On the Influence of Multiplication Operators on the Ill-Posedness of Inverse Problems

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Aufgabenstellung

Ziel der Arbeit ist ein Beitrag zur Analyse des lokalen Inkorrektheitsverhaltens nichtlinearer Operatorgleichungen

$$F(x) = y \qquad (x \in D(F) \subset X,\ y \in Y)$$

mit einem Operator F, der auf einem konvexen Definitionsbereich $D(F)$ definiert ist und zwischen unendlich-dimensionalen Hilberträumen X and Y wirkt.

Speziell soll der Fall betrachtet werden, dass die Fréchet-Ableitung $F'(x_0)$ von F im Punkt $x_0 \in D(F)$ eine Komposition eines kompakten linearen Integraloperators J von X in Y mit einer wohlbekannten Abklingrate der absteigend geordneten Singulärwerte $s_i(J)$ of J gegen Null für $i \to \infty$ und eines Multiplikationsoperators M ist. Solche Gleichungen treten als Anwendungen in den Naturwissenschaften und der Finanzmathematik auf.

Wenn für $X = Y = L^2(0,1)$ der Operator $F = N \circ J$ auf dem Halbraum $D(F) = \{x \in L^2(0,1) : x(t) \geq \underline{c} \geq 0$ f.ü. auf $[0,1]\}$ definiert ist als

$$[F(x)](t) = k(t, [J(x)](t)) \qquad (0 \leq t \leq 1)$$

mit dem einfachen Faltungsoperator $[J(x)](t) = \int_0^t x(\tau)d\tau$ $(0 \leq t \leq 1)$ und ein Nemytskii-Operator $[N(z)](t) = k(t, z(t))$ durch die hinreichend glatte Kernfunktion $k(t,s)$ $((t,s) \in [0,1] \times [0,\infty))$ erzeugt wird, so liegt die oben beschriebene Situation vor. Die Fréchet-Ableitung $F'(x_0)$ nimmt im Punkt $x_0 \in D(F)$ die Gestalt $F'(x_0) = M \circ J$ bzw.

$$\left[F'(x_0)(h)\right](t) = m(t)\,[J(h)](t) \quad (0 \leq t \leq 1,\ h \in X)$$

mit einem Multiplikationsoperator M an, der über die Multiplikatorfunktion

$$m(t) = \frac{\partial k(t, [J(x_0)](t))}{\partial s} \quad (0 \leq t \leq 1)$$

definiert wird. Dann überträgt sich für $m \in L^\infty(0,1)$ die Kompaktheit von J auf den beschränkten linearen Operator $F'(x_0)$. Mehr noch, im Falle eines positiven wesentlichen Infimums von m gilt $s_i(F'(x_0)) \sim 1/i$ für die Asymptotik der singulären Werte, welche die lokale Inkorrektheit von $F(x) = y$ im Punkt x_0 charakterisiert. Falls jedoch die Multiplikatorfunktion m eine wesentliche Nullstelle aufweist, z.B. bei $t = 0$ und für ein $\nu > 0$ in der Form $m(t) = t^\nu$ oder $m(t) = \exp\left(-\frac{1}{t^\nu}\right)$ $(0 < t \leq 1)$ darstellbar ist, so ist die Bestimmung des lokalen Grades der Inkorrektheit im Punkt x_0 kompliziert. Ausgehend von einer weitreichenden Literaturrecherche zu diesem Problemkreis sollen die Möglichkeiten und Grenzen der Analysis für solche Fragen ausgelotet werden. Weiter sollte die Hypothese untersucht werden, dass derartige Multiplikationsoperatoren mit isolierten Nullstellen in m nur einen begrenzten Einfluss auf Chancen der Regularisierung derartiger nichtlinearer Operatorgleichungen haben. In diesem Kontext spielen Quelldarstellungen gewiss eine Schlüsselrolle. Durch eine Serie systematischer numerischer Fallstudien sollen auch Antworten auf den Teil der Fragen gefunden werden, der sich nicht analytisch behandeln lässt.

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Abstract

In this thesis we deal with the degree of ill-posedness of linear operator equations $Bx = y$ in Hilbert spaces $X = Y = L^2(0,1)$, where B may be decomposed into a compact linear integral operator J with a well-known decay rate of singular values and a multiplication operator M.

This case occurs for example for nonlinear operator equations $F(x) = y$, where $F = N \circ J$. Then the local degree of ill-posedness is investigated via the Fréchet derivative in $x_0 \in D(F)$ which has the form $F'(x_0) = M \circ J$ providing the situation described above.

If the multiplier function has got zeroes, the determination of the local degree of ill-posedness is not trivial. We are going to investigate this situation, provide analytical tools as well as their limitations. By using several numerical approaches for computing the singular values of $F'(x_0)$ we find that the degree of ill-posedness does not change through those multiplication operators. We even provide a conjecture, verified by several numerical studies, how these multiplication operators influence the singular values of $F'(x_0) = M \circ J$.

Finally we analyze the influence of those multiplication operators on the opportunities of Tikhonov regularization and corresponding convergence rates. In this context we also provide a short summary on the relationship between nonlinear problems and their linearizations.

Zusammenfassung

Diese Arbeit beschäftigt sich mit dem Grad der Inkorrektheit linearer Operatorgleichungen der Form $Bx = y$ in Hilberträumen $X = Y = L^2(0,1)$, wobei B als Komposition eines vollstetigen linearen Integraloperators J mit bekannter Abklingrate der Singulärwerte und eines Multiplikationsoperators M dargestellt werden kann.

Dieser Fall tritt beispielsweise bei nichtlinearen Operatorgleichungen $F(x) = y$, wobei $F = N \circ J$. Dann wird der lokale Inkorrektheitsgrad über die Fr*é*chet-Ableitung in $x_0 \in D(F)$ bestimmt, welche mit $F'(x_0) = M \circ J$ die oben beschriebene Form hat.

Falls die Multiplikatorfunktion Nullstellen hat, ist die Bestimmung des lokalen Grades der Inkorrektheit nicht einfach. Möglichkeiten und Grenzen der Analyis für diese Situation werden betrachtet. Unterschiedliche numerische Ansätze für die Bestimmung der Singulärwerte von $F'(x_0)$ liefern das Ergebnis, dass der Grad der Inkorrektheit durch die Multiplikatorfunktionen nicht beeinflusst wird. Es wird sogar ein Zusammenhang gefunden, wie diese Multiplikationsoperatoren die Singulärwerte von $F'(x_0) = M \circ J$ beeinflussen.

Schließlich werden noch die Möglichkeiten der Tikhonov Regularisierung unter Einfluss der Multiplikationsoperatoren untersucht. In diesem Zusammenhang wird auch eine kurze Zusammenfassung zur Beziehung von nichtlinearen Problemen und ihren Linearisierungen gegeben.

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Chapter 1

Introduction

The work described in this diploma thesis deals with the ill-posedness of inverse problems and especially how specific multiplication operators influence the degree of ill-posedness of an integral operator.

Inverse problems are very old and ubiquitous in science, finance and engineering and have received a great deal of attention by applied mathematicians, engineers and statisticians. Most inverse problems cannot be solved analytically, hence numerical methods have to be used. We are interested in the question, how ill-posed a certain problem, namely the inverse problem of an operator $B = M \circ J$, which can be decomposed into a multiplication operator M and an integral operator J, is.

The first three chapters of this thesis contain background material related to inverse problems and the concept of ill-posedness. Chapter 2 deals with inverse problems and ill-posedness in general. In this chapter linear and nonlinear operators are introduced and the origin of multiplication operators in the context of nonlinear operators and its linearizations is described.

Chapter 3 provides properties of linear operators in Hilbert spaces. The singular value decomposition of compact operators is explained which is important for the determination of the degree of ill-posedness. Several examples for integral equations and their degree of ill-posedness are given.

Chapters 4 and 5 provide the main analytical results obtained for the singular value decomposition (and therefore the degree of ill-posedness) of integral operators. Chapter 4 contains a characterization of multiplication operators and several analytical tools for the determination of the singular value decomposition, whereas in chapter 5 the integral operator is transformed into a Sturm-Liouville problem and an overview of relevant results for such problems is given.

Chapter 6 covers the important topic of computing the singular value decomposition for the integral operator numerically. Several approaches are used in order to achieve this. Firstly, finite difference methods are applied to the Sturm-Liouville problem. Then Galerkin and Rayleigh-Ritz methods are used in order to determine the singular value decomposition of the integral equation with multiplication operators. All computations were carried out in MATLAB[1], version 6.1 (R12). Approximation properties are introduced for all three methods and several types of multiplier functions are considered, containing polynomial type as well as exponential type functions. Finally, very good approximations are produced for the singular values and their asymptotic behaviour, which is important for the determination of the degree of ill-posedness.

Chapter 7 examines the influence of multiplication operators M on the regularization of an ill-posed problem. Therefore we consider the nonlinear problem $F(x) = y$ in addition to the linearized problem, i.e. the integral equation with operator $B = M \circ J$. Tikhonov regularization applied to the nonlinear as well as the linear problems and convergence rates properties are considered.

Finally chapter 8 presents a summary of all our analytical and numerical results.

[1] MATLAB is a registered trademark of The MathWorks, Inc.

Chapter 2

Motivation

In this first chapter we want to give a short introduction on inverse problems, and ill-posedness of linear and nonlinear operators. Furthermore we want to describe how multiplication operators develop from nonlinear operator equations and mention some applications in science and finance.

2.1 Inverse Problems and Hadamard's definition of well-posedness

Inverse problems arise in a variety of important applications in science and technology, industry and finance. In this chapter we want to give a short introduction to inverse problems and the concept of well-posedness.

The following explanation of an inverse problem can be found in [33]. An inverse problem is a problem, which is posed in a way that is inverted from that in which most direct problems are posed. The so-called direct problem we have in mind is that of determining the effect y from given causes and conditions x, when a definite mathematical model

$$F(x) = y$$

is given. Hence, in an inverse problem we are looking for special parameters x, i.e. a special cause or a special condition of our mathematical model. The solution of a direct problem is the precise mathematical description of a mathematical problem whereas the solution of an inverse problem can be described as the construction of x from data y. In studying inverse problems we often face a lack of information as it can be seen through many examples (see for example [4], [33], [43], [45] and [80]). An inverse problem does not necessarily have a solution or even a unique solution. Furthermore, if a solution exists, it does not continuously depend on the data, i.e. it is unstable. This leads to the following definition of *well-posedness* for identification problems taken from [4], which was first introduced by Hadamard:

Definition 2.1. *Let X and Y be two Banach spaces and $F : X \mapsto Y$ be a mapping from X into Y. According to Hadamard the operator equation*

$$F(x) = y, \quad x \in D \subset X, \quad y \in Y \tag{2.1}$$

is said to be ***well-posed****, if the following three conditions hold:*

(i) For every $y \in Y$ there exists at least one $x \in D$ satisfying $F(x) = y$ (existence).

(ii) The element x satisfying $F(x) = y$ is uniquely determined in D (uniqueness).

(iii) The solution x depends continuously on the right hand side y (stability).

If one of these conditions is not satisfied, then the problem (2.1) is called ***ill-posed*** *in the sense of Hadamard.*

If equation (2.1) is well-posed, then F has a well-defined, continuous inverse operator F^{-1}. Direct problems are usually well-posed whereas the nature of inverse problems leads to ill-posed problems as a characteristic property. Further explanations, theory and examples on inverse and ill-posed problems can be found in [4], [33], [45], [61] and [81].

2.2 Linear and nonlinear operator equations

In addition we want to introduce the class of bounded linear operators A that plays an important role in understanding direct and inverse problems (see [45]):

Definition 2.2. *An operator $A : X \mapsto Y$ is said to be* ***linear****, if*

$$A(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 A x_1 + \lambda_2 A x_2$$

holds for all $x_1, x_2 \in X$, $\lambda_1, \lambda_2 \in \mathbb{R}$. If there exists a constant $K \geq 0$ such that

$$\|Ax\|_Y \leq K \|x\|_X$$

holds for all $x \in X$, then the linear operator A is called ***bounded****.*

Bounded linear operators A between X and Y form a Banach space denoted by $\mathcal{L}(X,Y)$ (see [41]). The norm in this space is defined by

$$\|A\|_{\mathcal{L}(X,Y)} := \frac{\|Ax\|_Y}{\|x\|_X}.$$

If the operator A is not linear, it is said to be a nonlinear operator. For nonlinear operators we will need the definition of local ill-posedness.

For nonlinear operator equations (2.1) conditions (i) and (ii) of definition 2.1 are generally not satisfied in the global sense. Hence, we need a local definition of ill-posedness, in which we focus on the stable dependence of the solution on the right hand side. This definition is taken from [45]:

Definition 2.3. *The operator equation (2.1) is said to be* ***locally ill-posed*** *in x_0, if for any small radius $r > 0$ there exists an infinite sequence $\{x_n\}$ of elements from the ball $B_r^D(x_0)$, that does not converge to x_0 and whose image set $\{F(x_n)\} \subset Y$ does converge to $F(x_0)$, i.e.*

$$F(x_n) \to F(x_0) \quad in \quad Y, \quad but \quad x_n \not\to x_0 \quad in \quad X \quad for \quad n \to \infty. \tag{2.2}$$

Otherwise (2.1) is called ***locally well-posed****.*

We see, that for local well-posedness, there exits a radius r_0, such that

$$F(x_n) \to F(x_0) \quad \text{in} \quad Y, \quad x_n \in B_{r_0}^D(x_0) \quad \text{but} \quad x_n \to x_0 \quad \text{in} \quad X \quad \text{for} \quad n \to \infty.$$

holds. This corresponds exactly to the condition (iii) in Hadamard's definition of well-posedness. The local ill-posedness is a very disadvantageous property that often appears for nonlinear identification problems for the reconstruction of a solution.

Some examples (see [45]) of nonlinear ill-posed problems, which we will also derive in the next section, are given by operators of the form

$$[[F(x)](s) = c_0 \exp\left(c_1 \int_0^s x(t)dt\right), \quad (0 \le s \le T), \tag{2.3}$$

with $c_0 \neq 0$ and $c_1 \neq 0$. Both in spaces $X = Y = L^2(0,T)$ and $X = Y = C[0,T]$ the operator $F : X \to Y$ with $D = X$ is defined and continuous everywhere in X. But the nonlinear operator equation (2.1) with the operator F and spaces given above are locally ill-posded in all points $x_0 \in X$ according to definition 2.3. We can show this by considering $X = Y = C[0,T]$, $c_0 = c_1 = 1$ and $x_0(t) = 0$, $(0 \le t \le T)$ with $[F(x_0)](s) = 1$, $(0 \le s \le T)$. Then there are sequences of continuous functions

$$x_n(t) = \frac{r \cos nt}{2(1 + \frac{r \sin nt}{2n})}, \, (0 \le t \le T),$$

for arbitrarily small radii $r > 0$, with

$$\|x_n\|_{C[0,T]} \le \frac{r}{2(1 + \frac{r}{2n})} \le r$$

and $x_n \in B_r^D(x_0)$ for n large enough. Furthermore

$$\lim_{n\to\infty} \|x_n\|_{C[0,T]} = \frac{r}{2}$$

and therefore $x_n \not\to x_0$ in $C[0,1]$ for $n \to \infty$. However, the sequence of image functions

$$[F(x_n)](s) = 1 + \frac{r \sin ns}{2n}, \quad (0 \le s \le T)$$

converges in $C[0,1]$ to $F(x_0)$, since

$$\lim_{n\to\infty} \|F(x_n) - F(x_0)\|_{C[0,1]} = \lim_{n\to\infty} \frac{r}{2n} = 0.$$

This corresponds to the definition of local ill-posedness of the operator equation (2.1) in x_0 and therefore the nonlinear operator equation (2.3) is locally ill-posed.

