Transcritical flow modelling with the Box Scheme

Melina Freitag
Supervisor: Bill Morton

Forschungsseminar: Numerik partieller Differentialgleichungen

TU Chemnitz

13.01.2004

– Typeset by FoilTEX –

Introduction

- The Saint Venant Equations
- The Preissmann Box Scheme
- Transcritical Flow
- Model Problems and Steady State Solutions
- Accuracy, Stability and time-step Constraint
- Extension of the Box Scheme to Transcritical Flow
- Conclusions

The Saint Venant Equations I - Assumptions

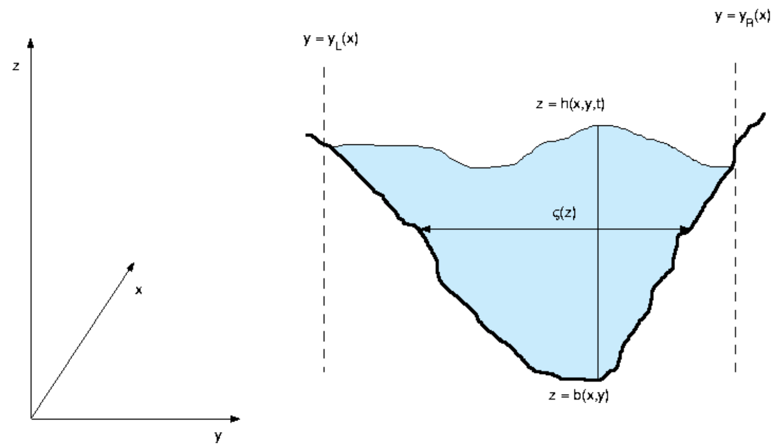


Figure 1: *Channel cross section*

- flow is essentially 1-D, incompressible ideal fluid, constant density
- all forces due to gravity and friction, channel bed does not change
- volumetric inflow due to rain, evaporation in negligible

The Saint Venant Equations II

- Mass Conservation

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

- Momentum Conservation

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\beta Q^2}{A} + gI_1\right) = gI_2 + gA(S_0 - S_f)$$

$A(x,t) \ldots$ wetted cross-sectional area,
$Q(x,t) \ldots$ discharge,
$S_0(x) \ldots$ bed slope,
$S_f(x,A,Q) \ldots$ frictional slope,
$\beta \ldots$ momentum coefficient (≈ 1),
$I_1(x,t) \ldots$ cross-sectional moment integral,
$I_2(x,t) \ldots$ pressure force acting on the channel bed

Channel with Trapezoidal Cross-Section

System may be written in conservative vector form

$$\mathbf{u}_t + \mathbf{f}_x = \mathbf{s},$$

where

$$\begin{aligned} \mathbf{u} &= [A, Q]^T, \\ \mathbf{f} &= \left[Q, \frac{Q^2}{A} + g\left(\frac{D^2 B}{2} + \frac{D^3 S_T}{3}\right)\right]^T, \\ \mathbf{s} &= \left[0, gA(S_0 - S_f) + gD^2\left(\frac{1}{2}\frac{\partial B}{\partial x} + \frac{D}{3}\frac{\partial S_T}{\partial x}\right)\right]^T. \end{aligned}$$

Hyperbolic System

Jacobian of $\mathbf{f}$ is given by

$$\mathcal{A}(\mathbf{u}) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \begin{bmatrix} 0 & 1 \\ c^2 - v^2 & 2v \end{bmatrix},$$

where $v = Q/A$ average velocity and $c = \sqrt{gA/T}$ wave celerity.

- eigenvalues: $a_1 = v - c$, $a_2 = v + c$,
- eigenvectors: $v_1 = [1, v - c]^T$, $v_2 = [1, v + c]^T$

2 real distinct eigenvalues and linearly independent eigenvectors, if $c \neq 0$, system is *strictly hyperbolic*

The Preissmann Box Scheme

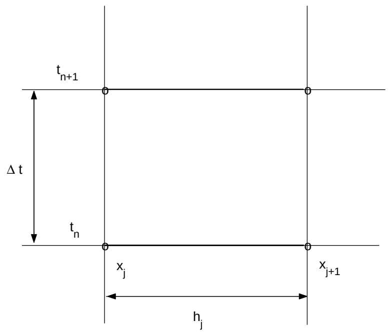


Figure 2: *The Box Scheme Stencil*

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} &\approx \frac{\mathbf{u}_{j+1}^{n+1} - \mathbf{u}_{j+1}^{n} + \mathbf{u}_{j}^{n+1} - \mathbf{u}_{j}^{n}}{2\Delta t}, \\
\frac{\partial \mathbf{f}}{\partial x} &\approx \frac{\theta(\mathbf{f}_{j+1}^{n+1} - \mathbf{f}_{j}^{n+1}) + (1-\theta)(\mathbf{f}_{j+1}^{n} - \mathbf{f}_{j}^{n})}{h_j}, \\
\mathbf{s} &\approx \frac{1}{2}\theta(\mathbf{s}_{j+1}^{n+1} + \mathbf{s}_{j}^{n+1}) + \frac{1}{2}(1-\theta)(\mathbf{s}_{j+1}^{n} + \mathbf{s}_{j}^{n}),
\end{aligned}
$$

Implementation and Solution Procedure I

Differential form

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{s}(\mathbf{u}, x),$$

$$\begin{aligned} &\frac{\mathbf{u}_{j+1}^{n+1} - \mathbf{u}_{j+1}^{n}}{2} + \frac{\mathbf{u}_{j}^{n+1} - \mathbf{u}_{j}^{n}}{2} + \lambda_j \theta (\mathbf{f}_{j+1}^{n+1} - \mathbf{f}_{j}^{n+1}) + \lambda_j (1-\theta)(\mathbf{f}_{j+1}^{n} - \mathbf{f}_{j}^{n}) \\ &\qquad - \Delta t \left(\frac{1}{2}\theta (\mathbf{s}_{j+1}^{n+1} + \mathbf{s}_{j}^{n+1}) + \frac{1}{2}(1-\theta)(\mathbf{s}_{j+1}^{n} + \mathbf{s}_{j}^{n}) \right) = \mathbf{0}, \end{aligned} \tag{1}$$

for $j = 0, \ldots, N-1$, $\lambda_j = \Delta t / h_j$.

