Calculating the H_∞-norm and the Real Stability Radius

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joint work with Alastair Spence and Paul Van Dooren

UNIVERSITY OF BATH

"Take-home-message" - a new method for computing eigenvalues

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Bordered system for $\operatorname{rank}(T(\lambda)) = n - 1$

$$\underbrace{\begin{bmatrix} T(\lambda) & w \\ v^H & 0 \end{bmatrix}}_{M(\lambda)} \begin{bmatrix} x(\lambda) \\ f(\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$M(\lambda)$ is nonsingular if $v^H x \neq 0$ and $w^H y \neq 0$.

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Solve $f(\lambda) = 0$ using Newton's method $\lambda^+ = \lambda - \dfrac{f(\lambda)}{f_\lambda(\lambda)}$.

Continuous-time linear dynamical system

Consider

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t),$$

where $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times p}$, $C \in \mathbb{C}^{m \times n}$ and $D \in \mathbb{C}^{m \times p}$, input $u(t) \in \mathbb{C}^p$, output $y(t) \in \mathbb{C}^m$.

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- Assume A is stable
- H_∞-norm is important quantity for measuring robust stability, error in model order reduction ...

H_∞-norm of the transfer function

H_∞-norm of the transfer function for continuous time systems

$$\|G\|_\infty := \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(i\omega)),$$

- $\sigma_{\max}$: maximum singular value
- $G(s) = C(sI - A)^{-1}B + D$ is the transfer function

- reciprocal of the H_∞-norm: complex stability radius.

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- reciprocal of the H_∞-norm: complex stability radius.
- For $B = C = I$ and $D = 0$: distance to instability (Van Loan 1984)

$$\beta(A) = \min_{\omega\in\mathbb{R}} \sigma_{\min}(A - \omega i I)$$

$\sigma_{\min}$: minimum singular value

Distance to instability - definition

- Stability of A: spectral abscissa $\eta(A) := \max\{\mathrm{Re}(\lambda) \,|\, \lambda \in \Lambda(A)\} < 0$.

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Better measure of stability: Distance to instability

Distance of a stable matrix A to instability

$$\beta(A) = \min\{\|E\| \mid \eta(A+E) = 0,\ E \in \mathbb{C}^{n\times n}\}$$

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$$(A + E - \omega i I)z = 0,$$

for some $\omega \in \mathbb{R}$ and $z \in \mathbb{C}^n$.

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- SVD of $A - \omega i I$:

$$A - \omega i I = U\Sigma V^H.$$

  Minimising destabilising perturbation: $E_{\min} = -\sigma_{\min} u_n v_n^H$.

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Distance to instability ($B = C = I$, $D = 0$)

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Consider the singular values of $A - \omega i I$:

$$(A - \omega i I)v = \alpha u \quad \text{and} \quad (A - \omega i I)^H u = \alpha v.$$

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$$\underbrace{\begin{bmatrix} A & -\alpha I \\ \alpha I & -A^H \end{bmatrix}}_{H(\alpha)} \begin{bmatrix} v \\ u \end{bmatrix} = \omega i \begin{bmatrix} v \\ u \end{bmatrix}$$

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Theorem (Byers 1988)

The $2n \times 2n$ Hamiltonian matrix

$$H(\alpha) = \begin{bmatrix} A & -\alpha I \\ \alpha I & -A^H \end{bmatrix}.$$

has an eigenvalue on the imaginary axis if and only if $\alpha \geq \beta(A)$.

Results on $H(\gamma)$ (for general B, C, D)

Theorem (Boyd, Balakrishnan, Kabamba 1989)

Let A be stable, $\gamma > \sigma_{\max}(D)$, $G(s) = C(sI-A)^{-1}B + D$. The $2n \times 2n$ Hamiltonian matrix

$$H(\gamma) = \begin{bmatrix} A - BR(\gamma)^{-1}D^HC & -\gamma BR(\gamma)^{-1}B^H \\ \gamma C^H S(\gamma)^{-1}C & -A^H + C^H DR(\gamma)^{-1}B^H \end{bmatrix}$$

where $R(\gamma) = D^HD - \gamma^2 I$ and $S(\gamma) = DD^H - \gamma^2 I$ has eigenvalues on the imaginary axis if and only if $\gamma \le \|G\|_\infty$.