In the next section we are going to state some more examples of nonlinear ill-posed problems and how so-called multiplication operators arise from those nonlinear equations. This motivates the interest in such multiplication operators which we want to examine in this thesis.

2.3 On the origin of multiplication operators

In this work we want to analyze the local ill-posedness properties of nonlinear operator equations

$$F(x) = y, \quad x \in D(F) \subset X, y \in Y \tag{2.4}$$

with an operator F, defined on convex $D(F)$ and mapping between infinite dimensional Hilbert spaces X and Y. We will show, how multiplication operators emerge from this nonlinear operator. To this end we are going to introduce three examples for nonlinear ill-posed problems arising in science and technology as well as in finance. The first example concerning growth rates is mentioned in Groetsch [33], the second example concerning the heat equation can be found in Anger [4]. The last example from financial mathematics was discussed by Hofmann/Hein [38].

Chemical reaction This example can be found in the book by Groetsch [33]. From the kinetics of chemical reactions we know that the change of concentration of a certain material in time is proportional to the current concentration $u(t)$ of this special material for all considered times $t \geq 0$. Let the alteration rate $x(t)$ be variable in time. Then the linear first order ordinary differential equation

$$u'(t) = x(t)u(t)$$

with initial condition

$$u(0) = u_0$$

is valid, where $u_0 > 0$ is the given initial concentration. Notice that this kind of problem does not only appear in chemistry but also in many other applications. The direct problem is to determine the concentration $u(t)$ for $0 \leq t \leq T$, if the initial concentration u_0 and the parameter function $x(t)$, $(0 \leq t \leq T)$ is given. The nonlinear operator F of the direct problem may be written in the form

$$[F(x)](s) = u_0 \exp\left(\int_0^s x(t)dt\right), \quad (0 \leq s \leq T), \tag{2.5}$$

as a composition $F = N \circ J$ as a composition of a nonlinear Nemytskii operator N and the linear operator J given by

$$[J(x)](s) := \int_0^s x(t)dt, \quad (0 \leq s \leq T).$$

Now, the inverse problem is given by the identification of the variable parameter function $x(t)$ in time interval $[0,T]$ on the basis of given measurements of the concentration $u(t)$. This problem regularly arises in applications. We may determine the Fréchet derivative $F'(x_0)$ of F for all points $x_0 \in L^2(0,1)$ (see [45]):

Definition 2.4. *A bounded linear operator $A \in \mathcal{L}(X,Y)$ is called* ***Fréchet derivative*** *of operator $F: D \subseteq X \mapsto Y$ in $x_0 \in int(D)$ if there exists an open ball $B_r(x_0) \subseteq D$ and a positive real functional $\varepsilon : B_r(x_0) \subset X \mapsto \mathbb{R}$ with $\lim_{x \to x_0} = 0$ such that*

$$\|F(x) - F(x_0) - A(x - x_0)\|_Y \leq \varepsilon(x)\|x - x_0\|_X$$

for all $x \in B_r(x_0)$. The operator F is called Fréchet differentiable on the open set $S \subseteq D$, if it has a Fréchet derivative $F'(x_0) := A$ in all points $x_0 \in S$.

The Fréchet derivative of operator F from (2.5), which is defined on $L^2(0,1)$ for $X = Y = L^2(0,1)$ is given by the following composition

$$[F'(x_0)(h)](t) = [F(x_0)](t)[J(h)](t), \quad (0 \leq t \leq 1, h \in X = L^2(0,1)), \tag{2.6}$$

where the integral operator $J \in \mathcal{L}(L^2(0,1), L^2(0,1))$ of first order differentiation is given by

$$[J(h)](t) = \int_0^t h(\tau)d\tau. \tag{2.7}$$

Hence $F'(x_0)$ may be decomposed into $F'(x_0) = M \circ J$ with the convolution operator J and a *multiplication operator* M, which attains the form

$$m(t) := [F(x_0)](t).$$

Notice that $m(t) > 0$ because of the exponential function in (2.5). Hence

$$0 < c \leq |m(t)| \leq C < \infty$$

is valid for all t. We are going to examine another problem with multiplication operators arising in engineering.

Heat conduction This example can be found in the book by Anger [4]. Consider a one dimensional heat conduction problem on the interval $[0,1]$, where $x(t)$, $(0 \leq t \leq T)$ is the coefficient of thermal conductivity, which is variable in time. Then the second order partial differential equation which describes this problem has got the form

$$\frac{\partial u(z,t)}{\partial t} = x(t)\frac{\partial^2 u(z,t)}{\partial z^2}, \quad (0 < z < 1, 0 < t < T),$$

where $u(z,t)$, $(0 \leq z \leq 1, 0 \leq t \leq T)$ describes the temperature field. Obviously this assumption is justified, because the thermal conductivity may vary in time due to temperature and material changes. The initial condition is assumed to be

$$u(z,0) = \sin(\pi z), \quad (0 \leq z \leq 1).$$

Furthermore we assume homogenous boundary conditions given by

$$u(0,t) = u(1,t) = 0 \quad (0 \le t \le T).$$

Let the temperature be measurable in the middle of the rod (i.e. at $z = \frac{1}{2}$), for example with the help of a sensor. Hence, we may observe

$$y(t) := u\left(\frac{1}{2}, t\right) \tag{2.8}$$

Then the operator of the direct problem is given by the nonlinear mapping $x(t) \to y(t)$. Using the initial values and the boundary conditions we may find the solution u to the above problem, which is given by

$$u(z,t) = \sin(\pi z) \exp\left(-\pi^2 \int_0^t x(\tau) d\tau\right).$$

From (2.8) it follows that

$$y(t) = \exp\left(-\pi^2 \int_0^t x(\tau) d\tau\right).$$

The inverse problem is then given by the identification of the thermal conductivity as a function in time on the basis of a time-dependent temperature measuring which was provided by a single sensor in the middle of the temperature field. This problem corresponds to the solution of the nonlinear problem

$$[F(x)](t) = \exp\left(-\pi^2 \int_0^t x(\tau) d\tau\right), \quad (0 \le t \le T), \tag{2.9}$$

which may also be written as a composition $F = N \circ J$ (see previous example). The Fr*é*chet derivative of operator F from (2.9), which is defined on $L^2(0,1)$ for $X = Y = L^2(0,1)$ is given by the following composition

$$[F'(x_0)(h)](t) = -\pi^2 [F(x_0)](t)[J(h)](t), \quad (0 \le t \le 1, h \in X = L^2(0,1)), \tag{2.10}$$

where the integral operator J is given by (2.7). Therefore $F'(x_0)$ may be decomposed into $F'(x_0) = M \circ J$ with the convolution operator J and a multiplication operator M, which attains the form

$$m(t) := -\pi^2 [F(x_0)](t).$$

Notice that $m(t) < 0$ because of the exponential function in (2.9). Therefore, as in the previous example,

$$0 < c \le |m(t)| \le C < \infty$$

is valid for all t. Finally we are going to examine multiplication operators arising in financial mathematics.

Option pricing This example is taken from [38]. We analyze the inverse problem of identifying a purely time-dependent volatility function where the data is given by a maturity-dependent option price function with varying remaining term. This problem is formulated in the following discussion. Let $X = Y = L^2(0,1)$ and the nonlinear operator $F = N \circ J$ defined on

$$D(F) = \{x \in L^2(0,1) : x(t) \geq c \geq 0 \text{ a.e. in } [0,1]\} \tag{2.11}$$

be given by

$$[F(x)](t) = k(t,[J(x)](t)), \quad (0 \leq t \leq 1), \tag{2.12}$$

where J is a simple inner *linear* convolution operator, $J = 1 * x$ given by

$$[J(x)](t) := \int_0^t x(\tau)d\tau, \quad (0 \leq t \leq 1) \tag{2.13}$$

and N is an outer *nonlinear* Nemytskii operator given by

$$[N(z)](t) = k(t,z(t)),$$

where $k(t,s)$, $(t,s) \in [0,1] \times [0,\infty)$ is a sufficiently smooth kernel function. This situation arises for example in financial mathematics in option pricing (see [38]), where the Nemytskii operator is given by the Black-Scholes function. This formula arises in financial mathematics. The fair option prices $u(t)$ on arbitrage-free markets are explicitely given by the Black-Scholes-type formula

$$u(t) = u_{BS}(X,K,r,t,S(t)), \quad (0 \leq t \leq T),$$

where $S(t)$ is given by

$$S(t) := \int_0^t x(\tau)d\tau. \tag{2.14}$$

and 0 is the starting time and T the upper time limit. This formula is based on the Black-Scholes function u_{BS} which is defined as

$$u_{BS}(X,K,r,\tau,s) := \begin{cases} X\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), & (s > 0) \\ \max 0, X - Ke^{-r\tau}, & (s = 0), \end{cases}$$

for a current asset price $X := X(0) > 0$, a fixed strike $K > 0$, a fixes rise-free interest rate $r \geq 0$, the time $\tau \geq 0$ and $s \geq 0$ with

$$d_1 = \frac{\ln\frac{X}{K} + r\tau + \frac{s}{2}}{\sqrt{2}}, \quad d_2 = d_1 - \sqrt{2}$$

and the cumulative density function of the standard normal distribution

$$\Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{\xi^2}{2}} d\xi.$$

The inverse problem is now given by finding the volatility function $x(t)$ which is not directly observable. Then the Black-Scholes function u_{BS} allows to define the Nemytskii operator by

$$[N(v)](t) := u_{BS}(X,K,r,t,v(t)), \quad (0 \leq t \leq T).$$

Therefore the nonlinear operator F is given by the above form (2.12).

The Fr*é*chet derivative of operator F from (2.12) is given by the composition $F'(x_0) = M \circ J$ of the convolution operator J with a *multiplication operator* M described by a multiplier function m in the form

$$[F'(x_0)(h)](t) = m(t)[J(h)](t), \quad (0 \leq t \leq 1, h \in X = L^2(0,1)).$$

The multiplier function attains the form

$$m(t) = \frac{\partial k(t,[J(x_0)](t))}{\partial s}, \quad (0 \leq t \leq 1).$$

We may calculate $m(t)$ for the special kernel function $k(t,z(t)) = [N(z)](t)$ for the Nemytskii operator if $X \neq K$, that means if the current asset price X is not equal to the strike K, i.e. at-the-money-options are excluded, since only for $X \neq K$ the Fr*é*chet derivative has got reasonable characteristics. Then this multiplier function m is given by

$$m(0) = 0, \quad m(t) = \frac{\partial u_{BS}(X,K,r,t,S(t))}{\partial s} \quad (0 < t \leq T).$$

In [38] it has been shown that the linear operator $F'(x_0)$ maps continuously into $L^2(0,1)$ with $m \in L^\infty(0,1)$, but only if $X \neq K$. For this special problem we may derive (see [38])

$$m(t) = \frac{X}{2\sqrt{2\pi S(t)}} \exp\left(-\frac{(v+rt)^2}{2S(t)} - \frac{(v+rt)}{2} - \frac{S(t)}{8}\right) > 0,$$

where $v = \ln\frac{X}{K} \neq 0$. Based on this formula we get

$$\frac{1}{m(t)} = K\sqrt{S(t)}\exp(\beta(t)), \quad (0 < t < T),$$

where

$$\beta(t) = \frac{v^2}{2S(t)} + \frac{r^2t^2}{2S(t)} + \frac{vrt}{S(t)} + \frac{v}{2} + \frac{rt}{2} + \frac{S(t)}{8}, \quad v := \left(\frac{X}{K}\right) \neq 0 \quad \text{for} \quad X \neq K.$$

For $S \in I(D(F))$ from (2.14) with (2.11) we have

$$ct \leq S(t) \leq d\sqrt{t}, \quad (0 < t \leq T),$$

where $d := \|x\|_{L^2(0,1)}$. Then we may estimate

$$\underline{C}\frac{\exp\left(-\frac{v^2}{2ct}\right)}{\sqrt[4]{t}} \leq m(t) \leq \overline{C}\frac{\exp\left(-\frac{v^2}{2d\sqrt{t}}\right)}{\sqrt{t}}, \quad (0 < t \leq T)$$

for some positive constants $\underline{C}$ and $\overline{C}$. Hence, for fixed $v \neq 0$ (i.e. $X \neq K$) the multiplier function $m(t)$ falls exponentially as $t \to 0$ and $m(t) \to 0$ as $t \to 0$. We may find several more examples for multiplication operators, by considering the Fréchet derivative of further nonlinear operators.

We have seen through various examples that the Fréchet derivative of operator F in $x_0 \in D(F)$ is given by a composition of a compact linear integral operator J from (2.13) with well-known singular value asymptotics

$$\sigma_i(J) \sim \frac{1}{i} \quad \text{as} \quad i \to \infty \tag{2.15}$$

and a multiplication operator M. We will explain the concept of *compactness* and *singular values* in the next chapter. We will also determine the singular value expansion of the standard convolution operator J and therefore show that (2.15) is valid. Furthermore we are going to see that for $m \in L^\infty(0,1)$ the compactness of J transfers to the operator $F'(x_0)$. We will also see that the decay rate of singular values of an operator determines its *degree of ill-posedness*. The nature of local ill-posedness of F in x_0 arises from the decay rate of singular values $\sigma_i(F'(x_0))$ of the linear integral operator $F'(x_0)$. If

$$0 < c \leq |m(t)| \leq C < \infty$$

holds (as in the first two examples which we have seen), i. e. if m has got a positive essential infimum, it can be shown that for all points $x_0 \in L^2(0,1)$ the Fréchet derivative $F'(x_0)$ from (2.6) and (2.10) and the compact linear operator J from (2.7) are spectrally equivalent (see [44, Proposition 3] for a proof). Then the asymptotic of the singular values does not change, i.e.

$$\sigma_i(F'(x_0)) \sim \frac{1}{i} \quad \text{as} \quad i \to \infty$$

and we find ill-posedness of degree $\nu = 1$.

Problems arise, if the multiplier function $m(t)$ has got a zero, for example at $t = 0$. This situation occurs for example for

$$m(t) = t^\alpha \quad \text{or} \quad m(t) = \exp\{-\frac{1}{t^\alpha}\}, \quad \alpha > 0.$$

This is the case in the third example which we have seen above. The degree of ill-posedness, which we will define in the next chapter, is very important for measuring the difficulties in solving a linear operator equation. Therefore it is necessary to determine the degree of ill-posedness for the above problem. We will try to find this degree of ill-posedness by considering analytical and numerical methods.

Finally we are going to consider the influence of multiplication operators on the opportunities of regularization of such a nonlinear ill-posed problem.