- $2N+2$ unknowns Q_j, A_j, $2N$ equations $\rightarrow$ boundary conditions
- subcritical flow: Q upstream, D downstream
- supercritical flow: Q, D upstream

Implementation and Solution Procedure II

- need nonlinear iteration technique to solve $\mathbf{R}_{j+\frac{1}{2}} = 0 \quad \forall\, j = 0, \ldots, N-1$
- Newton system

$$\begin{bmatrix} x & x & \cdots & \cdots & \cdots & \cdots & \cdots \\ x & x & x & x & \cdots & \cdots & \cdots \\ x & x & x & x & \cdots & \cdots & \cdots \\ \cdots & \cdots & x & x & x & x & \cdots \\ \cdots & \cdots & x & x & x & x & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \Delta \mathbf{u}^{(k)} = -\mathbf{R}$$

Newton update $\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + \Delta\mathbf{u}^{(k)}$.

- stopping criterion $\frac{\|\mathbf{u}^{(k+1)} - \mathbf{u}^{(k)}\|_1}{\|\mathbf{u}^{(k)}\|_1} <$ tol,

Thomas Algorithm

- extend Thomas Algorithm for tridiagonal matrices to block-tridiagonal matrices
- Tridiagonal system $A_i \mathbf{W}_{i-1} + D_i \mathbf{W}_i + C_i \mathbf{W}_{i+1} = \mathbf{Z}_i, \quad i = 0, \ldots, N,$
  Solution given by backward recursion

$$\begin{aligned} \mathbf{W}_N &= \mathbf{F}_N, \\ \mathbf{W}_j &= E_j \mathbf{W}_{j+1} + \mathbf{F}_j, \quad j = N-1, \ldots, 0, \end{aligned}$$

  where E_j and $\mathbf{F}_j$ are given by the forward recursion

$$\begin{aligned} E_j &= -(D_j + A_j E_{j-1})^{-1} C_j, \quad j = 0, \ldots, N-1, \\ \mathbf{F}_j &= (D_j + A_j E_{j-1})^{-1} (\mathbf{Z}_j - A_j \mathbf{F}_{j-1}), \quad j = 0, \ldots, N, \end{aligned}$$

- stable procedure, if inverses exist and $\|E_j\| \leq 1, \forall j$

Transcritical Flow

- subcritical and supercritical regions
- *Froude Number:* $F = \frac{|v|}{c}$
- subcritical $F < 1$, supercritical $F > 1$
- eigenvalues of the Jacobian $\approx$ velocities at which disturbances propagate; opposite sign: subcritical flow, same sign: supercritical flow
- steep slopes or rapidly changing channel widths
- counting problem, stability problems
- typical situation: subcritical flow with an interior supercritical region