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where $R(\gamma) = D^H D - \gamma^2 I$ and $S(\gamma) = DD^H - \gamma^2 I$ has eigenvalues on the imaginary axis if and only if $\gamma \leq \|G\|_\infty$.

- γ is a singular value of $G(i\omega)$ if and only if $H(\gamma) - i\omega I$ is singular. Find the largest possible value of $\gamma = \gamma^*$ such that $H(\gamma)$ has imaginary eigenvalues.
- Assume $\gamma^* = \|G\|_\infty$ is simple.

Existing/other approaches

- Bisection approach by Byers (for distance to instability)
  - choose lower and upper bound on α (0 and $\sigma_{\min}(A)$)
  - take mean value s and calculate all the eigenvalues of $H(s)$, update lower and upper bound according to pure imaginary eigenvalues of $H(s)$

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- Boyd/Balakrishnan method (BBBS) for H_∞-norm
  - given an lower bound $\gamma \leq \|G\|_\infty$, compute all pure imaginary eigenvalues $iw_1, iw_2, \ldots, iw_l$ of $H(\gamma)$ ordered so that $w_1 \leq w_2 \leq \ldots \leq w_l$
  - set $s_k = \frac{w_k + w_{k+1}}{2}$, $k = 1, \ldots l-1$ and update $\gamma = \max_k \sigma_{\max} G(s_k iI)$

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- Spectral value sets (Guglielmi/Gürbüzbalaban/Overton)
  - relies on calculating the rightmost point in spectral value sets/structured pseudospectra
  - approximation of the H_∞-norm using a Newton-bisection method
  - Kressner/Vandereycken combined this method with subspace acceleration
  - Benner/Voigt developed a similar idea, but for descriptor systems $\rightarrow$ Poster tonight

Results on $H(\gamma^*)$

Hamiltonian matrix:

$$H(\gamma) = \begin{bmatrix} A & 0 \\ C^H C & -A^H \end{bmatrix} + \begin{bmatrix} B \\ C^H D \end{bmatrix} R(\gamma)^{-1} \begin{bmatrix} -D^H C & B^H \end{bmatrix}.$$

Assumption

$(\omega^* i, x)$ is a defective eigenpair of $H(\gamma^*)$ of algebraic multiplicity 2.

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$$(H(\gamma^*) - \omega^* iI)\hat{x} = x, \quad \text{and} \quad y^H \hat{x} \neq 0,$$

Jordan block of dimension 2 at the critical value of $\gamma = \gamma^*$

Parameter dependent matrix eigenvalue problem $H(\omega, \gamma)$

Problem

How do we find a 2-dimensional Jordan block in $H(\gamma)$?

$$\underbrace{(H(\gamma) - \omega i I)}_{H(\omega,\gamma)} x = 0, \quad x \neq 0, \quad x^H J x = 0$$

Converting the problem into a Hermitian eigenvalue problem

$$(H(\gamma) - \omega i I)x = 0, \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

$$H(\gamma) = \begin{bmatrix} A & 0 \\ C^H C & -A^H \end{bmatrix} + \begin{bmatrix} B \\ C^H D \end{bmatrix} R(\gamma)^{-1} \begin{bmatrix} -D^H C & B^H \end{bmatrix}.$$

Converting the problem into a Hermitian eigenvalue problem

$$(H(\gamma)J - \omega i J)Jx = 0, \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

$$H(\gamma)J = \begin{bmatrix} 0 & A \\ A^H & C^H C \end{bmatrix} - \begin{bmatrix} B \\ C^H D \end{bmatrix} R(\gamma)^{-1} \begin{bmatrix} B^H & D^H C \end{bmatrix}.$$

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- $(H(\gamma)J - \omega i J)$ is the Schur complement of

$$\mathcal{H}(\gamma,\omega) = \begin{bmatrix} 0 & A - i\omega I & B \\ A^H + i\omega I & C^H C & C^H D \\ B^H & D^H C & \underbrace{D^H D - \gamma^2 I}_{R(\gamma)} \end{bmatrix} \in \mathbb{C}^{2n+p,2n+p},$$