Chapter 3

Linear operator equations in Hilbert spaces

In this section we consider linear operator equations (2.1) mapping between two Hilbert spaces X and Y:

$$Ax = y, \quad x \in D \subset X, \quad y \in Y \quad A \in \mathcal{L}(X,Y) \tag{3.1}$$

In each Hilbert space we have an inner product $\langle \cdot, \cdot \rangle$ and an induced norm $\| \cdot \|$. In separable Hilbert spaces, which we want to consider, there exist finite or countable orthonormal systems (see for example [45]) which we will need for the lemma on singular value decomposition in the following section.

3.1 Singular value decomposition of compact operators and the Pseudo-Inverse

Many ill-posed problems arising in applications involve compact operators. Therefore we want to give the definition of a compact operator (see [47]):

Definition 3.1. *A bounded linear operator $A \in \mathcal{L}(X,Y)$ is **compact** if and only if the image of any bounded set $T \subset X$ is a relatively compact set $AT = \{y \in Y : y = Ax, x \in T\} \subset Y$.*

Since the operator A is compact it has a non-closed range in Y, i.e. $R(A) \neq \overline{R(A)}$, the inverse operator A^{-1} is unbounded and equation (3.1) is ill-posed. Nashed (see [67]) calls this situation ill-posedness of type II.

The singular value decomposition of compact operators plays an important role for studying the Pseudo-Inverse of those operators. In order to formulate a lemma on singular value decomposition we need the definition of *adjoint operators* (see [16]).

Definition 3.2. *The operator $A^* \in \mathcal{L}(Y,X)$ is called the **adjoint operator** of $A \in \mathcal{L}(X,Y)$ if*

$$\langle Au, v \rangle_Y = \langle u, A^* v \rangle_X$$

holds for all $u \in X$ and $v \in Y$.

The following considerations are taken from Hofmann [45]. If $A = A^*$ holds, the operator is called *self-adjoint*. It is easy to see, that for any operator $A \in \mathcal{L}(X,Y)$ the products $A^*A \in \mathcal{L}(X,X)$ as well as $AA^* \in \mathcal{L}(Y,Y)$ are self-adjoint.

If we consider eigenvalues $\lambda \in \mathbb{R}$ and eigenelements $u \in X$ with $u \neq \mathbf{0}$ of a self-adjoint operator A, which satisfy the eigenvalue equation

$$Au = \lambda u$$

then eigenelements u_1 and u_2 belonging to different eigenvalues λ_1 and λ_2 are orthogonal (see for example Heuser [41]). Each operator $A \in \mathcal{L}(X,Y)$ with finite dimensional image space, $\dim(R(A)) = m$, is compact. Then the operators A^*A and AA^* have a finite sequence of non-negative eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0,$$

and orthonormal systems of eigenelements

$$\{u_j\}_{j=1}^m \subset X \quad \text{with} \quad A^*Au_i = \lambda_i u_i \quad (i = 1, \ldots, m)$$

and

$$\{v_j\}_{j=1}^m \subset Y \quad \text{with} \quad AA^*v_i = \lambda_i v_i \quad (i = 1, \ldots, m).$$

On the other hand, for a compact operator with $\dim(R(A)) = \infty$, A^*A as well as AA^* have an infinite sequence of positive eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq \ldots \to 0, \quad \text{for} \quad n \to \infty$$

and corresponding infinite orthonormal systems of eigenelements

$$\{u_j\}_{j=1}^\infty \subset X \quad \text{and} \quad \{v_j\}_{j=1}^\infty \subset Y,$$

where $\{u_j\}_{j=1}^\infty$ is a complete orthonormal system in $\overline{R(A^*)} = \overline{R(A^*A)} = N(A)^\perp$, a subspace of the Hilbert space X. Similarly $\{v_j\}_{j=1}^\infty$ is a complete orthonormal system in $\overline{R(A)} = \overline{R(AA^*)} = N(A^*)^\perp$, a subspace of the Hilbert space Y. From those considerations we can follow, that, if $A \in \mathcal{L}(X,Y)$ is compact then the adjoint operator $A^* \in \mathcal{L}(X,Y)$ is compact, too. For further details on spectral analysis of compact operators, see [1].

Using these properties of compact self-adjoint operators we can formulate the following lemma about singular value decomposition which is taken from [45]:

Lemma 3.1. *Let $A \in \mathcal{L}(X,Y)$ be a compact operator between the separable Hilbert spaces X and Y. Then there exist sets of indices $J = \{1, \ldots, m\}$ for $\dim(R(A)) = m$ and $J = \mathbb{N}$ for $\dim(R(A)) = \infty$, orthonormal systems $\{u_j\}_{j \in J}$ in X and $\{v_j\}_{j \in J}$ in Y and a sequence $\{\sigma_j\}_{j \in J}$ of positive real numbers with the following properties:*

$$\{\sigma_j\}_{j \in J} \quad \textit{is non-increasing and} \quad \lim_{j \to \infty} \sigma_j = 0 \quad \textit{for} \quad J = \mathbb{N} \tag{3.2}$$

$$Au_j = \sigma_j v_j \, (j \in J) \quad \textit{and} \quad A^* v_j = \sigma_j u_j \, (j \in J). \tag{3.3}$$

For all $x \in X$ there exists an element $x_0 \in N(A)$ with

$$x = x_0 + \sum_{j \in J} \langle x, u_j \rangle_X u_j \quad \textit{and} \quad Ax = \sum_{j \in J} \sigma_j \langle x, u_j \rangle_X v_j. \tag{3.4}$$

Furthermore

$$A^* y = \sum_{j \in J} \sigma_j \langle y, v_j \rangle_Y u_j \tag{3.5}$$

holds for all $y \in Y$.

For a proof of this lemma, see for example [16]. The lemma states, that for each compact operator there exists a singular system according to the following definition taken from [45]:

Definition 3.3. *Let X and Y be separable Hilbert spaces and $A \in \mathcal{L}(X,Y)$ be a compact operator. Then a* ***singular system*** *for this compact operator is a countable set of triples*

$$\{\sigma_j, u_j, v_j\}_{j \in J}, \tag{3.6}$$

where the index set J is given by $J = \{1, \ldots, m\}$ for dim$(R(A)) = m$ and $J = \mathbb{N}$ for dim$(R(A)) = \infty$. It holds $\sigma_j > 0$ and $u_j \in X$, $v_j \in Y$ as defined in the previous lemma, satisfying conditions (3.2)-(3.5). Furthermore the positive numbers

$$\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_n(A) \geq \ldots \to 0, \quad \textit{for} \quad n \to \infty$$

are called ***singular values*** *or s-numbers of the compact operator A and the decomposition, defined in formula (3.4), is called* ***singular value decomposition*** *of the operator.*

We are going to summarize some properties of s-numbers or singular values, taken from [56] and [58]: Any singular value of a compact operator A satisfies

1. $\|A\| = \sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq 0, \quad A \in \mathcal{L}(X,Y)$,
2. $\sigma_{n+m-1}(A+B) \leq \sigma_n(A) + \sigma_m(B), \quad A, B \in \mathcal{L}(X,Y), \quad m, n \in \mathbb{N}$,
3. $\sigma_n(CAB) \leq \|C\| \sigma_n(A) \|B\|$ for $B \in \mathcal{L}(X_0, X), A \in \mathcal{L}(X,Y), C \in \mathcal{L}(Y, Y_0)$, $n = 1, 2, \ldots$,
4. $\sigma_n(A) = 0$ for $\operatorname{rank}(A) < n$,
5. $\sigma_n(I) = 1$, where I denotes the identical map and
6. the s-numbers are continuous functions, namely

$$|\sigma_n(A) - \sigma_n(B)| \leq \|A - B\|, \quad \forall A, B \in \mathcal{L}(X,Y).$$

From

$$A^*Au_j = \sigma_j^2 u_j \quad \text{and} \quad AA^*v_j = \sigma_j^2 v_j \tag{3.7}$$

and the previous consideration follows, that that the singular values σ_j are the square roots of the positive eigenvalues of A^*A as well as AA^* and

$$\sigma_j \to 0, \quad \text{as} \quad j \to \infty. \tag{3.8}$$

The orthonormal systems $\{u_j\}_{j\in J}$ and $\{v_j\}_{j\in J}$ are complete in $\overline{R(A^*)}$ and $\overline{R(A)}$. We can use these results in order to find expressions for the solution of the linear operator equation (3.1) and to construct a pseudo-inverse of the compact operator A. Expressing the right hand side $y \in Y$ of the operator equation as a Fourier series and using the singular value decomposition (3.4) we get the following form of (3.1):

$$\sum_{j\in J} \sigma_j \langle x, u_j\rangle_X v_j = y_0 + \sum_{j\in J} \langle y, v_j\rangle_Y v_j, \tag{3.9}$$

where $y_0 \in N(A^*)$. As a necessary condition for the existence of a solution we therefore have $y_0 = \mathbf{0}$ or $y \in N(A^*)^\perp$. Furthermore

$$\sigma_j \langle x, u_j\rangle_X = \langle y, v_j\rangle_Y \tag{3.10}$$

must hold for all $j \in J$ and therefore, using (3.4) we get

$$x = x_0 + \sum_{j\in J} \frac{\langle y, v_j\rangle_Y}{\sigma_j} u_j, \tag{3.11}$$

where $x_0 \in N(A)$ is arbitrary. The element x from (3.11) must be an element of the Hilbert space X which corresponds to the *Picard condition*

$$\sum_{j=1}^{\infty} \frac{\langle y, v_j\rangle_Y^2}{\sigma_j^2} < \infty.$$

So there is a solution to (3.1) if and only if $y \in N(A^*)^\perp$ and for $\dim(R(A)) = \infty$ the Picard condition holds. For $x_0 = \mathbf{0}$ in (3.11) we have a unique solution of minimum norm (see [45]),

$$x_{mn} = A^\dagger y = \sum_{j\in J} \frac{\langle y, v_j\rangle_Y}{\sigma_j} u_j, \quad y \in R(A) \oplus R(A)^\perp, \tag{3.12}$$

where $A^\dagger$ is the so-called *Moore-Penrose* inverse or generalized inverse of A. Note, that $R(A)$ is the smaller the faster σ_j tends to zero as $j \to \infty$, since the Picard condition has to be satisfied. In addition the influence of a perturbed right-hand side y is the stronger the faster the σ_j tend to zero. An important class of compact operators are the so-called *Hilbert-Schmidt operators* [43]:

Definition 3.4. *An operator $A \in \mathcal{L}(X,Y)$ mapping between separable Hilbert spaces X and Y is called a* ***Hilbert-Schmidt operator*** *if, for an arbitrarily chosen complete orthonormal system $\{e_j\} \subset X$,*

$$S(A) := \sum_{j=1}^{\infty} \|Ae_j\|_Y^2 < \infty$$

holds.

From this definition we immediately get the following lemma (see [16], [43] and [45]):

Lemma 3.2. *Any Hilbert-Schmidt operator is compact. Furthermore for all complete orthonormal systems $\{e_j\} \subset X$ we have*

$$S(A) = \sum_{j=1}^{\infty} \|Ae_j\|_Y^2 = \sum_{j=1}^{\infty} \sigma_j^2 < \infty.$$

As a consequence of lemma 3.2 we obtain that for any Hilbert-Schmidt operator the sum $S(A)$ is independent of the choice of the orthonormal system $\{e_j\}$. Therefore we could define a so-called Hilbert-Schmidt norm

$$\|A\|_{HS} := S(A) = \sum_{j=1}^{\infty} \sigma_j^2 < \infty,$$

which we will need in a later section. The so-called *linear Fredholm integral operator* A mapping from $X = L^2(0,1)$ into $Y = L^2(0,1)$, defined by

$$[Ax](s) = \int_0^1 k(s,t)x(t)dt \tag{3.13}$$

with a quadratically integrable kernel $k(s,t) \in L^2((0,1) \times (0,1))$ is a Hilbert-Schmidt operator, where

$$S(A) = \int_0^1 \int_0^1 (k(s,t))^2 dtds = \sum_{j=1}^{\infty} \sigma_j^2 < \infty$$

and the kernel can be expressed by

$$k(s,t) = \sum_{j=1}^{\infty} \sigma_j u_j(t) v_j(s).$$

For further details, see [16]. We are specifically interested in the singular values of an operator, in order to regularize ill-posed problems. Unfortunately, it is not always easy to determine the singular values (or their decay rate, see next section). We can use the results on singular values in order to characterize the degree of ill-posedness of a linear operator equation.

3.2 The degree of ill-posedness of linear operator equations

The level of difficulties in solving a linear operator equation (3.1) with compact operator A in Hilbert spaces may be expressed by the *degree of ill-posedness* of this operator. We will consider injective compact linear operators. From (3.12) we see, that the problem of calculating the inverse (or Pseudo-Inverse) of an operator A gets worse, the faster the σ_j tend to zero. Let the operator equation $Ax = y$ be given and the Pseudo-Inverse $A^\dagger$ be applied to a perturbed right hand side $y_\delta = y + \delta$ for a small δ. Then the error in the solution is given by

$$\begin{aligned} \|x^\delta_{mn} - x_{mn}\|_X &= \|A^\dagger y_\delta - A^\dagger y\|_X \\ &\leq \|A^\dagger\|_{\mathcal{L}(Y,X)} \cdot \delta. \end{aligned}$$

So even for small values of δ, there might be a huge error, since $\|A^\dagger\|_{\mathcal{L}(Y,X)}$ may become very large, depending on the convergence of the singular values σ_j.

Hence, the degree of ill-posedness is given by the rate of convergence of the singular values σ_j to zero as $j \to \infty$. The faster σ_j tends to zero, the more instable is the problem. Therefore we get the following definition of the *degree of ill-posedness* (see [45]):

Definition 3.5. *A linear operator equation (3.1) with compact operator $A \in \mathcal{L}(X,Y)$ and separable Hilbert spaces X and Y has got a **degree of ill-posedness** of $\nu > 0$ if there exist constants $0 \leq \underline{C} \leq \overline{C} < \infty$, such that*

$$\underline{C} n^\nu \leq \frac{1}{\sigma_n} \leq \overline{C} n^\nu, \quad n = 1, 2, \ldots \tag{3.14}$$

This means that the decay rate of singular values is proportional to $n^{-\nu}$,i.e. $\sigma_n \sim n^{-\nu}$. If only the lower bound in (3.14) is satisfied, i.e. $\sigma_n = O(n^{-\nu})$, then the operator equation is called *at least* ill-posed of degree $\nu > 0$. If only the upper bound is satisfied, the equation is said to be *at most* ill-posed of degree $\nu > 0$. Later, especially for our problem, we will see, that it is hard to find upper bounds on the degree of ill-posedness analytically, whereas it is much easier to determine lower bounds.

Moreover, an operator equation is called mildly ill-posed if $\nu \leq 1$, moderately ill-posed for $1 < \nu < \infty$ and severely ill-posed for infinite ν.

Unfortunately it is not always possible to characterize the decay rate of singular values by a single constant ν. Therefore Hofmann and Tautenhahn [51] introduced the so-called *interval of ill-posedness*:

Definition 3.6. *We call the finite or infinite interval*

$$[\underline{\mu}(A), \overline{\mu}(A)] := \left[\liminf_{n\to\infty} \frac{-\log \sigma_n(A)}{\log n}, \limsup_{n\to\infty} \frac{-\log \sigma_n(A)}{\log n} \right]$$

*the **interval of ill-posedness** of the ill-posed linear operator equation (3.1).*

We now want to find bounds on the degree of ill-posedness for our problem. In order to illustrate the degree of ill-posedness, we consider several examples of integral equations of the first kind, where the asymptotics of singular values is known.