Model Problem 1

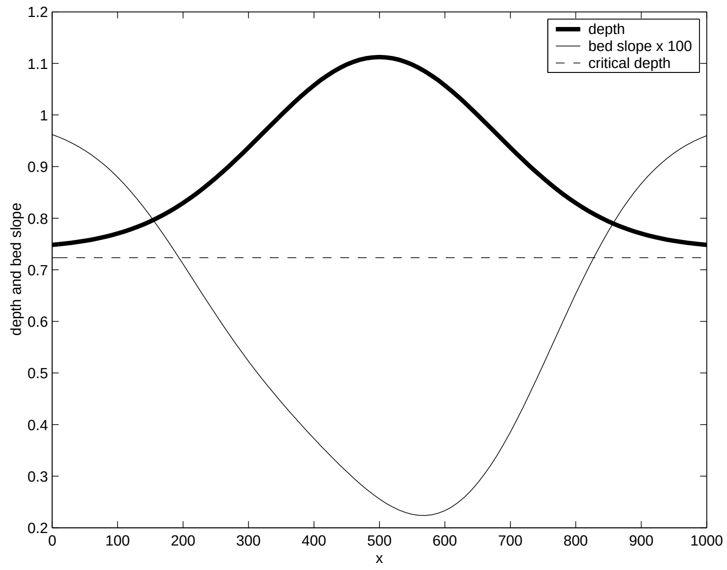


Figure 3: *Depth and bed slope*

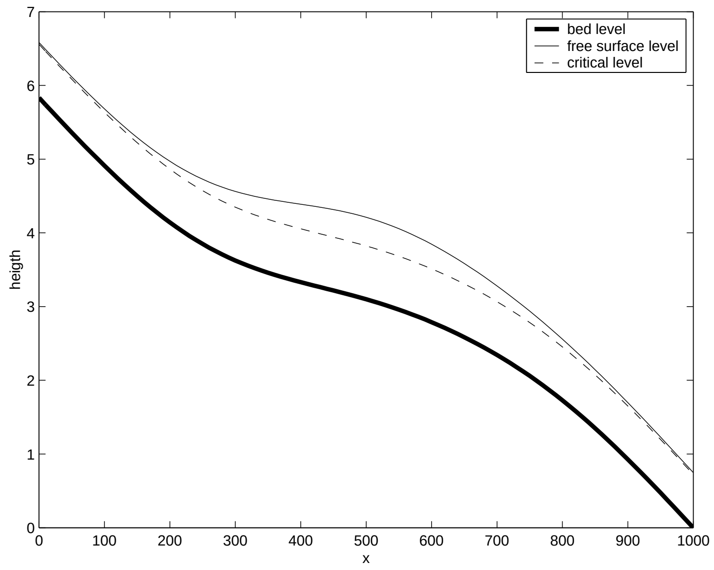


Figure 4: *Bed level and surface level*

- channel length $1km$, width $10m$
- upstream boundary condition $Q = 20m^3/s$ at inflow,
- downstream boundary condition $h = 0.75m$ at outflow

Model Problem 1 - Results

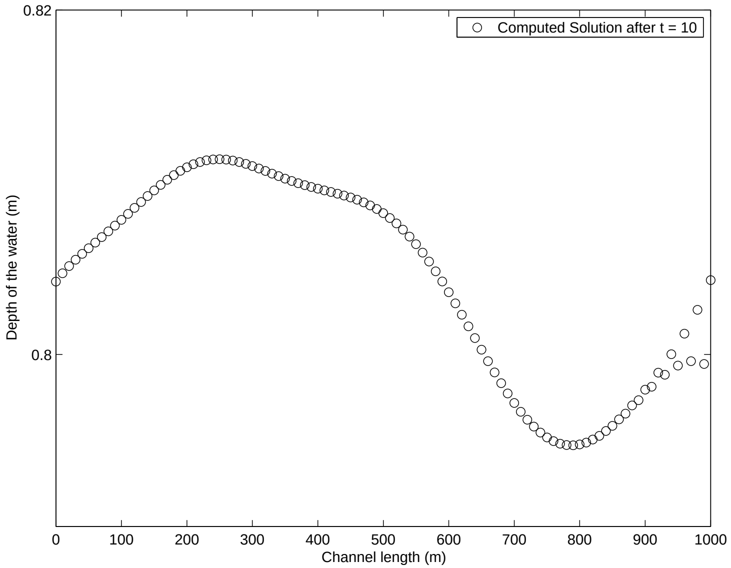


Figure 5: *Near-critical problem after one time-step,* $t = 10$

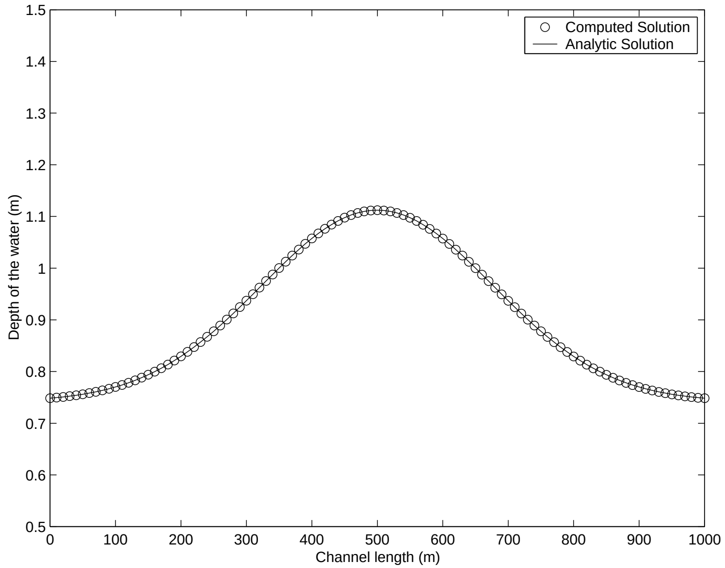
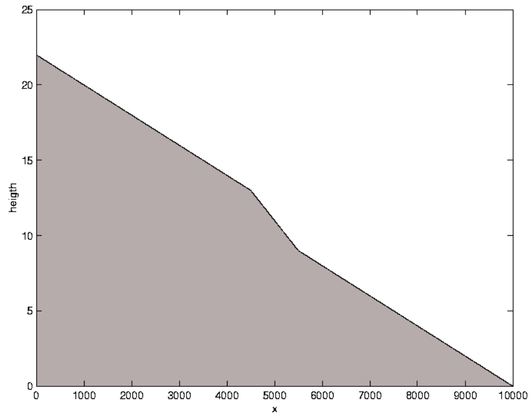


Figure 6: *Near-critical problem at steady state*

- $\Delta x = 10$, $\Delta t = 10$, $\theta = \frac{2}{3}$
- $3 - 5$ iterations per time step (tolerance 10^{-10}), Froude Number: $F = 0.95$

Model Problem 2



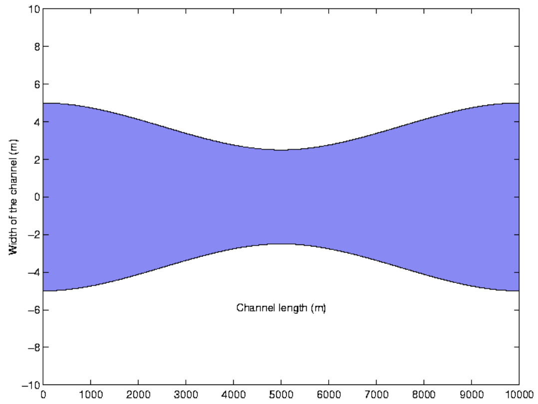


Figure 7: *Problem with changing channel width*

- channel length $10km$,
- upstream boundary condition $Q = 20m^3/s$ at inflow,
- downstream boundary condition $h = 1.3m$ at outflow, $S_0(x) = 0.002$