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- With our assumptions

$$\det \mathcal{H}(\gamma,\omega) = 0 \quad \Leftrightarrow \quad \det(H(\gamma)J - \omega iJ) = 0.$$

$$\mathcal{H}(\gamma,\omega)z = 0 \quad \text{where} \quad z = \begin{bmatrix} Jx \\ \star \end{bmatrix}$$

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$$\mathcal{H}(\gamma,\omega)z = 0, \quad z \neq 0, \quad z = \begin{bmatrix} Jx \\ \star \end{bmatrix} \quad \text{where} \quad x^H Jx = 0.$$

The implicit determinant method

Two-parameter problem

$$\mathcal{H}(\gamma, \omega) z = 0 \quad \text{or} \quad \det(\mathcal{H}(\gamma, \omega)) = 0 \quad \dim \ker \mathcal{H}(\gamma, \omega) = 1$$

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Solve

$$f(\gamma,\omega) = 0 \quad \text{instead of} \quad \det(\mathcal{H}(\gamma,\omega)) = 0,$$

where $f(\gamma,\omega)$ is real.

The implicit determinant method

- Differentiate $\begin{bmatrix} \mathcal{H}(\gamma,\omega) & v \\ v^H & 0 \end{bmatrix} \begin{bmatrix} z(\gamma,\omega) \\ f(\gamma,\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with respect to ω,

  where $\mathcal{H}(\gamma,\omega) = \begin{bmatrix} 0 & A - i\omega I & B \\ A^H + i\omega I & C^H C & C^H D \\ B^H & D^H C & D^H D - \gamma^2 I \end{bmatrix}$:

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- $$\begin{bmatrix} \mathcal{H}(\gamma,\omega) & v \\ v^H & 0 \end{bmatrix} \begin{bmatrix} z_\omega(\gamma,\omega) \\ f_\omega(\gamma,\omega) \end{bmatrix} = i \begin{bmatrix} J \begin{bmatrix} z_1(\gamma,\omega) \\ z_2(\gamma,\omega) \end{bmatrix} \\ 0 \\ 0 \end{bmatrix}.$$

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- First row multiplied by left eigenvector $z = \begin{bmatrix} Jx \\ \star \end{bmatrix}$ from the left

  $$f_\omega(\gamma,\omega) = ix^H Jx = 0,$$

  because of Jordan block of dimension 2.

Newton's method for *real* function g in two real variables

- Solve

$$g(\gamma,\omega) = \begin{bmatrix} f(\gamma,\omega) \\ f_\omega(\gamma,\omega) \end{bmatrix} = 0.$$

using quadratically convergent Newton's method.

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$$g(\gamma,\omega) = \begin{bmatrix} f(\gamma,\omega) \\ f_\omega(\gamma,\omega) \end{bmatrix} = 0.$$
using quadratically convergent Newton's method.
- Jacobian elements: differentiate the system
$$\begin{bmatrix} \mathcal{H}(\gamma,\omega) & v \\ v^H & 0 \end{bmatrix} \begin{bmatrix} z(\gamma,\omega) \\ f(\gamma,\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
with respect to ω and γ.

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- Solve
$$g(\gamma,\omega) = \begin{bmatrix} f(\gamma,\omega) \\ f_\omega(\gamma,\omega) \end{bmatrix} = 0.$$
using quadratically convergent Newton's method.
- Jacobian elements: differentiate the system
$$\begin{bmatrix} \mathcal{H}(\gamma,\omega) & v \\ v^H & 0 \end{bmatrix} \begin{bmatrix} z(\gamma,\omega) \\ f(\gamma,\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
with respect to ω and γ.
- Implementation: one (sparse) LU factorisation of
$$\begin{bmatrix} \mathcal{H}(\gamma,\omega) & v \\ v^H & 0 \end{bmatrix}$$