3.3 Integral equations of the first kind and their degree of ill-posedness

In this section we want to give a short introduction to integral equations of the first kind and, since this type of equation is a common mathematical model for inverse problems, we want to discuss their degree of ill-posedness according to the definition given in the previous section.

3.3.1 Linear Volterra and Fredholm integral operators

We consider the Hilbert spaces $X = Y = L^2(0,1)$. A linear Volterra integral equation of the first kind is given by

$$[Ax](s) = \int_0^s k(s,t)x(t)dt = y(s) \quad (0 \le s \le 1), \tag{3.15}$$

which is a subset of linear Fredholm integral equations of the first kind

$$[Ax](s) = \int_0^1 k(s,t)x(t)dt = y(s) \quad (0 \le s \le 1), \tag{3.16}$$

where the kernel $k(s,t)$ defined in $(s,t) \in [0,1] \times [0,1]$ vanishes in the triangle $\{(s,t) : 0 \le s < t \le 1\}$ for the Volterra integral equations. For a Fredholm operator equation the integral is taken over a fixed interval, whereas the integral in a Volterra integral equation is taken over a variable interval, which changes in s. Analytical and numerical methods on Volterra equations can be found in [59].

Connected with the above equations is an eigenvalue problem, as stated generally in the previous section. Here we are going to write it down especially for integral equations.

$$\lambda x(s) = \int_0^1 k(s,t)x(t)dt \quad (0 \le s \le 1)$$

is an eigenvalue problem for integral equations. A value of λ for which this equation has got a non-null solution will be called an eigenvalue of the kernel $k(s,t)$, and the corresponding function $x(s)$ will be called an eigenfunction or characteristic function. The set of eigenvalues of a kernel, together with zero, is called the spectrum of $k(s,t)$. If $k(s,t)$ is not identically zero and $k(s,t) = k(t,s)$, i.e. a symmetric kernel in the real case or $k(s,t) = \overline{k(t,s)}$, i.e. a Hermitian kernel in the complex case, then we know, that two eigenfunctions $x_1(t)$ and $x_2(t)$ corresponding to two different eigenvalues are orthogonal, that is

$$\int_0^1 x_1(t)x_2(t)dt = 0.$$

Furthermore a kernel $k(s,t)$ and the corresponding operator A are said to be normal , if $A^*A = AA^*$. Hermitian kernels, which we will see later, are normal.

In general, the set of non-zero eigenvalues of a kernel $k(s,t)$ is countable and there is no non-zero limit point (or cluster point) for the eigenvalues. Furthermore we know that the eigenvalues of a Hermitian kernel are real. We may obtain bounds on any eigenvalue, which continuous kernel may possess (see [6]:

$$\lambda \leq \sup_{0 \leq s,t \leq 1} |k(s,t)|.$$

For square integrable kernel functions we obtain

$$\lambda \leq \left(\int_0^1 \int_0^1 |k(s,t)|^2 ds dt \right)^{\frac{1}{2}}.$$

Finally we are interested in the degree of smoothness of any eigenfunction. Suppose that $x(t)$ is an eigenfunction corresponding to eigenvalue λ. Consider the function

$$F(s) = \int_0^1 k(s,t)p(t)dt \quad (0 \leq s \leq 1),$$

where $p(t)$ is absolutely integrable, that is,

$$\int_0^1 |p(t)|dt < \infty.$$

If we suppose that $k(s,t)$ is continuous we may show that

$$\begin{aligned} |F(s_1) - F(s_2)| &\leq \sup_{0 \leq t \leq 1} |k(s_1,t) - k(s_2,t)| \int_0^1 |p(t)|dt \\ &\leq \varepsilon \int_0^1 |p(t)|dt =: \varepsilon_1. \end{aligned}$$

and ε_1 may be made as small as required. Thus $F(s)$ is continuous. If we now set $p(s) = \frac{1}{\lambda}x(s)$, then $F(s)$ becomes the eigenfunction $x(s)$ and $x(s)$ is continuous if it is absolutely integrable. Furthermore, a function which is square-integrable over a finite range is also absolutely integrable.

If we proceed further, we may show that $x'(s)$ exists if the eigenfunction $x(s)$ is absolutely integrable and $\lambda \neq 0$, whenever the derivative $k_s(s,t)$ is continuous. If $k(s,t)$ has got a finite jump at $s = t$ but is continuous for $0 \leq s \leq t$ and $t \leq s \leq 1$, we consider both areas separately and get the same results.

It is sometimes possible to investigate the higher differentiability of eigenfunctions of kernels in a similar fashion.

Moreover, there are integral equations of the second kind, where the unknown function does not only appear in the integral, i.e.

$$y(s) = x(s) + \int_0^s k(s,t)x(t)dt \quad (0 \leq s \leq 1), \tag{3.17}$$

is a Volterra integral equation of the second kind and

$$y(s) = x(s) + \int_0^1 k(s,t)x(t)dt \quad (0 \le s \le 1), \tag{3.18}$$

is a Fredholm integral equation of the second kind. Usually equations of the first kind represent ill-posed problems and equations of the second kind are well-posed and therefore easier to solve (see [16]). For more details on integral equations, see [16], [20] and [42]. For integral equations of the first kind in special, see [8].

3.3.2 Some examples

We are going to consider some special Fredholm and Volterra integral operators of the first kind and we are going to determine their singular value decomposition according to (3.4) and therefore their degree of ill-posedness.

A Volterra integral operator with kernel $k(s,t) = 1$

We consider the compact operator $A : L^2(0,1) \to L^2(0,1)$

$$[Ax](s) = \int_0^s x(t)dt, \quad (0 \le s \le 1). \tag{3.19}$$

The adjoint operator A^* is given by

$$[A^*y](t) = \int_t^1 y(s)ds, \quad (0 \le t \le 1),$$

and hence we can state the eigenvalue equation

$$[A^*Au](\tau) = \sigma^2 u(\tau) = \int_\tau^1 \int_0^s u(t)dtds, \quad (0 \le \tau \le 1),$$

which leads to the boundary value problem

$$-u(t) = \sigma^2 u''(t) \quad \text{with} \quad u(1) = u'(0) = 0.$$

For details, we refer to [71]. Then we find the following solution for the singular system

$$\sigma_n = \frac{2}{(2n-1)\pi}, \quad n = 1,2,\ldots$$

and

$$u_n(t) = \sqrt{2}\cos\left(n - \frac{1}{2}\right)\pi t \quad \text{and} \quad v_n(t) = \sqrt{2}\sin\left(n - \frac{1}{2}\right)\pi t.$$

Since

$$\frac{1}{\pi}\frac{1}{n} \le \sigma_n \le \frac{2}{\pi}\frac{1}{n}$$

the degree of ill-posedness of this problem is $\nu = 1$.

A Fredholm integral operator with continuous kernel

We will consider the Fredholm integral operator $A : L^2(0,\pi) \to L^2(0,\pi)$, given by

$$[Ax](s) = \int_0^\pi k(s,t)x(t)dt, \quad (0 \le s \le \pi), \tag{3.20}$$

where

$$k(s,t) = \begin{cases} \frac{1}{\pi}(\pi - s)t, & (0 \le t \le s \le \pi) \\ \frac{1}{\pi}(\pi - t)s, & (0 \le s \le t \le \pi). \end{cases} \tag{3.21}$$

Since the kernel (3.21) is real-valued, continuous and symmetric, A is a self-adjoint compact operator. We can transform this integral equation into a boundary value problem (as it was done in [71]) in order to get the eigenvalue equation $Ax = \lambda x$. The boundary value problem is then given by

$$\lambda x''(t) + x(t) = 0, \quad \text{with} \quad x(0) = x(\pi) = 0.$$

From that we may calculate the eigenvalues and eigenfunctions and, since A is a symmetric operator, the singular system, which is given by

$$\lambda_n = \sigma_n = \frac{1}{n^2}, \quad n = 1, 2, \dots$$

and

$$u_n(t) = v_n(t) = \sqrt{\frac{2}{\pi}}\sin(nt).$$

Therefore, the degree of ill-posedness of this problem is $\nu = 2$. Considering the Volterra integral operator from the previous example again, we see, that we can express that operator as a Fredholm integral operator with kernel

$$k(s,t) = \begin{cases} 1, & (0 \le t \le s \le 1) \\ 0, & (0 \le s \le t \le 1), \end{cases} \tag{3.22}$$

which is not continuous along the diagonal $s = t$. The decay rate of its singular values is $\frac{1}{n}$. Now, the kernel (3.21) that we have considered in this part is continuous, but yields a discontinuity in the first derivative for $s = t$. Since the asymptotics of the singular values of this operator is $\frac{1}{n^2}$, we conjecture a connection between the smoothness of the kernel and the decay rate of the singular values of an integral operator. The smoother the kernel, the worse is the degree of ill-posedness. This conjecture is partly verified by Chang's proposition (4.2) in the next chapter.

An integral operator with infinite degree of ill-posedness

Finally we are going to consider an example for an integral operator equation with $\nu = \infty$ as degree of ill-posedness. It is taken from [61]. Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad (x,t) \in [0,\pi] \times [0,1],$$

with homogeneous boundary conditions

$$u(0,t) = u(\pi,t) = 0.$$

Let $u(x,1) = f(x)$ be the given temperature at time $t = 1$. The task is to determine the initial temperature $u(x,0) = g(x)$, hence the operator $A : L^2(0,\pi) \to L^2(0,\pi)$ assigns the solution of the heat equation at time $t = 0$ to the solution at time $t = 1$. The solution of the heat equation is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin(nx)$$

with

$$a_n = \frac{2}{\pi} \int_0^{\pi} u(x,0) \sin(ny) dy.$$

Then $u(x,1)$ is given by

$$u(x,1) = f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2} \sin(nx) \int_0^{\pi} g(x) \sin(ny) dy := [Ag](x).$$

This is an integral operator and from its kernel we can determine the singular value decomposition singular values

$$\sigma_n = e^{-n^2}, \quad n = 1,2,\ldots$$

and singular functions

$$u_n(t) = v_n(t) = \sqrt{\frac{2}{\pi}} \sin(nt).$$

The degree of ill-posedness of this operator is $\nu = \infty$. We are now going to consider a special operator for which we want to determine the decay rate of singular values, i.e. its degree of ill-posedness.

Chapter 4

Asymptotics of singular values of integral equations with multiplication operators

We are looking for the singular value asymptotics of the composite operator $B : L^2(0,1) \to L^2(0,1)$, defined by

$$[Bx](s) := m(s) \left(\int_0^s x(t)dt \right), \quad (0 \le s \le 1) \tag{4.1}$$

for special multiplier functions $m(s)$. We can write (4.1) as an operator equation $B := M \circ J$, with the integral operator J as in (3.19) or (2.13). We are going to state some definitions and results for these multiplication operators.

4.1 Characterization of multiplication operators

Multiplication operators occur in various fields of mathematics, typically as coefficient operators for differential or integral equations. The following definition is taken from [21] (see also [23] and [22]).

Definition 4.1. *Let X and Y be Hilbert spaces and Φ be a Banach space over the same set Ω. Then for every $m \in \Phi$ the operator*

$$\mathcal{M} : X \to Y \tag{4.2}$$

with

$$[\mathcal{M}x](s) := m(s) \cdot x(s), \quad s \in \Omega \tag{4.3}$$

is called a ***multiplication operator*** *for all $x \in X$, for which the product is defined.*

In the following we want to consider the case that Ω is a bounded subset of $\mathbb{R}^1$. Furthermore we set $X = Y = L^2(0,1)$. Then the operator of multiplication $\mathcal{M} : L^2(0,1) \to$

$L^2(0,1)$ with essentially bounded m, i.e. $m \in L^\infty(0,1)$, is defined by

$$[\mathcal{M}x](s) := m(s) \cdot x(s), \quad s \in (0,1) \quad a.e. \tag{4.4}$$

A function f belongs to the Hilbert space $L^p(0,1)$ if

$$\int_0^1 |f(x)|^p dx < \infty$$

holds. Multiplier function m has to be chosen from $\Phi = L^\infty(0,1)$, which is a Banach space with norm

$$\|m\|_{L^\infty(0,1)} := \operatorname{ess}\sup_{s\in(0,1)} |m(s)|,$$

in order to guarantee the boundedness of the operator $\mathcal{M}$ and that the product $\mathcal{M}x$ is in $L^2(0,1)$ for all $x \in L^2(0,1)$. This is a consequence from the following proposition (see [39]).

Proposition 4.1. *Let $\mathcal{M} : L^p(0,1) \to L^q(0,1)$, $1 \le q \le p < \infty$ be given by*

$$[\mathcal{M}x](s) := m(s) \cdot x(s) \quad s \in [0,1]. \tag{4.5}$$

If

$$m \in L^r(0,1) \quad with \quad r := \begin{cases} \frac{pq}{p-q}, & q < p \\ \infty, & q = p \end{cases} \tag{4.6}$$

holds, then $\mathcal{M} \in \mathcal{L}(L^p(0,1), L^q(0,1))$ with $\|\mathcal{M}\| \le \|m\|_{L^r(0,1)}$.

Proof. Let $q < p$ and r as chosen above. Then

$$\begin{aligned} \|\mathcal{M}x\|_{L^q}^q &= \|mx\|_{L^q}^q \\ &= \int_0^1 |m(t)|^q |x(t)|^q dt \\ &\le \||m|^q\|_{L^{\hat{r}}} \||x|^q\|_{L^{\frac{p}{q}}} \end{aligned}$$

with

$$\frac{1}{\hat{r}} + \frac{1}{\frac{p}{q}} = 1, \quad \text{i.e.}\, \hat{r} = \frac{p}{p-q}.$$

Furthermore

$$\||x|^q\|_{L^{\frac{p}{q}}} = \left(\int_0^1 |x|^{q\frac{p}{q}}\right)^{\frac{q}{p}} = \|x\|_{L^p}^q$$

and

$$\||m|^q\|_{L^{\hat{r}}} = \left(\int_0^1 |m|^{\frac{pq}{p-q}} dt\right)^{\frac{p-q}{p}} = \|m\|_{L^r}^q.$$

Therefore we get

$$\|\mathcal{M}x\|_{L^q}^q \le \|m\|_{L^r}^q \|x\|_{L^p}^q,$$

i.e. $\mathcal{M}$ is a continuous operator from $L^p(0,1)$ into $L^q(0,1)$ and $\|\mathcal{M}\| \leq \|m\|_{L^r}$. If $p=q$ then $m \in L^\infty(0,1)$ and

$$\|\mathcal{M}x\|_{L^p} \leq \|m\|_{L^\infty}\|x\|_{L^p}.$$

□

Hence, $\mathcal{M}$ is a bounded linear operator with

$$\|\mathcal{M}\|_{\mathcal{L}(L^p(0,1),L^q(0,1))} = \|m\|_{L^\infty(0,1)}.$$

In our case we have $p=q=2$ and therefore $\mathcal{M}$ is bounded. If m is a piecewise continuous function, then $m \in L^\infty(0,1)$ always holds. For $m \in L^\infty(0,1)$ the compactness of J is passed on to the bounded linear operator B in $B = M \circ J$. This follows from the following theorem.