Model Problem 2 - Results I

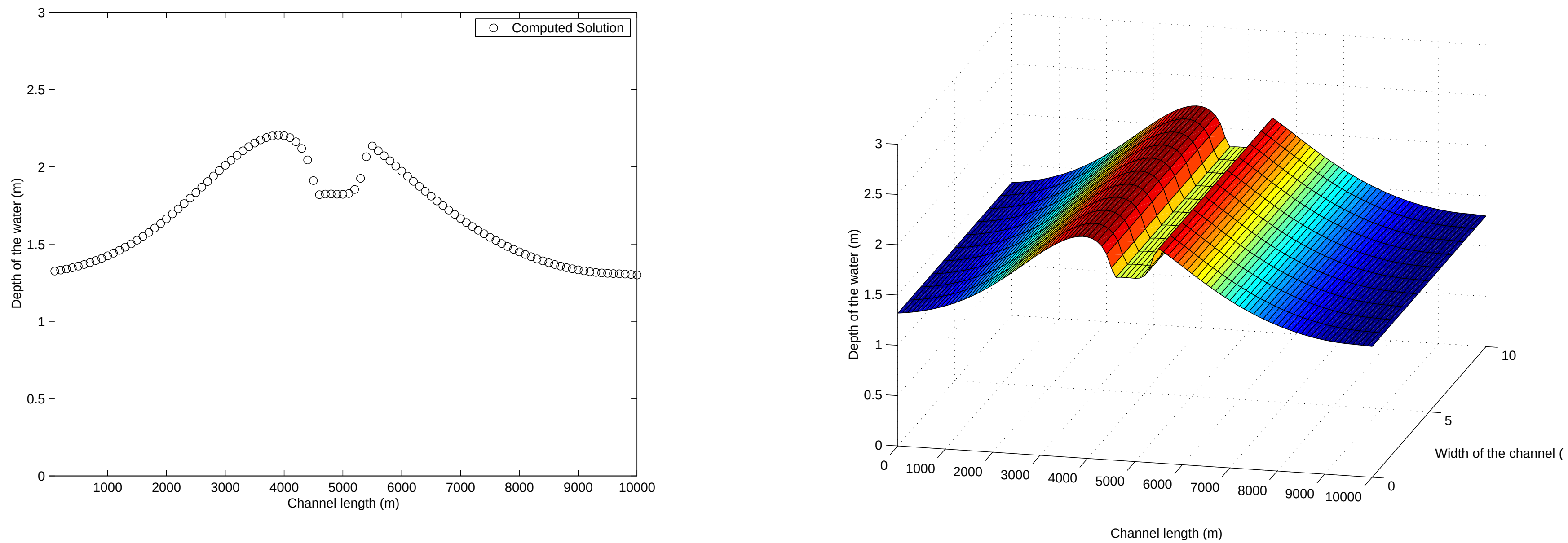
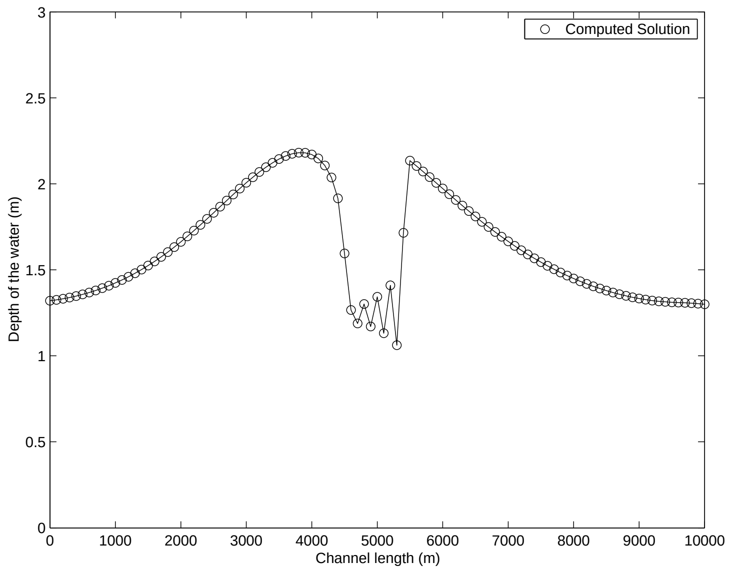


Figure 8: *Subcritical Problem*

- $\Delta x = 100$, $\Delta t = 100$, $\theta = \frac{2}{3}$
- more than 5 iterations per time step (tolerance 10^{-10}), Froude Number: $F = 0.52$

Model Problem 2 - Results II



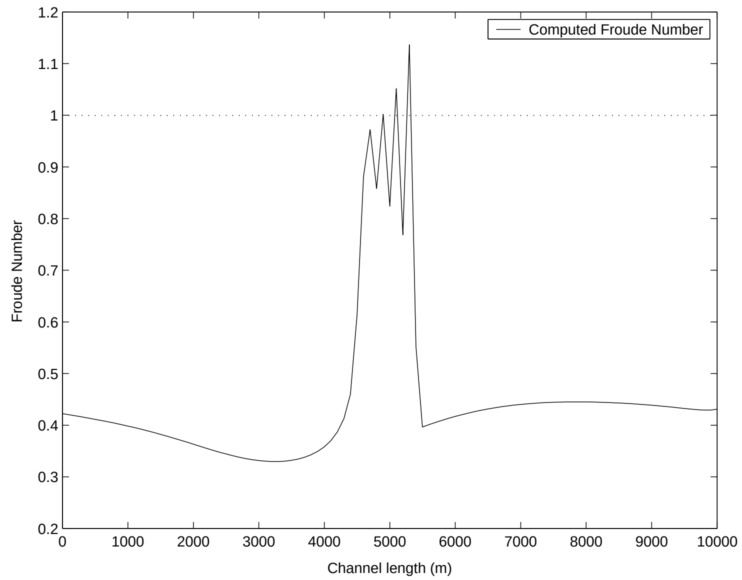


Figure 9: *Near-crititcal problem at steady state, Froude Number*

- $\Delta x = 100$, $\Delta t = 100$, $\theta = \frac{2}{3}$, Froude Number: $F = 1.05$