Newton's method for *real* function g in two real variables

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$$g(\gamma,\omega) = \begin{bmatrix} f(\gamma,\omega) \\ f_\omega(\gamma,\omega) \end{bmatrix} = 0.$$
using quadratically convergent Newton's method.
- Jacobian elements: differentiate the system
$$\begin{bmatrix} \mathcal{H}(\gamma,\omega) & v \\ v^H & 0 \end{bmatrix} \begin{bmatrix} z(\gamma,\omega) \\ f(\gamma,\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
with respect to ω and γ.
- Implementation: one (sparse) LU factorisation of
$$\begin{bmatrix} \mathcal{H}(\gamma,\omega) & v \\ v^H & 0 \end{bmatrix}$$
- Jacobian of g at critical point: $\begin{bmatrix} 0 & f_\gamma(\gamma^*,\omega^*) \\ f_{\omega\omega}(\gamma^*,\omega^*) & f_{\omega\gamma}(\gamma^*,\omega^*) \end{bmatrix}$ is nonsingular and $f_{\omega\omega}(\gamma^*,\omega^*) > 0$ and $f_\gamma(\gamma^*,\omega^*) > 0$.

Remarks on solving linear system

- Transform the pair (A, B) into upper block staircase form, that is $(U^H AU, U^H BV) =$ (◥, ◥), where U, V are unitary matrices

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- Transform the pair (A, B) into upper block staircase form, that is $(U^H AU, U^H BV) = (\diagdown\!\!|, \diagdown\!\!|)$, where U, V are unitary matrices
- With $Q = \begin{bmatrix} UP & 0 & 0 \\ 0 & 0 & U \\ 0 & V & 0 \end{bmatrix}$, $P = \begin{bmatrix} 0 & \cdots & 1 \\ & \iddots & \\ 1 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{n \times n}$ and

$$\mathcal{H}(\gamma, \omega) = \begin{bmatrix} 0 & A - i\omega I & B \\ A^H + i\omega I & C^H C & C^H D \\ B^H & D^H C & D^H D - \gamma^2 I \end{bmatrix}:$$

$$Q^H \mathcal{H}(\gamma, \omega) Q = \underbrace{\begin{bmatrix} 0 & \triangle & x \\ \triangle & x & x \\ x & x & x \end{bmatrix}}_{\text{lower anti-}p+1 \quad \text{Hessenberg}} + \begin{bmatrix} 0 & 0 & -i\omega P \\ 0 & -\gamma^2 I & 0 \\ i\omega P & 0 & 0 \end{bmatrix}.$$

Distance to instability

Orr-Sommerfeld operator

$$\frac{1}{\gamma R}L^2 v - i(UL - U'')v = \lambda L v, \quad \text{where} \quad L = \frac{d^2}{dx^2} - \gamma^2 \quad \text{and} \quad U = 1 - x^2.$$

Discretise the operator on $v \in [-1, 1]$ using finite differences with $\gamma = 1$, $R = 1000$ and $n = 1000$.

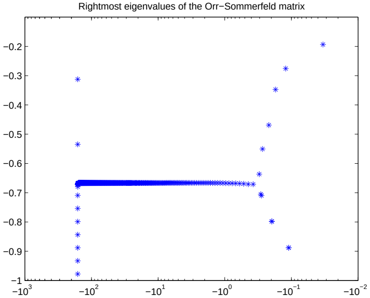

Figure: Eigenvalues of the Orr-Sommerfeld matrix.

Distance to instability

Convergence to $\omega = 0.199756$ and $\alpha = 0.001978$ within 5 iterations.

Table: CPU times.

Algorithm	"Inner" iterations		"Outer" iterations (Eigenvalue computation for Hamiltonian matrix)		Total CPU time
	quantity	CPU time	quantity	CPU time	
Boyd/Balakrishnan	6	3.49 s	6	63.28 s	66.77 s
Newton	5	5.67 s	1	10.33 s	16.00 s

Distance to instability

Tolosa matrix `tols340.mtx`

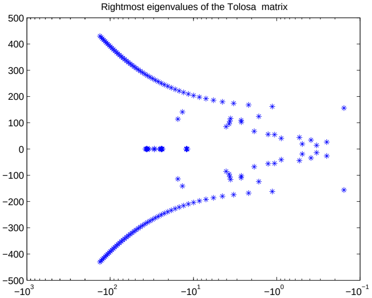

Figure: Eigenvalues of the Tolosa matrix.