Theorem 4.1. *Let $B : L^2(0,1) \to L^2(0,1)$ be given by $B = M \circ J$. Then B is a compact operator, if J is compact and $m \in L^\infty(0,1)$.*

Proof. The compactness of B is shown by proving

$$x_n \rightharpoonup x \Longrightarrow Bx_n \to Bx \quad \text{in} \quad L^2(0,1).$$

Let $\{x_n\} \in L^2(0,1)$ be a sequence that converges weakly against an element $x \in L^2(0,1)$. From the compactness of $J : L^2(0,1) \to L^\infty(0,1)$, which was shown in [29], follows

$$\|Jx_n - Jx\|_{L^\infty(0,1)} \to 0 \quad \text{as} \quad n \to \infty.$$

Now, choose $0 < r < 2$ arbitrarily and consider

$$\begin{aligned}
\|Bx_n - Bx\|^2_{L^2(0,1)} &= \int_0^1 m^2(t)(J(x_n-x)(t))^2 dt \\
&\leq \|Jx_n - Jx\|^r_{L^\infty(0,1)} \int_0^1 m^2(t)(J(x_n-x)(t))^{2-r} dt \\
&\leq \|Jx_n - Jx\|^r_{L^\infty(0,1)} \int_0^1 m^2(t) t^{\frac{2-r}{2}} \|x_n - x\|^{2-r}_{L^2(0,1)} dt \\
&= \|Jx_n - Jx\|^r_{L^\infty(0,1)} \|x_n - x\|^{2-r}_{L^2(0,1)} \int_0^1 m^2(t) t^{\frac{2-r}{2}} dt.
\end{aligned}$$

Since $\|x_n - x\|_{L^2(0,1)}$ is bounded we may follow $Bx_n \to Bx$ in $L^2(0,1)$ if and only if

$$\int_0^1 m^2(t) t^{\frac{2-r}{2}} dt < \infty. \tag{4.7}$$

Using the same arguments as in the proof of proposition 4.1 (with $p=q=2$) we get that

$$\int_0^1 m^2(t) t^{\frac{2-r}{2}} dt \leq \|m\|^2_{L^\infty(0,1)} \|t^{\frac{2-r}{4}}\|^2_{L^2(0,1)}.$$

Hence compactness follows for $m \in L^\infty(0,1)$. □

Hence, with proposition 4.1 and theorem 4.1 the operator $B = M \circ J$ is a compact bounded linear operator, if the multiplier function m satisfies $m \in L^\infty(0,1)$. This holds especially for continuous multiplier functions $m \in C[0,1]$ and square integrable multiplier functions $m \in L^2(0,1)$ (see [60]).

Furthermore we may remark that if $B : C[0,1] \to C[0,1]$ and $m \in C[0,1]$, then B is a compact Volterra integral operator. This may be proved via the theorem of Arzela-Ascoli, for which we refer to [60, page 180].

In the case of a positive essential infimum of m we can easily divide by m and the degree of ill-posedness of operator B is the same as for operator J, i.e. $\nu = 1$ (see section 3.3.2 for details). If $m(s)$ has got zeros, we cannot just divide by $m(s)$ and say that the decay rate of singular values is the same as the one for operator J. Problems are going to occur during the calculation of the singular values and therefore for the determination of the degree of ill-posedness.

A sequence of special multiplier functions $m \in L^\infty(0,1)$, which we want to consider is given by

$$m(s) = s^\alpha \quad \text{or} \quad m(s) = \exp\left(-\frac{1}{s^\alpha}\right), \quad \alpha > 0. \tag{4.8}$$

The second, exponential type of multiplier function with $\alpha = 1$ arises in option 03 (see [38]).

We have illustrated both types of multiplier functions for different values of α in figures (4.1) and (4.2). We want to find out, how these multiplier functions influence the

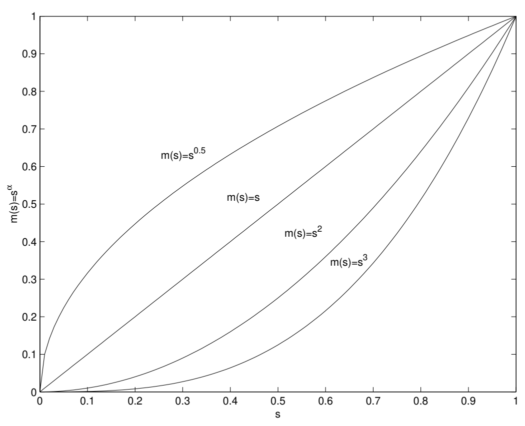

Figure 4.1: *Illustration of* $m(s) = s^\alpha$

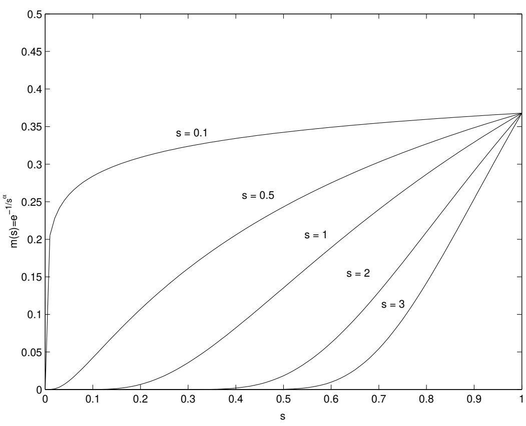

Figure 4.2: *Illustration of* $m(s) = e^{-\frac{1}{s^\alpha}}$

decay rate of singular values of an integral operator. Obviously both types of multiplier functions satisfy $m \in L^\infty(0,1)$ for $\alpha > 0$.

4.2 Lower bounds on the degree of ill-posedness

4.2.1 Approach using Chang's theorem

We know the singular value decomposition of the operator J, given by

$$[Jx](s) = \int_0^s x(t)dt, \quad (0 \leq s \leq 1) \tag{4.9}$$

in $X = Y = L^2(0,1)$. The problem corresponds to finding the first derivative of an observed function $y(t)$, i.e. $x(t) = y'(t)$. The singular system of this operator is given by

$$\left\{ \frac{2}{(2n-1)\pi}; \sqrt{2}\cos\left(n - \frac{1}{2}\right)\pi t; \sqrt{2}\sin\left(n - \frac{1}{2}\right)\pi t \right\}_{n=1}^{\infty}, \quad (0 \leq s \leq 1).$$

For the derivation we use (3.7) and the corresponding eigenvalue problem (see previous chapter). Hence, the operator J from (4.9) satisfies

$$\sigma_n(J) \sim n^{-1}$$

and is therefore ill-posed of degree $\nu = 1$. We want to examine, whether the degree of ill-posedness, i.e. the decay rate of the singular values can be destroyed by a multiplication operator, such as given in (4.1)-(4.8).

We therefore are going to consider some previous results. For a so-called *fractional integral operator* J_r (see [29]), which is given by

$$[J_r x](s) := \int_0^s \frac{(s-t)^{r-1}}{\Gamma(r)} x(t)dt, \quad (0 \leq s \leq 1), \tag{4.10}$$

in $X = Y = L^2(0,1)$, a special Volterra integral operator equation of the first kind, Faber and Wing [19], Dostanić and Milinković [14] and Vu Kim Tuan and Gorenflo [78] have proved, that the asymptotics of the singular values of this operator yield

$$\sigma_n(J_r) \sim n^{-r}, \quad \text{for all} \quad r > 0 \quad \text{as} \quad n \to \infty. \tag{4.11}$$

Notice that for $r = 1$ we get our operator J from (4.9). In [78] and in [28] we may even found results on singular value asymptotics of fractional integral operators with weight functions. Moreover, Vu Kim Tuan and Gorenflo [78] have even found that the fractional integral operator J_r multiplied with a pole function $m(s) = s^{-\alpha}$ in $L^2(0,1)$,

$$[s^{-\alpha} J_r x](s) := s^{-\alpha} \int_0^s \frac{(s-t)^{r-1}}{\Gamma(r)} x(t)dt, \quad (0 \leq s \leq 1) \tag{4.12}$$

in $X = Y = L^2(0,1)$ has also got the asymptotics given in (4.11), if $r > 2\alpha \geq 0$. Hence, this pole function does not change the singular value asymptotics of J_r and J.

But we want to know, what happens to the degree of ill-posedness, if we multiply J (or J_r) by smooth functions like the ones given in (4.8). As we have already seen in the examples in section 3.3.2 the degree of ill-posedness usually becomes larger, the smoother the kernel function is.

We want to use some results on Fredholm integral equations and therefore are going to transform our Volterra operator equation of the first kind (4.1), given by

$$[Bx](s) = \int_0^s m(s)x(t)dt, \quad (0 \leq s \leq 1) \tag{4.13}$$

into a Fredholm integral equation

$$[Bx](s) = \int_0^1 k(s,t)x(t)dt, \quad (0 \leq s \leq 1) \tag{4.14}$$

with a kernel function $k(s,t)$ that vanishes for values of t that are greater than s, i.e.

$$k(s,t) = \begin{cases} m(s), & (0 \leq t \leq s \leq 1) \\ 0, & (0 \leq s \leq t \leq 1). \end{cases} \tag{4.15}$$

Therefore the kernel $k(s,t)$, defined on the square $[0,1] \times [0,1]$ is in $L^2((0,1)^2)$, since it has got a finite jump on the diagonal $s = t$. For Fredholm integral operators with quadratically integrable kernels we have a well-known connection between the kernel smoothness and the decay rate of the singular values, which we will state below. It is due to Chang (see [84] or [9] and [48]) and follows from a classical theorem by Hille and Tamarkin on eigenvalues of integral operators, stated in [2].

Proposition 4.2 (Chang). *Consider the Fredholm integral equation*

$$[Ax](s) = \int_0^1 k(s,t)x(t)dt, \quad (0 \leq s \leq 1), \tag{4.16}$$

with kernel $k \in L^2((0,1)^2)$. Then for the singular values of A,

$$\sigma_n(A) = o(n^{-\frac{1}{2}}), \quad for \quad n \to \infty \tag{4.17}$$

holds. Furthermore, if

$$k, \frac{\partial k}{\partial s}, \frac{\partial^2 k}{\partial s^2}, \ldots, \frac{\partial^{l-1} k}{\partial s^{l-1}}$$

are continuous in s for almost all t and

$$\frac{\partial^l k(s,t)}{\partial s^l} = \int_0^s g(\tau,t)d\tau + h(t),$$

where $g \in L^2((0,1)^2)$ and $h \in L^1(0,1)$, then we even have

$$\sigma_n(A) = o(n^{-(l+\frac{3}{2})}), \quad for \quad n \to \infty \tag{4.18}$$

for the singular values of A.

Since our kernel $k(s,t)$ given in (4.15) is quadratically integrable, but not continuous, we can follow from the previous theorem, that for the decay rate of singular values of B,

$$\sigma_n(B) = o(n^{-\frac{1}{2}}), \quad \text{for} \quad n \to \infty$$

holds, i.e. the operator equation is *at least* ill-posed of degree $\nu = \frac{1}{2}$. Moreover, we know that

$$\sigma_n(B) \leq C(n) n^{-\frac{1}{2}} \quad \text{with} \quad \lim_{n\to\infty} C(n) = 0,$$

by the definition of the Landau symbols. Therefore the degree of ill-posedness might actually be higher, but unfortunately we do not know any statement on the highest possible degree of ill-posedness for this problem, since Chang's theorem does not provide upper bounds on the decay rate of singular values.

4.2.2 Approach using Minimax principle

Another approach to our problem is the *Minimum-maximum principle by Poincaré and Fischer* (see [45]):

Lemma 4.1 (Poincaré and Fischer). *Let $A \in \mathcal{L}(X,Y)$ be a compact linear operator in separable Hilbert spaces X and Y with a singular system $\{\sigma_j; u_j; v_j\}_{j\in J}$. Then*

$$\sigma_n = \max_{X_n \subset X} \min_{x \in X_n \setminus \{0\}} \frac{\|Ax\|_Y}{\|x\|_X} = \min_{x \in span(u_1,\ldots,u_n)\setminus\{0\}} \frac{\|Ax\|_Y}{\|x\|_X}$$

holds for all $n \in J$, where X_n is the set of all n-dimensional linear subspaces of the space X.

Hence, from the obvious inequality for $m \in L^\infty(0,1)$ with $X = Y = L^2(0,1)$ and the discussion in section 4.1 we may follow

$$\begin{aligned}
\|Bx\|_{L^2(0,1)} &= \|M(Jx)\|_{L^2(0,1)} && (4.19)\\
&\leq \|M\|_{\mathcal{L}(L^2(0,1),L^2(0,1))} \|Jx\|_{L^2(0,1)} && (4.20)\\
&= \|m\|_{L^\infty(0,1)} \|Jx\|_{L^2(0,1)}. && (4.21)
\end{aligned}$$

Then, using the above lemma we obtain

$$\sigma_n(B) = \sigma_n(M \circ J) \leq \|m\|_{L^\infty(0,1)} \sigma_n(J) \tag{4.22}$$

for the singular values of B. For the operator B from (4.1) with our *special* multiplier functions m from (4.8) with $\|m\|_{L^\infty(0,1)} = 1$ we get

$$\sigma_n(B) \leq \sigma_n(J)$$

Since we know from its singular value decomposition, that the operator J from (4.9) is ill-posed of degree $\nu = 1$, i.e. there exists a constant C such that the singular values of J satisfy

$$\sigma_n(J) \leq C n^{-1}, \quad n \to \infty,$$

the same inequality holds for the singular values of the linear integral operator $B = M \circ J$:

$$\sigma_n(B) \leq Cn^{-1}, \quad n \to \infty. \tag{4.23}$$

Hence, the operator B has got at least the same degree of ill-posedness, i.e. $\nu = 1$. We do not even have to consider special multiplier functions m, since from inequality (4.22) we immediately obtain inequality (4.23), if $m \in L^\infty(0,1)$. Then the constant C is given by $C := \|m\|_{L^\infty(0,1)}$.

The following lemma is a consequence of the minimum-maximum principle of Poincaré and Fischer:

Lemma 4.2 (Spectral equivalence). *Let $B : X \to Y$ and $J : X \to Z$ be compact linear operators mapping between Hilbert spaces X, Y and Z such that*

$$c\|Jx\|_Z \leq \|Bx\|_Y \leq C\|Jx\|_Z \quad \forall x \in X$$

with some constants $0 < c \leq C < \infty$. Then the ordered singular values of J and B satisfy

$$c\sigma_n(J) \leq \sigma_n(B) \leq C\sigma_n(J), \quad n = 1, 2, \ldots.$$

Hence the degrees of ill-posedness of both operators J and B coincide.

By setting $X = Y = Z = L^2(0,1)$ we could apply this lemma to our operator B from (4.1). As we have just seen in (4.21) we may provide an upper bound on $\|Bx\|_{L^2(0,1)}$ and therefore for the singular values of B, where C is given by $\|m\|_{L^\infty(0,1)}$. For multiplication operators with

$$0 < c \leq |m(t)| \leq C < \infty$$

as we have seen in (2.6) and (2.10) in Chapter 2, where $B = F'(x_0)$ we may also find a lower bound on $\|Bx\|_{L^2(0,1)}$ and therefore B and J are spectrally equivalent. This is not the case for this special multiplication operator with zeros.