Model Problem 3

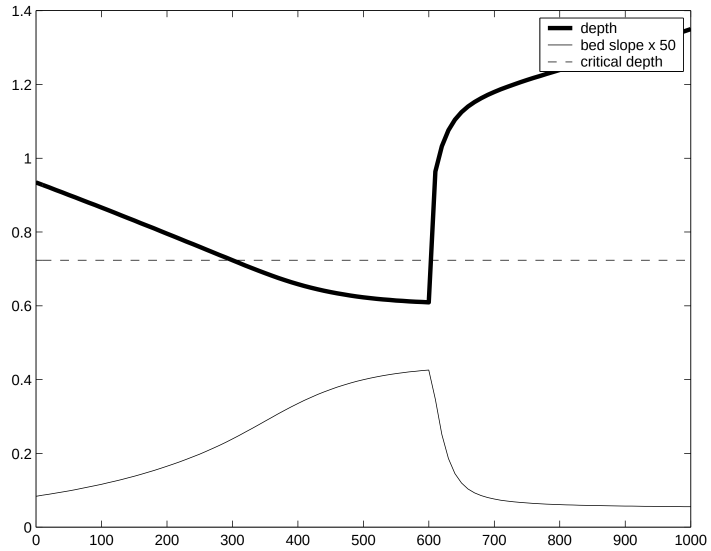


Figure 10: *Depth and bed slope*

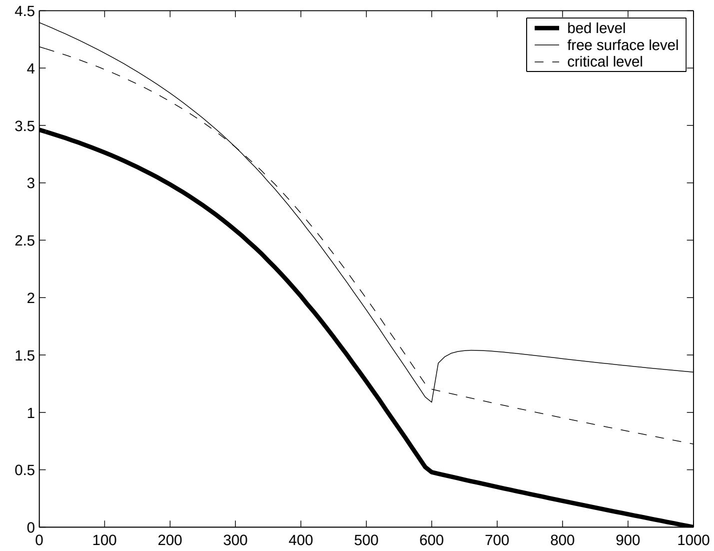


Figure 11: *Bed level and surface level*

- channel length $1km$, width $10m$
- upstream boundary condition $Q = 20m^3/s$ at inflow,
- downstream boundary condition $h = 1.35m$ at outflow