Distance to instability

Convergence to $\omega = 1.559998$ and $\alpha = 0.0000199$ within 4 iterations.

Table: CPU times.

Algorithm	"Inner" iterations		"Outer" iterations (Eigenvalue computation for Hamiltonian matrix)		Total CPU time
	quantity	CPU time	quantity	CPU time	
Boyd/Balakrishnan	3	67.52 s	3	5.27 s	72.79 s
Newton	4	2.01 s	1	1.69 s	3.7 s

H_∞-norm - Examples

	n	$m,\ p$	$\|\|G\|\|_\infty^{IDM}$	NI	$\|\|G\|\|_\infty^{SVS}$	CPU(IDM)	CPU(SVS)
BBK	4	2 , 2	$6.44051653e+00$	5	$6.44051653e+00$	3.000e-01	**2.100e-01**
CBM	351	2 , 1	$2.62976526e-01$	5	$2.62976526e-01$	**3.100e+00**	1.106e+02
CSE2	63	32 , 1	$2.03391753e-02$	4	$2.03391753e-02$	**5.400e-01**	8.238e+01
CM1	23	3 , 1	$8.16496496e-01$	5	$8.16496589e-01$	**2.100e-01**	5.700e-01
CM3	123	3 , 1	$8.16094047e-01$	7	$8.21443671e-01$	**3.900e-01**	1.687e+01
CM4	243	3 , 1	$1.58866664e+00$	5	$1.44542364e+00$	**1.360e+00**	6.475e+02
HE6	23	16 , 6	$4.92937305e+02$	1	$4.92937159e+02$	**4.800e-01**	8.430e+00
HE7	23	16 , 9	$3.46529860e+02$	5	$3.46529860e+02$	**3.100e-01**	7.800e-01
ROC1	12	2 , 2	$1.21658902e+00$	5	$1.21658902e+00$	**2.100e-01**	4.200e-01
ROC2	13	1 , 4	$1.33366743e-01$	2	$1.33366743e-01$	**3.600e-01**	3.900e-01
ROC3	14	11 , 11	$1.72310471e+04$	4	$1.72310471e+04$	**3.000e-01**	3.600e-01
ROC4	12	2 , 2	$2.95650873e+02$	8	$2.95650873e+02$	**1.900e-01**	3.500e-01
ROC5	10	2 , 3	$9.79995384e-03$	1	$9.79995186e-03$	**2.800e-01**	6.200e-01
ROC6	8	3 , 3	$2.57633040e+01$	1	$2.57633040e+01$	3.400e-01	**2.200e-01**
ROC7	8	3 , 1	$1.12196212e+00$	9	$1.12196215e+00$	**1.000e-01**	1.420e+00
ROC8	12	7 , 1	$6.59896425e+00$	5	$6.59896425e+00$	2.800e-01	**2.700e-01**
ROC9	9	5 , 1	$3.29185918e+00$	6	$3.29404544e+00$	**1.300e-01**	5.000e-01
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ROC10	9	2 , 2	$1.01446326e-01$	4	$1.01480433e-01$	**3.300e-01**	4.500e-01
NN18	1006	2 , 1	$1.02336052e+00$	5	$1.02336052e+00$	**3.607e+01**	4.604e+01
dwave	2048	6 , 4	$3.80199635e+04$	1	$3.80199635e+04$	**1.516e+01**	1.077e+02

Final remarks

Conclusions

- fast algorithm for computing the H_∞-norm and distance to unstable matrix
- relies on finding a 2-dimensional Jordan block in 2-parameter matrix
- only one LU decomposition per Newton step of bordered matrix
- computation time can further be reduced using transformation into staircase form
- numerical results show that new method outperforms other algorithms
- extension to descriptor systems and discrete time problems
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Further work and extensions

- iterative solvers for linear system
- calculation of larger Jordan blocks

A (rather incomplete) list of references

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Thank you.