4.2.3 Calculating the eigenvalues of the operator B^*B

From previous considerations we know, that the square roots of the eigenvalues of B^*B (as well as BB^*) are the singular values of B.

Hence, we consider the linear Fredholm integral operator $B : L^2(0,1) \to L^2(0,1)$ given by (4.14) with the kernel $k(s,t) \in L^2((0,1)^2)$ given by (4.15) again:

$$[Bx](s) = \int_0^1 k(s,t)x(t)dt, \quad (0 \leq s \leq 1), \quad k(s,t) = \begin{cases} m(s), & (0 \leq t \leq s \leq 1) \\ 0, & (0 \leq s \leq t \leq 1). \end{cases}.$$

We can determine the adjoint operator B^* for real valued functions:

$$\begin{aligned} \langle Bx, y\rangle_{L^2(0,1)} &= \int_0^1 \left(\int_0^1 k(s,t)x(t)dt \right) y(s)ds \\ &= \int_0^1 x(t) \left(\int_0^1 k(s,t)y(s)ds \right) dt = \langle x, B^*y\rangle_{L^2(0,1)} \end{aligned}$$

Hence, for real-valued functions $k(s,t)$ the adjoint operator is given by

$$[B^*y](t) = \int_0^1 k(s,t)y(s)ds, \quad (0 \le t \le 1), \tag{4.24}$$

and finally, the operator B^*B which is needed for the calculation of the singular values is given by

$$\begin{aligned}[B^*Bx](\tau) &= \int_0^1 k(s,\tau)\left(\int_0^1 k(s,t)x(t)dt\right)ds \quad (0 \le \tau \le 1), && (4.25)\\ &= \int_0^1 \left(\int_0^1 k(s,\tau)k(s,t)ds\right)x(t)dt. && (4.26)\end{aligned}$$

Therefore we find the kernel $K(t,\tau)$ of the operator B^*B as a function of the kernel of the operator B:

$$K(t,\tau) = \int_0^1 k(s,\tau)k(s,t)ds. \tag{4.27}$$

Using the definition of the kernel function (4.15) we get

$$K(t,\tau) = \int_{\max(t,\tau)}^1 m^2(s)ds. \tag{4.28}$$

The eigenvalue problem associated with equation (4.26) is given by

$$\lambda x(\tau) = \int_0^1 K(t,\tau)x(t)dt, \tag{4.29}$$

where λ is the eigenvalue of the kernel $K(t,\tau)$ for a non-null solution $x(\tau)$ that is the corresponding eigenfunction. We may obtain bounds on any eigenvalue applying the Cauchy-Schwartz inequality to (4.29) and knowing that the eigenfunctions of a self-adjoint operator are orthogonal (see section 3.1) and may be orthonormalized. Then we get (see [6])

$$\lambda \le \left(\int_0^1 \int_0^1 |K(t,\tau)|^2 dt d\tau\right)^{\frac{1}{2}},$$

which we may determine for special kernels $K(t,\tau)$. We may obtain further bounds by using Bessel's inequality for an orthonormal system $\phi_1, \phi_2, \ldots$, which is given by

$$\sum_{i=1}^{\infty} \langle f, \phi_i \rangle^2_{L^2(0,1)} \le \langle f, f \rangle_{L^2(0,1} = \|f\|^2_{L^2(0,1)}$$

for some function f. Setting $f = K(t,\tau)$ and knowing that the eigenfunctions of a self-adjoint operator form an orthonormal system, we may state, that

$$\sum_{i=1}^{\infty} \left(\int_0^1 K(t,\tau), \phi_i(\tau)d\tau\right)^2 \le \int_0^1 K(t,\tau)^2 d\tau,$$

or

$$\sum_{i=1}^{\infty} \lambda_i^2 \phi_i(\tau)^2 \leq \int_0^1 K(t,\tau)^2 d\tau.$$

Integrating with respect to t and using $\langle \phi_i, \phi_i \rangle_{L^2(0,1)} = 1$ yields

$$\sum_{i=1}^{\infty} \lambda_i^2 \leq \int_0^1 \int_0^1 K(t,\tau)^2 d\tau dt.$$

Hence the sum of the squares of the eigenvalues converge and the eigenvalues cannot heap at a finite point. This examination confirms that there is an infinite number of eigenvalues accumulating at infinity.

We know from the considerations in the previous chapter, that the eigenvalues of a compact operator (as the one in the present case) converge to zero. We need statements on the decay rate of the eigenvalues (and therefore the asymptotics of the singular values) of Fredholm integral operators with kernels as the one given in (4.28). Obviously $K(t,\tau) \in L^2((0,1)^2)$ holds and

$$K(t,\tau) = K(\tau,t),$$

i.e. B^*B is a compact self-adjoint operator in $L^2(0,1)$. Notice that an integral equation with symmetric kernel has got a complete system of eigenfunctions. (see [54]). Furthermore the kernel (4.28) is positive definite, i.e.

$$\int_0^1 \int_0^1 K(t,\tau) f(t) f(\tau) dt d\tau \geq 0$$

holds for all $f \in L^2(0,1)$. In order to show this we use (4.27)

$$\begin{aligned}
\int_0^1 \int_0^1 K(t,\tau) f(t) f(\tau) dt d\tau &= \int_0^1 \int_0^1 \left(\int_0^1 k(s,\tau) k(s,t) ds \right) f(t) f(\tau) dt d\tau \\
&= \int_0^1 \left(\int_0^1 k(s,\tau) f(\tau) d\tau \right) \left(\int_0^1 k(s,t) f(t) dt \right) ds \\
&= \langle Bf, Bf \rangle_{L^2(0,1)} \\
&= \|Bf\|^2_{L^2(0,1)} \geq 0.
\end{aligned}$$

Now, we are going to use results by Reade (see [73] and [74]) on the eigenvalues of operators with positive definite kernels:

Proposition 4.3 (Reade). *Let K be a Fredholm integral operator as given in (3.16)*

$$[Kx](s) = \int_0^1 k(s,t) x(t) dt, \quad (0 \leq s \leq 1)$$

with kernel function $k(s,t)$. Then the eigenvalues of any continuously differentiable positive definite kernel of this compact self-adjoint operator K in $L^2(0,1)$ satisfy

$$\lambda_n(K) = o(n^{-2}) \quad as \quad n \to \infty. \tag{4.30}$$

Moreover, if the kernel $k(s,t)$ is p times continuously differentiable, then the eigenvalues of K have the asymptotics

$$\lambda_n(K) = o(n^{-(p+1)}) \quad as \quad n \to \infty. \tag{4.31}$$

For details and proofs, see [73] and [74]. Note, that Reade's theorem is an improvement of an older result by Weyl (see [83]), stating that for an operator K with kernel $k(s,t) \in L^2(0,1)$ which is in addition p times continuously differentiable, the eigenvalues have the asymptotics

$$\lambda_n(K) = o(n^{-(p+\frac{1}{2})}) \quad \text{as} \quad n \to \infty. \tag{4.32}$$

This result again follows from a theorem by Hille and Tamarkin (see [16]). Since our kernel (4.28) does *not* even satisfy $K(t,\tau) \in C^1((0,1)^2)$ (because of the involved function $\max(t,\tau)$, which is only Lipschitz continuous), we cannot apply Reade's results. However, we can show that $K(t,\tau)$ is *Lipschitz continuous* in t (and also in τ due to symmetry), i.e. there exists a constant $L > 0$ such that

$$|K(\bar{t},\tau) - K(t,\tau)| \leq L|\bar{t} - t|, \quad \forall \quad \bar{t}, t, \tau.$$

The Lipschitz condition for (4.28) is obvious for $t \neq \tau$. Therefore we are going to consider the critical case $t = \tau$. We also assume (refering to section 4.1) that $m \in L^\infty(0,1)$.

$$\begin{aligned} |K(\bar{t},\tau) - K(t,\tau)| &= |K(\bar{t},t) - K(t,t)| \\ &= \left| \int_t^{\bar{t}} m^2(s) ds \right| \\ &\leq \int_t^{\bar{t}} |m^2(s)| ds \\ &\leq M|\bar{t} - t|, \end{aligned}$$

using

$$M := \text{ess} \sup_{s \in (0,1)} |m^2(s)|.$$

Since $K(t,\tau)$ is Lipschitz continuous we can use an extension of Reade's [73] results for operators with Lipschitz continuous symmetric positive definite kernels:

Proposition 4.4 (Reade). *The eigenvalues of operators K as given in proposition 4.3 with Lipschitz continuous self-adjoint positive definite kernels $k(s,t)$ satisfy*

$$\lambda_n(K) = O(n^{-2}). \tag{4.33}$$

Note, that Reade also states that these results are best possible in the sense that if $K \in C^1[0,1]$ instead of $K \in Lip[0,1]$, then $\lambda_n(K) = o(n^{-2})$. Therefore, for the eigenvalues of the operator B^*B we have

$$\lambda_n(B^*B) = O(n^{-2})$$

and for the singular values of B, we get (4.23) again

$$\sigma_n(B) \leq Cn^{-1}, \quad n \to \infty, \tag{4.34}$$

i.e. only a lower bound on the degree of ill-posedness. Hence, as a summary, we see, that for continuously differentiable kernels the eigenvalues have the asymptotics $o(n^{-2})$, whereas, if the kernel is only *Lipschitz continuous* we get $O(n^{-2})$ for the decay rate of the eigenvalues. So we only know, that the problem is *at least* ill-posed of degree $\nu = 1$.

Here, we may also state a result for continuous kernels from Pietsch [72]:

Proposition 4.5 (Eigenvalue theorem for Fredholm kernels). *Let μ be any finite Borel measure on a compact Hausdorff space X. Then $K \in [C(X), C(X)]$ implies*

$$\lambda_n(K) \in l_2.$$

We need some explanation on this proposition. A Hausdorff space X is a space, where any two points $x \in X$ and $y \in X$ are separated by a neighbourhood. Obviously any metric space is a Hausdorff space and therefore $X = [0,1]$ is a Hausdorff space. A kernel K is said to be of *Fredholm type*, if $K(t,\tau) \in C([0,1]^2)$, i.e. the kernel is continuous.

Therefore, for integral operators with continuous kernels $K(t,\tau) \in C((0,1)^2)$, as the one given in (4.28) for the self-adjoint operator B^*B, the eigenvalues satisfy

$$\lambda_n(B^*B) \in l_2.$$

From previous sections we know that the singular values are the square roots of the positive eigenvalues of B^*B or BB^*. Therefore

$$\sigma_n(B) = \sqrt{\lambda_n(B^*B)}$$

holds and

$$\sum_{n=1}^{\infty} \sigma_n^4(B) = \sum_{n=1}^{\infty} \lambda_n^2(B^*B) < \infty.$$

Hence, we get

$$\sigma_n(B) \in l_4,$$

a better result as we got by using the theory of Hilbert-Schmidt operators (see section 3.1), where we found that the singular values of those Hilbert-Schmidt operators satisfy

$$\sigma_n(B) \in l_2.$$

But this result does not help to get a better estimation for the singular values or a better lower bound on the degree of ill-posedness than stated in (4.34) or even an upper bound.

4.2.4 Approach using results for special kernel functions

In addition to the results in [78], that are stated in (4.11), in a further paper (see [79]) Vu Kim Tuan and Gorenflo have proved, that the singular values $\sigma_n(J_r^k)$ of the operator

$$[J_r^k x](s) := \int_0^s \frac{(s-t)^{r-1}}{\Gamma(r)} k(s,t)x(t)dt, \quad (0 \le s \le 1), \tag{4.35}$$

in $L^2(0,1)$ with $r > 0$ have got the asymptotics

$$\sigma_n(J_r^k) \sim (n\pi)^{-r}, \quad \text{as} \quad n \to \infty,$$

when the kernel can be expressed in the form

$$k(s,t) = 1 + (s-t)h(s,t), \tag{4.36}$$

and $h(s,t)$ is a smooth enough function (see following proposition) with respect to s and square integrable. For the case of a positive integer r, Vu Kim Tuan and Gorenflo [79] have even shown the following proposition:

Proposition 4.6 (Vu Kim Tuan and Gorenflo). *Let r be a positive integer and $\frac{\partial^j}{\partial s^j}h(s,t)$, $j = 0,1,\ldots,r$ be continuous in s and almost everywhere with respect to t, and $\frac{\partial^r}{\partial s^r}h(s,t) \in L^2(0,1)$. Then*

$$\sigma_n(J_r^k) = (n\pi)^{-r}(1+o(1))$$

holds.

This result can be extended to fractional integral operator J_r^k multiplied with a pole function $m(s) = s^{-\alpha}$,

$$[s^{-\alpha}J_r^k x](s) := s^{-\alpha}\int_0^s \frac{(s-t)^{r-1}}{\Gamma(r)} k(s,t)x(t)dt, \quad (0 \le s \le 1) \tag{4.37}$$

in $X = Y = L^2(0,1)$ has also got the asymptotics given in proposition 4.6 when $r > 2\alpha \ge 0$. See also [30] for details.

Hence, if for the special value $r = 1$, that we are considering, the special kernel $m(s)$ of the operator (4.13) can be expressed in the way described above, we can use the theorem by Vu Kim Tuan and Gorenflo, in order to state the degree of ill-posedness. We can do so and will get

$$h(s,t) = \frac{m(s)-1}{s-t}.$$

This function has got a discontinuity for $s = t$ and therefore we cannot apply the above proposition.

4.3 Upper bounds on the degree of ill-posedness

So far we have seen that it is easy to find lower bounds on the degree of ill-posedness of problem (4.1). Through various ways we have therefore found that the problem is at least ill-posed of degree $\nu = 1$.

Only for some special kernels, such as kernels of fractional integral operators (4.10), fractional integral operators multiplied with a pole function (4.12) or kernels of the form (4.35) with (4.36), upper bounds on the degree of ill-posedness were found. Since the kernel $k(s,t)$ of the integral operator (4.1) cannot be represented by such a special kernel function, we will have to use other approaches.