Model Problem 3 - Results

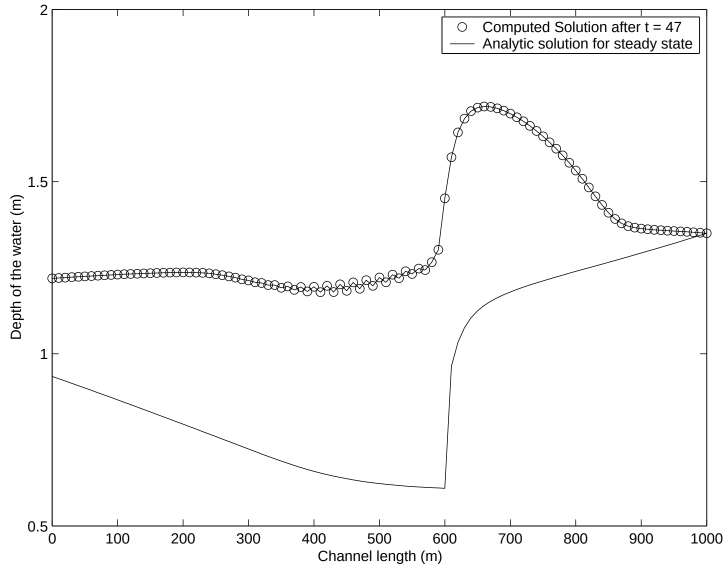


Figure 12: *Transcritical problem at time* $t = 47$

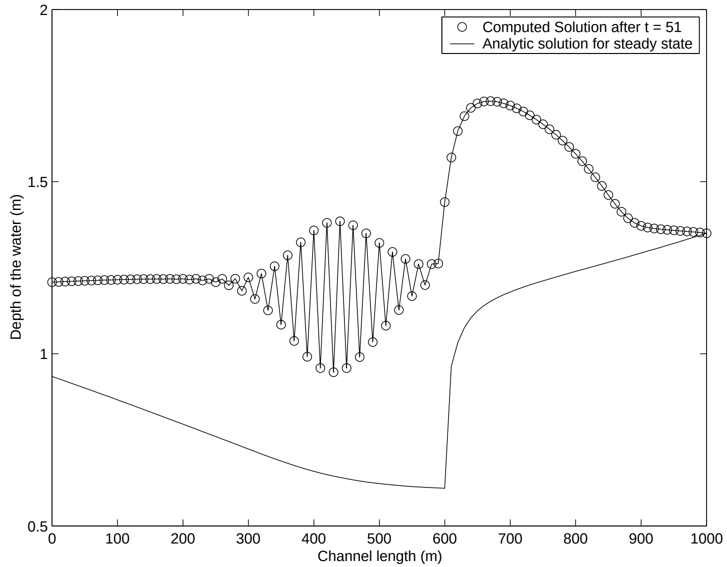


Figure 13: *Transcritical problem at time* $t = 51$

- $\Delta x = 10$, $\Delta t = 1$, $\theta = \frac{2}{3}$
- oscillations, scheme breaks down

Accuracy and Stability

- *truncation error:* $\mathcal{O}(\Delta x^2, \Delta t)$, for $\theta = \frac{1}{2}$: $\mathcal{O}(\Delta x^2, \Delta t^2)$
- *Stability via Fourier analysis:* unconditional stability for $\theta \geq \frac{1}{2}$, but
- for $\theta = \frac{1}{2}$ or $\nu = a_{1/2}\frac{\Delta t}{\Delta x} = 0$ scheme is non-dissipative $\rightarrow$ critical case
- convergence of the Box Scheme
- restriction on time-step due to convergence theory of Newton's Method (*Newton-Kantorovich*)

More Stability Analysis ...

- *Stability of Boundary Conditions:* solution at time t^{n+1} may be written as a recurrence relation

$$\mathbf{w}_N = (-1)^N \left(\prod_{j=0}^{N-1} G_j \right) \mathbf{w}_0 + \Delta t \mathbf{T}_N,$$

with amplification matrix G_j

$$G_j = \begin{bmatrix} \frac{1-2\lambda_j\theta(v_j-c_j)}{1+2\lambda_j\theta(v_j-c_j)} & 0 \\ 0 & \frac{1-2\lambda_j\theta(v_j+c_j)}{1+2\lambda_j\theta(v_j+c_j)} \end{bmatrix}.$$

- if one of the eigenvalues of the Jacobian $\mathcal{A}$ passes through zero, i.e. $v_j \approx c_j$, perturbations not damped

Invalidity of the Box Scheme for Transcritical Flow

- as critical limit is approached, one of the eigenvalues become small and spurious Fourier modes are allowed to propagate uninhibited
- counting problem: Box Scheme needs exactly two boundary conditions, but for transcritical problems there may be too few or too many boundary conditions
- critical problem results in locally underdetermined/overdetermined systems:

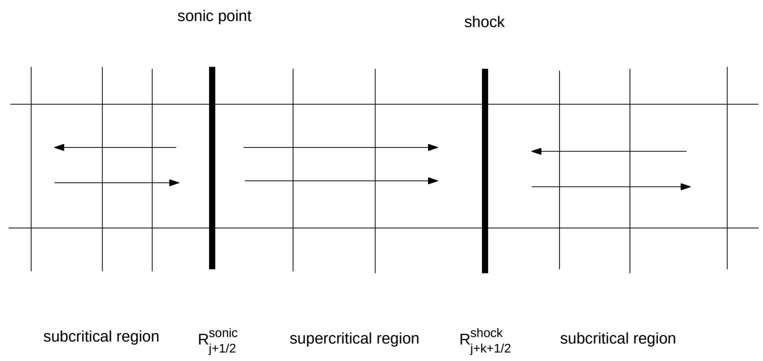


Figure 14: *Cell residuals and transcritical flow*

Cell and Nodal Residuals I

- introduce mapping between cell residuals and nodal unknowns

$$\mathbf{N}_j = D^+_{j-\frac{1}{2}}\mathbf{R}_{j-\frac{1}{2}} + D^-_{j+\frac{1}{2}}\mathbf{R}_{j+\frac{1}{2}} + \mathbf{B}_j = 0,$$

$\mathbf{B}_j$ accounts for boundary conditions.

- distribution matrices for subcritical case:

$$D^-_{j+\frac{1}{2}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad D^+_{j-\frac{1}{2}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

for subcritical case:

$$D^-_{j+\frac{1}{2}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad D^+_{j-\frac{1}{2}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Cell and Nodal Residuals II

- more sophisticated alternative as a generalization of those used in upwinding schemes; characteristic decomposition $\mathcal{A} = V \Lambda V^{-1}$

$$
\begin{aligned}
D^-_{j+\frac{1}{2}} &= \tilde{V}_{j+\frac{1}{2}} \text{diag} \left\{ \frac{1}{2} - \frac{1}{2} \text{sign}(\tilde{a}^{(k)}_{j+\frac{1}{2}}) \, : \, k = 1, 2 \right\} \tilde{V}^{-1}_{j+\frac{1}{2}}, \\
D^+_{j-\frac{1}{2}} &= \tilde{V}_{j-\frac{1}{2}} \text{diag} \left\{ \frac{1}{2} + \frac{1}{2} \text{sign}(\tilde{a}^{(k)}_{j-\frac{1}{2}}) \, : \, k = 1, 2 \right\} \tilde{V}^{-1}_{j-\frac{1}{2}},
\end{aligned}
$$

where $\tilde{\mathcal{A}}_{j\pm\frac{1}{2}}$ an average value of $\mathcal{A}$ for the cell (Roe average matrix) and $\tilde{a}^{(k)}_{j\pm\frac{1}{2}}$, $k = 1, 2$, its eigenvalues.

