
0	1.0000e-01	5.8081e+00	5.0000e-01	-
1	1.2271e-01	5.7679e+00	4.0236e-01	8.6013e+00
2	1.9308e-01	5.7749e+00	4.3597e-01	3.9972e+00
3	3.7590e-01	5.8399e+00	7.8038e-01	1.9974e+00
4	3.5831e-01	5.7968e+00	8.7316e-01	3.0862e-01
5	3.6035e-01	5.8002e+00	8.4257e-01	1.0443e-01
6	3.6118e-01	5.8034e+00	8.5126e-01	6.5223e-02
7	3.6120e-01	5.8036e+00	8.5294e-01	6.8681e-03
8	3.6120e-01	5.8036e+00	8.5295e-01	2.9883e-05
9	3.6120e-01	5.8036e+00	8.5295e-01	1.8938e-09
10	3.6120e-01	5.8036e+00	8.5295e-01	1.2108e-14

Quadratic convergence is obtained to the value $\alpha^* = 0.3612$ for the real stability radius.

Final remarks

Conclusions

- new algorithm for computing the distance to unstable matrix
- relies on finding a 2-dimensional Jordan block in 2-parameter matrix
- only one linear system solve decomposition per Newton step of bordered matrix M necessary
- numerical results show that new method outperforms earlier algorithms

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Ongoing work

- extension to structured stability radius
- extension to discrete distance to instabilty (Gürbüzbalaban et al)
- extension to calculating the H_∞-norm

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Thank you.