Chapter 5

The Sturm-Liouville problem

5.1 Formulation as an eigenvalue problem for Sturm - Liouville equations

We are going to transform the Volterra integral operator $B : L^2(0,1) \to L^2(0,1)$ of the first kind (4.13) given by

$$[Bx](s) = \int_0^s m(s)x(t)dt, \quad (0 \leq s \leq 1), \tag{5.1}$$

into a boundary value problem, using the eigenvalue equation $B^*Bu = \sigma^2 u$. The adjoint operator of B in $L^2(0,1)$ is given by

$$[B^*y](t) = \int_t^1 m(s)y(s)ds, \quad (0 \leq t \leq 1). \tag{5.2}$$

Therefore

$$[B^*Bu](\tau) = \int_\tau^1 \int_0^s m^2(s)u(t)dtds, \quad (0 \leq \tau \leq 1), \tag{5.3}$$

holds and we can state the eigenvalue equation for the self-adjoint operator B^*B in integral form

$$B^*Bu = \sigma^2 u, \tag{5.4}$$

for $u \in L^2(0,1)$ and $m \in L^\infty(0,1)$ as given in section 4.1. From [69] we know that a function

$$\Phi(x) = \int_a^x f(t)dt$$

is absolutely continuous on $[a,b]$ for $f(t) \in L^1(a,b)$ and any absolutely continuous function is continuous (for a proof, see [69]). Then, considering (5.1) with $x \in L^2(0,1) \subset L^1(0,1)$ and $m \in C[0,1]$, which holds for our specific multiplier functions m from (4.8), we get $Bx \in C[0,1]$, i.e. Bx is continuous. Hence, any square integrable eigenfunction of a continuous kernel, corresponding to a non-zero eigenvalue, is continuous. Additionally we consider (5.3) and apply the same tools. With $u \in L^2(0,1) \subset L^1(0,1)$ and

$m^2 \in C^1[0,1]$, which holds for our specific multiplier functions m from (4.8) we obtain $B^*Bu \in C^1[0,1]$, i.e. the operator B^*Bu is continuously differentiable. With the eigenvalue equation (5.4), which has to be satisfied for eigenfunctions u, we furthermore obtain $u \in C^1(0,1)$. Now, using $u \in C[0,1]$, a solution to equation (5.3) has to satisfy the boundary condition $u(1) = 0$. In addition, with $u \in C^1[0,1]$, we get

$$\sigma^2 u'(\tau) = \frac{d}{d\tau}\int_\tau^1 \int_0^s m^2(s)u(t)dtds = -\int_0^\tau m^2(\tau)u(t)dt$$

after differentiation of equation (5.3) with respect to τ. Obviously, for a solution to this equation $u'(0) = 0$ has to hold. Assuming that $m(\tau) \neq 0$, we can divide by $m^2(\tau)$ (for our specific multiplier functions (4.8) this can be done for $\tau \neq 0$). If we expect an even stronger condition for u, i.e. that $u \in C^2[0,1]$ and if

$$a(\tau) = \frac{1}{m^2(\tau)} \in C^1[0,1],$$

we may differentiate again to get

$$\sigma^2 \left(\frac{u'(\tau)}{m^2(\tau)}\right)' = -u(\tau).$$

By setting $\lambda = \frac{1}{\sigma^2}$ and $a(\tau) = \frac{1}{m^2(\tau)}$ we get the following boundary value problem

$$-(a(\tau)u'(\tau))' = \lambda u(\tau), \quad u(1) = u'(0) = 0, \tag{5.5}$$

with $a(\tau) \in C^1[0,1]$ and $u \in C^2[0,1]$, i.e. *stronger* conditions on the smoothness of a and u as the ones given for the integral equations. If $u \notin C^2[0,1]$ we have to use a generalized formulation of (5.5), for which we refer to the next section. We are also going to consider some characteristics of this generalized eigenvalue problem formulation in the next section.

5.2 Eigenvalues of symmetric positive definite operators

The problem (5.5) is an eigenvalue problem for the linear symmetric positive definite operator

$$Au := -(a(\tau)u'(\tau))' \quad \text{in the Hilbert space} \quad L^2(0,1), \tag{5.6}$$

with

$$u \in D(A) = \{u | u \in C^2[0,1], u(1) = u'(0) = 0\}$$

and

$$a \in C^1[0,1].$$

Now, let $u, v \in D(A)$. Knowing that $D(A)$ is dense in $L^2(0,1)$ (see [66]), we determine the inner product of Au and v, i.e. we multiply the operator Au with v and integrate

$$\langle Au, v\rangle_{L^2(0,1)} = \int_0^1 -(a(\tau)u'(\tau))'v(\tau)d\tau$$

By applying partial integration we get

$$\langle Au, v\rangle_{L^2(0,1)} = -a(\tau)u'(\tau)v(\tau)|_{\tau=0}^{\tau=1} + \int_0^1 a(\tau)u'(\tau)v'(\tau)d\tau$$

Hence, actually we only need a continuous function u with a continuous first derivative and a quadratically integrable second derivative. Symmetry can be seen from equation (5.2) where not only $u'(0) = 0$ has to be satisfied but also

$$\lim_{\tau \to 0} a(\tau)u'(\tau) = 0,$$

in order to secure symmetry. Then this equation is invariant in exchange of u and v and therefore

$$\langle Au, v\rangle_{L^2(0,1)} = \langle u, Av\rangle_{L^2(0,1)}$$

holds for all $u, v \in D(A)$. This is the definition of a symmetric operator. Furthermore, positive definiteness is given by

$$\begin{aligned}\langle Au, u\rangle_{L^2(0,1)} &= \int_0^1 a(\tau)(u'(\tau))^2 d\tau \\ &\geq a_0 \int_0^1 (u'(\tau))^2 d\tau \\ &\geq a_0 \|u\|^2_{L^2(0,1)} \quad \forall u \in D(A),\end{aligned}$$

using $a_0 := \min_{\tau \in [0,1]} a(\tau) = \min_{\tau \in [0,1]} \dfrac{1}{m^2(\tau)} > 0$ and the inequality of Friedrichs (see [66]):

$$\|u\|^2_{L^2(a,b)} \leq (b-a)(u^2(a) + u^2(b)) + (b-a)^2\|u'\|^2_{L^2(a,b)}, \quad u \in C^1(a,b),$$

for $a = 0$ and $b = 1$. For our special multiplier functions m from (4.8) the function $a(\tau) = \dfrac{1}{m^2(\tau)}$ is monotonely decreasing and therefore

$$a(\tau) \geq a(1) = \frac{1}{m^2(1)} =: a_0.$$

We are now going to introduce the so-called *generalized eigenproblem:* The eigenproblem $Au = \lambda u$ for the linear symmetric positive definite operator A is multiplied by an arbitrary element $\eta \in H_A$, where H_A is the *energetic Hilbert space* with the inner product

$$[u, v]_A := \langle Au, v\rangle_{L^2(0,1)}, \quad u, v \in D(A). \tag{5.7}$$

The set $D(A)$, given by

$$D(A) = \{u | u \in C^2[0,1], u(1) = \lim_{\tau \to 0} a(\tau)u'(\tau) = 0\}$$

has to be dense in the Hilbert space $L^2(0,1)$, which is satisfied here. Equation (5.7) indeed yields an inner product (for a proof, see [66]), hence, the set $D(A)$ becomes a Hilbert space (if it is incomplete we have to complete it) with this inner product. This is the energetic Hilbert space H_A. Therefore we get the following definition of a generalized eigenproblem:

Definition 5.1. *The element $u \in H_A$, $u \neq \mathbf{0}$ and the number λ are called generalized eigenelement and generalized eigenvalue , if*

$$[u, \eta]_A = \lambda \langle u, \eta \rangle_{L^2(0,1)}, \quad \forall \eta \in H_A. \tag{5.8}$$

holds.

We will state some characteristics of this generalized eigenproblem (see [66] or [82] for further explanations):

- The eigenelements of a positive definite operator are orthogonal to each other in H_A and in $L^2(0,1)$.
- Any eigenvalue λ of a positive definite operator satisfies

  $$\lambda = \frac{[u,u]_A}{\langle u,u \rangle_{L^2(0,1)}} \geq a_0 > 0.$$

  Using the above definitions of λ and $a(\tau)$ we get

  $$\sigma_n(B) \leq \frac{1}{\sqrt{a_0}}$$

  for the singular values $\sigma_n(B)$ of (4.13). Note that for our special multiplier functions (4.8) we get

  $$\sigma_n(B) \leq m(1).$$

- Let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal system of eigenelements in H_A. Then $\{\frac{u_n}{\sqrt{\lambda_n}}\}_{n=1}^{\infty}$ is an orthonormal system in $L^2(0,1)$. Moreover $\{u_n\sqrt{\lambda_n}\}_{n=1}^{\infty}$ is an orthonormal system in H_A if $\{u_n\}_{n=1}^{\infty}$ is an orthonormal system in $L^2(0,1)$.
- If $\{u_n\}_{n=1}^{\infty}$ is a complete orthogonal system of eigenelements in $L^2(0,1)$ then it is a complete orthogonal system in H_A and vice versa.

Under certain conditions the generalized eigenproblem (5.8) for the symmetric operator A in the infinite Hilbert space H has got a discrete spectrum. We are going to define the discrete spectrum in the following.

Definition 5.2. *The generalized eigenproblem (5.8) is said to have a* ***discrete spectrum****, if*

- *The operator A has got an infinite countable sequence of eigenvalues*

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots,$$

  that accumulate only at infinity.

- *The system of eigenelements $\{u_n\}$ is complete in H.*

We get the following proposition on the discrete spectrum of a generalized eigenvalue problem.

Proposition 5.1. *If the operator A is symmetric and positive definite and furthermore, if any bounded set in H_A is relatively compact in $L^2(0,1)$, then the spectrum of this operator is discrete.*

For a proof of this theorem, see Michlin [66] or [75]. In order to show the discreteness of the spectrum of our elliptic (i.e. symmetric positive definite) operator A from (5.6), we need to show that any bounded set in H_A is relatively compact in $L^2(0,1)$. To this end we will use Sobolev's embedding theorems. For bounded Ω and piecewise Lipschitz continuous boundary $\partial\Omega$

$$H^1(\Omega) \hookrightarrow L^2(0,1), \quad \text{especially} \quad \|u\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)}$$

holds (see [34]), i.e. for $\Omega = (0,1)$, $H^1(0,1)$ is compactly embedded into $L^2(0,1)$. Hence, any bounded set in $H^1(0,1)$ is relatively compact in $L^2(0,1)$ and knowing that $H^1(0,1)$ and H_A have equivalent norms (see [82]), we get that any bounded set in H_A is relatively compact in $L^2(0,1)$. Therefore the operator A has got a *discrete spectrum*.

Hence, since the eigenvalues λ_j accumulate at infinity and since

$$\lambda(A) = \frac{1}{\sigma^2(B)},$$

we get $\sigma_j \to 0$, which has already been stated in (3.8). The completeness of the eigenfunctions u_j has already been stated as well.

Using the same assumptions as in proposition (5.1), we get results on the growth rate of the eigenvalues of the Sturm-Liouville problem. The following theorem is stated and proved in [66]. It uses the Minimum-maximum principle for eigenvalues of positive definite operators of a Sturm-Liouville problem and provides bounds on eigenvalues.

Proposition 5.2. *Let A and B be symmetric positive definite operators, satisfying the assumptions of proposition (5.1). Furthermore, let $H_A \subset H_B$ with*

$$\|u\|_B \leq \|u\|_A \quad \forall u \in H_A.$$

For the eigenvalues

$$\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le \ldots \quad \textit{of} \quad A,$$

and

$$\mu_1 \le \mu_2 \le \ldots \le \mu_n \le \ldots \quad \textit{of} \quad B,$$

we have

$$\mu_i \le \lambda_i \quad \forall i \in \mathbb{N}.$$

We may use this proposition in order to get bounds on the growth rate of the eigenvalues of (5.6). We know that for our *special* multiplier functions m from (4.8)

$$a(\tau) \ge a(1) = \frac{1}{m^2(1)} = a_0 \tag{5.9}$$

holds. We also know the eigenvalues of the *trivial* Sturm-Liouville eigenvalue problem for the operator $\tilde{A}$ which is given by

$$\tilde{A}u := -u'' \quad \text{in} \quad L^2(0,1) \quad \text{with} \quad u'(0) = u(1) = 0. \tag{5.10}$$

They are given by

$$\lambda_n(\tilde{A}) = \frac{\pi^2(2n-1)^2}{4}, \quad n \in \mathbb{N}. \tag{5.11}$$

Hence we may easily observe that the eigenvalues of operator A_0 with

$$A_0 u := -a_0 u'' \quad \text{in} \quad L^2(0,1)$$

are given by

$$\lambda_n(A_0) = a_0 \frac{\pi^2(2n-1)^2}{4}, \quad n \in \mathbb{N}.$$

Finally, since

$$a_0 \int_0^1 u'^2(\tau) d\tau \le \|u\|_A^2 = \int_0^1 a(\tau) u'^2(\tau) d\tau$$

we have

$$\|u\|_{A_0} \le \|u\|_A,$$

and therefore using proposition 5.2 we get

$$a_0 \frac{\pi^2(2n-1)^2}{4} \le \lambda_n(A). \tag{5.12}$$

Using the definition of λ and a_0 again, we get

$$\sigma_n(B) \le \frac{2}{\pi(2n-1)} m(1) = Cn^{-1}, \tag{5.13}$$

with some constant C for the singular values of the operator $B = M \circ J$. Inequality (5.13) yields only a lower bound on the degree of ill-posedness of the operator B again.

In order to get an upper bound on the degree of ill-posedness, we would need an upper bound for the eigenvalues of A in (5.12) and therefore an upper bound for $a(\tau)$ in (5.9). This is not possible, since

$$a(\tau) = \frac{1}{m^2(\tau)},$$

and $m(\tau)$ given by (4.8) has got a zero at $\tau = 0$. Notice that for functions $a(\tau)$ that are bounded from below *and* from above, we could state an inequality similar to (5.9), namely

$$a_1 \geq a(\tau) = \frac{1}{m^2(\tau)} \geq a_0 \tag{5.14}$$

and therefore (5.13) becomes

$$\frac{2}{\pi(2n-1)\sqrt{a_1}} \leq \sigma_n(B) \leq \frac{2}{\pi(2n-1)\sqrt{a_0}}. \tag{5.15}$$

Therefore, the degree of ill-posedness of operator $B = M \circ J$ that we are looking for, would be $\nu = 1$.

Since several analytical methods applied to both the integral operator $B = M \circ J$ (chapter 4) and the corresponding Sturm-Liouville problem with operator A from (5.6) (chapter 5) did only provide lower bounds on the degree of ill-posedness, but no upper bounds, we are now going to consider numerical approaches the the problem.

Chapter 6

Numerical approaches to the problem

We are now going to consider several numerical approaches to our problem in order to solve either the integral operator equation (4.1) or the boundary value problem (5.5). In this chapter we are only going to consider the special multiplication operators for multiplier functions $m(s) = s^\alpha$ and $m(s) = e^{-\frac{1}{s^\alpha}}$ with several values for $\alpha \geq 0$.

First we will look at the finite difference method applied to the Sturm-Liouville problem (5.5), after that we will examine expansion methods, such as the Galerkin or the Rayleigh-Ritz method applied to the integral eigenvalue problem for (4.1).

6.1 Finite difference methods for the Sturm - Liouville problem

6.1.1 Introduction

We are going to consider the boundary value problem for the Sturm-Liouville equation

$$-(a(\tau)u'(\tau))' = \lambda u(\tau), \quad \text{with} \quad \lambda = \frac{1}{\sigma^2} \quad \text{and} \quad a(\tau) = \frac{1}{m^2(\tau)} \quad \text{in} \quad \Omega = (0,1), \tag{6.1}$$

for $a \in C^1(0,1)$ and $u \in C^2(0,1)$ and boundary values

$$u(1) = 0 \quad \text{and} \quad \lim_{\tau \to 0} a(\tau)u'(\tau) = 0. \tag{6.2}$$

We will discretize this problem by using a finite difference method. Difficulties will arise in the discretization of the boundary condition at the left hand side, since $\lim_{\tau\to 0} a(\tau) = \infty$. We are going to use two different approaches, one with equidistant meshes and another one with non-equidistant meshes.