- problem, if

$$
\text{rank}(D^-_{j+\frac{1}{2}} + D^+_{j-\frac{1}{2}}) = 2,
$$

is not satisfied (sonic point)

Solutions

1. *Internal boundary conditions:*
   - treats only the sonic point
   - if $\text{rank}(D^{-}_{j+\frac{1}{2}} + D^{+}_{j-\frac{1}{2}}) \leq 1$, then by introducing an extra linearly independent equation

$$\mathbf{B}_j = (I - D^{-}_{j+\frac{1}{2}} - D^{+}_{j-\frac{1}{2}})\Delta \mathbf{u}_j.$$

2. *Splitting of the cell residual at the sonic point and combining of cell residuals at the shock (hydraulic jump)*
   - treats both sonic point and shock
   - calculate the Froude Number and determine subcritical and supercritical regions (at each time step)
   - find sonic point and split cell residual
   - find shock cell and combine two cell residuals either side
   - discrete conservation law still satisfied

Numerical Results - Model Problem 3

1. Internal boundary conditions

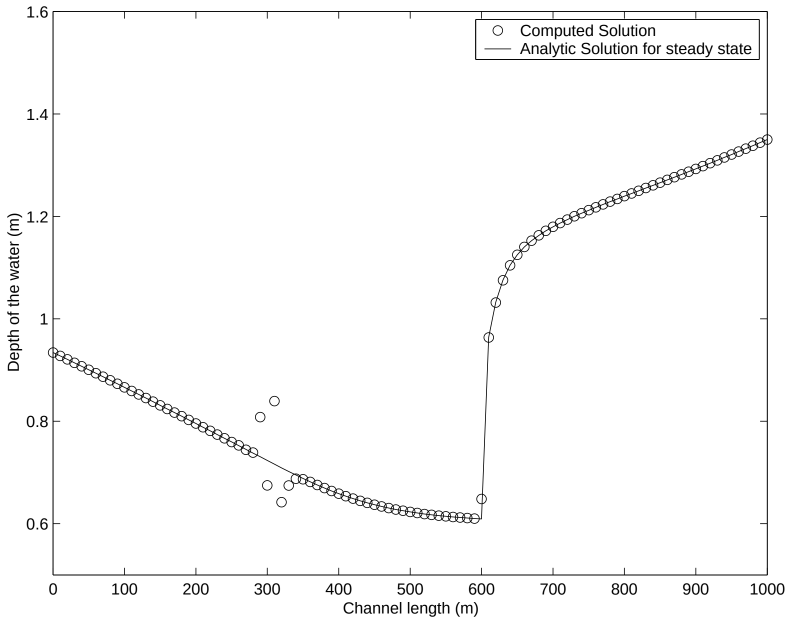


Figure 15: *Transcritical problem with internal boundary conditions*

slow convergence, sonic point not resolved

Numerical Results - Model Problem 3

2. Splitting and combining cell residuals

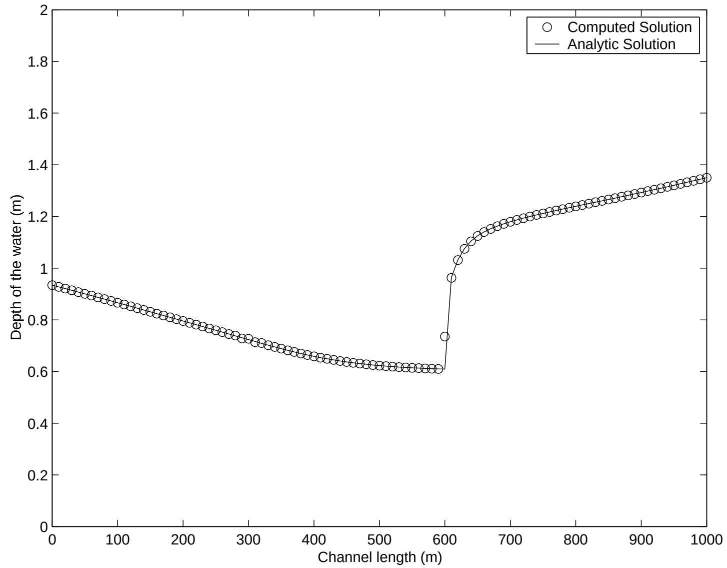


Figure 16: *Transcritical problem at steady state*

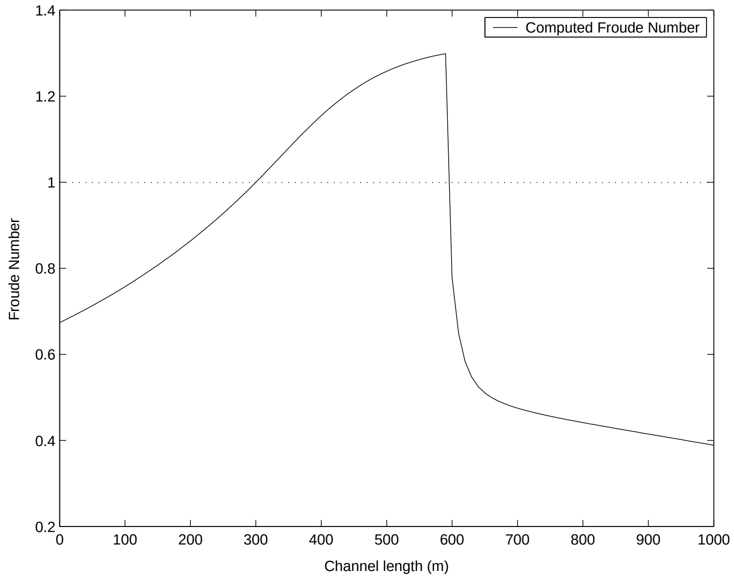


Figure 17: *Approximation to Froude Number at steady state*

$\Delta t = 1$, 3-5 iterations, Froude number 1.3

Local Post Processing and Shock Fitting - I

Box Scheme cannot resolve discontinuity properly → Shock Fitting

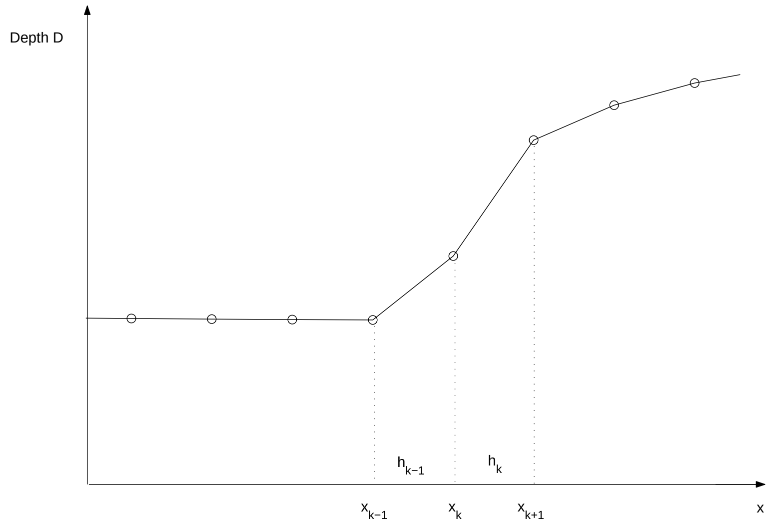


Figure 18: *Diagram showing depth function at the shock before introducing a discontinuity*

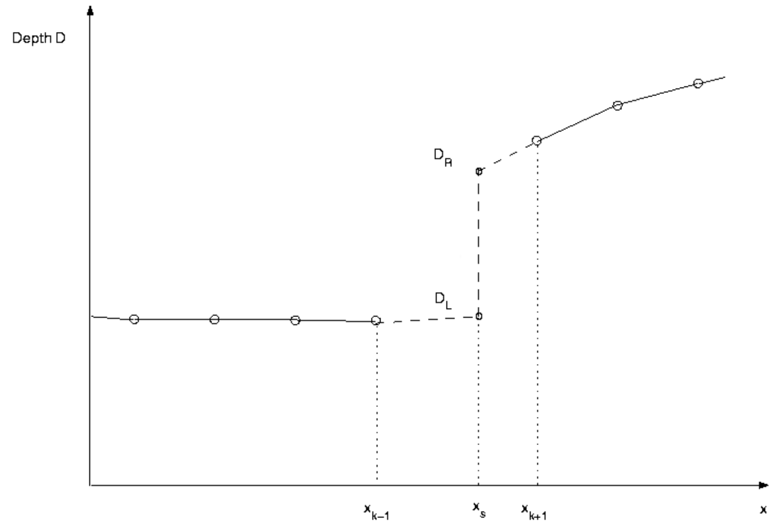


Figure 19: *Diagram showing depth and shock location x_s after introducing a discontinuity*

introduce shock position and values left and right of the hydraulic jump as unknowns

Local Post Processing and Shock Fitting - II

- apply discrete mass and momentum conservation laws locally
- use *Rankine Hugoniot jump condition:* for the shock speed

$$s = \frac{f(u_L) - f(u_R)}{u_L - u_R} = \frac{[f]}{[u]}.$$

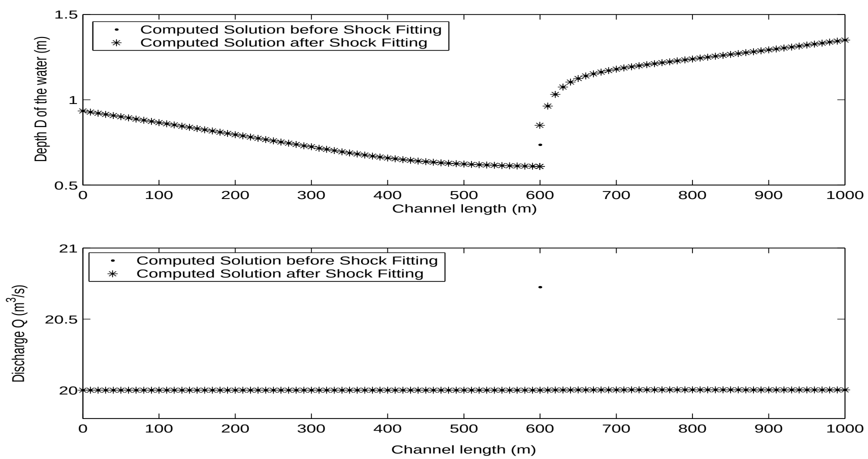


Figure 20: *Depth D and discharge Q at steady state after shock fitting*

Conclusion

- derivation of St Venant equations
- Box Scheme for test problems very adequate
- explanation for breakdown in the transcritical case
- *modified Box Scheme* for unsteady St Venant equations
- Shock fitting