First, we are going to find an approximate solution to the boundary value (6.1) problem by replacing the continuous interval $\Omega = [0,1]$ by a finite set of grid points

$$0 = \tau_0 < \tau_1 < \ldots \tau_{n-1} < \tau_n = 1.$$

Here, the first problem occurs, since $a(\tau)$ is not defined for $\tau = 0$. We will attempt this problem later. Then we have $x_{i+1} - x_i = h_{i+1}$ for $i = 0 \dots n-1$ and $h = \max\{h_i\}$ is the maximum step size.

6.1.2 Equidistant and non-equidistant meshes

First of all, we want to consider equidistant meshes, i.e. $h_i = h \quad \forall i$. We may write

$$-(a(\tau)u'(\tau))' = -a'(\tau)u'(\tau) - a(\tau)u''(\tau),$$

and approximate the derivatives a' and u' by the central difference quotient and the second derivative u'' by the second difference quotient (see for example [34] or [55]):

$$\begin{aligned} a'(\tau) &\approx (D^0 a)(\tau) := \frac{a(\tau+h) - a(\tau-h)}{2h}, && (6.3)\\ u'(\tau) &\approx (D^0 u)(\tau) := \frac{u(\tau+h) - u(\tau-h)}{2h}, && (6.4)\\ u''(\tau) &\approx (D^+ D^- u)(\tau) := \frac{u(\tau+h) - 2u(\tau) + u(\tau-h)}{h^2}. && (6.5) \end{aligned}$$

Since u is assumed to be continuous we may take the exact value $u(\tau_i)$ at step $\tau = \tau_i$ for the right hand side of the eigenvalue equation. Setting $u(\tau_i) = u_i$ and $a(\tau_i) = a_i$ we get the classical finite difference method for (6.1) for equidistant meshes

$$\left(\frac{a_{i+1} - a_{i-1}}{4h^2} - \frac{a_i}{h^2}\right) u_{i-1} + \frac{2a_i}{h^2} u_i + \left(\frac{a_{i-1} - a_{i+1}}{4h^2} - \frac{a_i}{h^2}\right) u_{i+1} = \lambda u_i \qquad (6.6)$$

for $i = 1 \dots n-1$. The boundary values are discretized by

$$u(1) \approx u_n = 0, \qquad (6.7)$$

on the right hand side and by a forward difference for the derivative on the left hand side boundary

$$\lim_{\tau \to 0} a(\tau)u'(\tau) \approx \frac{a_1 u_1 - a_0 u_0}{h} = 0. \qquad (6.8)$$

Here we face our problem again, since $a_0 = a(0)$ is not defined. We are therefore going to introduce a variable ε, which has to be close to zero and we define

$$a_0 := \varepsilon. \qquad (6.9)$$

Then, we can vary ε in order to take it as close to zero as possible.

Now we may formulate (6.6) together with the boundary conditions (6.7) and (6.8) as a matrix eigenvalue problem:

$$A_h u_h = \lambda u_h, \qquad (6.10)$$

where

$$A_h = \begin{bmatrix} -\frac{a_0}{h} & \frac{a_1}{h} & 0 & \dots & \dots \\ \frac{a_2-a_0}{4h^2} - \frac{a_1}{h^2} & \frac{2a_1}{h^2} & \frac{a_0-a_2}{4h^2} - \frac{a_1}{h^2} & 0 & \dots \\ 0 & \frac{a_3-a_1}{4h^2} - \frac{a_2}{h^2} & \frac{2a_2}{h^2} & \frac{a_1-a_3}{4h^2} - \frac{a_2}{h^2} & \dots \\ \ddots & \ddots & \ddots & \ddots & \dots \\ \dots & 0 & \frac{a_n-a_{n-2}}{4h^2} - \frac{a_{n-1}}{h^2} & \frac{2a_{n-1}}{h^2} & \frac{a_{n-2}-a_n}{4h^2} - \frac{a_{n-1}}{h^2} \\ \dots & \dots & \dots & \dots & 1 \end{bmatrix}$$

and

$$u_h = \begin{bmatrix} u_0 & u_1 & u_2 & \dots & u_{n-1} & u_n \end{bmatrix}^T.$$

The algebraic eigenproblem (6.10) is the discrete counterpart to the continuous one. Its solution gives us n approximate eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ and n corresponding eigenvectors $u_1, u_2, \dots, u_n$. This is summarized in the following algorithm.

Algorithm 6.1 (Finite difference method).

1. *Determine the matrix $A_h \in \mathbb{R}^{n+1,n+1}$ as given above in (6.10).*
2. *Compute the algebraic eigenvalues λ of this matrix*

$$A_h u_h = \lambda u_h.$$

We are going to investigate the convergence of the scheme (6.6). To this end we calculate the maximum difference between the true and the computed solution. The following definition is taken from [34].

Definition 6.1. *A finite difference scheme is said to be **convergent** of order k, if*

$$\max_i |u(\tau_i) - u_i| \leq Ch^k,$$

where u_i is the approximate solution and $u(\tau_i)$ is the exact solution at $\tau = \tau_i$ and C is independent of h.

The classical way for showing convergence is proving consistency and stability:

Definition 6.2. *A finite difference scheme is said to be **consistent** of order k, if*

$$\|A_h(R_h u - u_h)\|_\infty \leq Ch^k,$$

where u_h is the approximate solution and $R_h u$ is the restriction of the exact solution to the mesh points.

For our difference formulas

$$\begin{aligned}(D^0u)(x) &:= u'(x)+h^2R, \quad \text{with} \quad |R| \leq \frac{1}{6}\|u'''\|_{C(0,1)}, \quad \text{if} \quad u \in C^3(0,1),\\ (D^+D^-u)(x) &:= u''(x)+h^2R, \quad \text{with} \quad |R| \leq \frac{1}{12}\|u''''\|_{C(0,1)}, \quad \text{if} \quad u \in C^4(0,1),\end{aligned}$$

holds (see [34]). This can be proved by Taylor's series expansion. Therefore the scheme (6.6) is second order accurate if $u \in C^4(0,1)$. In addition we have to check stability (see [55]):

Definition 6.3. *A finite difference scheme is said to be* ***stable*** *with respect to the discrete maximum norm* $\|\cdot\|_\infty$ *, if there exists a constant C such that*

$$\|A_h^{-1}\|_\infty \leq C.$$

In order to check stability, we can check the three characteristics of an M-Matrix (see [34] or [40]):

- A_h is an L-Matrix, i.e. its entries satisfy $a_{ii}^h > 0$ and $a_{ij}^h \leq 0 \quad i \neq j \forall i,j$, which is easy to check, since $a(\tau)$ is positive and a monotone decreasing function. We have to change the first row of A_h into

  $$\begin{bmatrix}\frac{a_0}{h^2} & -\frac{a_1}{h^2} & 0 & \dots & \dots\end{bmatrix}$$

  in order to secure stability. Note that h is changed into h^2 in order to secure consistency. The same is done to the last row.

- The matrix A_h is diagonal dominant, i.e.

  $$|a_{ii}^h| \geq \sum_{j\neq i}|a_{ij}^h| \quad \forall i,$$

  which can be checked with the characteristics of the function $a(\tau)$

- The matrix A_h is irreducible.

Since A_h is an M-Matrix, the scheme is stable (see [34]). We also know (see [55]), that

Proposition 6.1. *A consistent and stable scheme is convergent.*

Therefore our finite difference scheme (6.6) is convergent of order 2.

To estimate the discretization errors that beset the eigenvector u and the eigenvalue λ, we make use of the regular pattern in the finite difference equations (6.6), as it was done in [25]. For convenience we use $a(\tau) \equiv 1$. Therefore (6.6) is observed to have the solution $u_j = cz^j$, where z is a constant and substituting this solution into (6.6) yields the characteristic equation

$$z^2-(2-\lambda h^2)z+1=0 \tag{6.11}$$

that is associated with the finite difference equation. The two roots of this equation are

$$z_{1,2} = \frac{1}{2}(2-\lambda') \pm \sqrt{\lambda'(\lambda'-4)}, \quad \lambda' = \lambda h^2.$$

Gershgorin's theorem (see for example [26]) assures that for matrix A_h, $0 \leq \lambda' \leq 4$ holds. For the limiting case $\lambda' = 0$ or $\lambda' = 4$, the two roots z_1 and z_2 are equal. Hence, the solution of the difference equation takes the form

$$u_j = (c_1 + c_2 j)z^j,$$

where c_1 and c_2 are determined from the boundary conditions $u'_0 = u_n = 0$. But these boundary conditions are only satisfied when $c_1 = c_2 = 0$ and so a nontrivial solution u does not exist for $\lambda' = 0$ or $\lambda' = 4$; they are not eigenvalues of A_h. We conclude, that the roots must be complex conjugates, denoted by $z_1 = e^{i\Phi}$ and $z_2 = e^{-i\Phi}$ and the general solution to (6.6) is therefore

$$u_j = c_1 \cos j\Phi + c_2 \sin j\Phi.$$

Because $u'_0 = 0$, $c_2 = 0$ and one boundary condition is met. For the second boundary condition we get

$$\cos n\Phi = 0, \quad \Phi = (2k-1)\frac{\pi}{2}h, \quad h = \frac{1}{n}, \quad k = 1, \dots n, \tag{6.12}$$

and therefore we get $u_j = c_2 \cos(2k-1)\frac{\pi}{2} jh$ for the computed eigenvectors. By putting $z = e^{i\Phi}$ back into (6.11) we get

$$\lambda h^2 = 2 - (e^{i\Phi} + e^{-i\Phi}) = 2(1-\cos\Phi),$$

and, with $1 - \cos\Phi = 2\sin^2 \frac{1}{2}\Phi$ we have

$$\lambda_k = \frac{4}{h^2}\sin^2\frac{1}{2}\Phi$$

for the eigenvalues. At the lower end of the discrete spectrum $\frac{1}{2}\Phi \ll 1$ holds and therefore $\sin\Phi \approx \Phi$ and approximately

$$\lambda_k = \frac{\pi^2(2k-1)^2}{4}\left(1 - \frac{1}{12}\frac{\pi^2(2k-1)^2}{4}h^2\right), \quad h = \frac{1}{n}.$$

Using (6.12) this holds for $2k \ll n$. Hence, by a finite difference method, the few exact eigenvalues (5.11) are approximated from below, i.e. for the kth eigenvalue

$$\lambda_k^{\text{approx}} \leq \lambda_k^{\text{exact}}, \quad k = 1, \dots, n \tag{6.13}$$

holds. As we climb in the spectrum the relative accuracy of λ_k declines, but it still converges $O(h^2)$. We can do a similar calculation for $a(\tau) \neq 1$, but the computation is not as trivial as this one, since the characteristic equation becomes

$$\left(\frac{a_{i+1}-a_{i-1}}{4} - a_i\right)z^2 + (2a_i - \lambda h^2)z + \left(\frac{a_{i-1}-a_{i+1}}{4} - a_i\right) = 0, \tag{6.14}$$

whose roots are either complex conjugates or real, depending on the a_i. We might even get complex values for λ, especially for strongly varying $a(\tau)$. Therefore we cannot state an inequality like (6.13).

Additionally we will consider non-equidistant meshes. For such meshes the finite difference method (6.6) becomes

$$\left(\frac{a_{i+1}-a_{i-1}}{(h_i+h_{i+1})^2}-\frac{2a_i}{(h_i+h_{i+1})h_i}\right)u_{i-1}+\frac{2a_i}{h_ih_{i+1}}u_i \\ +\left(\frac{a_{i-1}-a_{i+1}}{(h_i+h_{i+1})^2}-\frac{2a_i}{(h_i+h_{i+1})h_{i+1}}\right)u_{i+1}=\lambda u_i \tag{6.15}$$

for $i=1,\dots,n-1$. The boundary values are discretized as in the case of equidistant meshes with (6.7) and (6.8). The analysis is similar to the one in the equidistant case, but the scheme is only consistent of order 1 (see [34]). The sizes of the intervals are taken to be smaller near zero, i.e. where the function $a(\tau)$ changes rapidly. The non-equidistant mesh is generated by the following procedure: The interval $[0,1]$ is divided into n subintervals. We generate the grid points starting from the right boundary. The first grid point left of the right hand side boundary (i.e. left of $\tau_n=1$) is taken to be $\tau_{n-1}=\frac{p}{q}\tau_n=\frac{p}{q}$, where $p<q$. All other grid points are determined by the iteration

$$\tau_i=\frac{p}{q}\tau_{i+1},\quad i=n-1,\dots,0.$$

Note, that we have to choose p and q for each n in such a way, that $\tau_0=\left(\frac{p}{q}\right)^n$ is close enough to zero. Then we may set $\varepsilon:=\tau_0$.

We are now going to apply the finite difference methods with equidistant and non-equidistant meshes to several multiplier functions.

6.1.3 Multiplier functions of the form $m(s)=s^\alpha$

First, we consider multiplier functions $m(s)=s^\alpha$, i.e.

$$a(s)=\frac{1}{s^{2\alpha}}$$

for different values of α. Both equidistant and non-equidistant meshes are tested. We will try several values for $n=\frac{1}{h}$ and $\varepsilon=a_0$. Figure 6.1 shows plots for the first 30 eigenvalues $\lambda_1,\dots,\lambda_{30}$ and $n=10000$ for different values of α. We can easily compute more than 30 eigenvalues and get similar results. However, problems arise for larger values of n (both in the equidistant and the non-equidistant case), since the condition number of the matrix A_h becomes very large. The plots suggest that the computed eigenvalues λ_n are proportional to n^2 and therefore we are going to consider the logarithmic scale plot for all computed eigenvalues in the case of $n=500$ and for several values of α. The results are illustrated in figure 6.2. In this plot, we firstly recognize

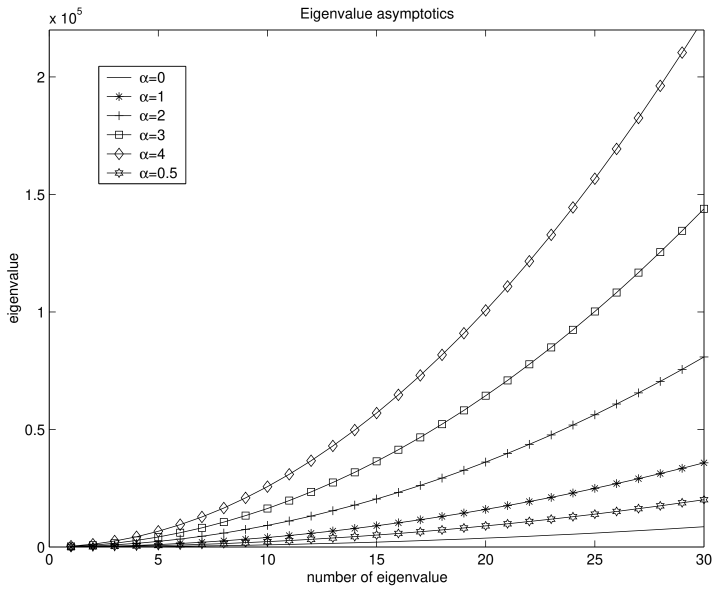

Figure 6.1: *First* 30 *computed eigenvalues of Sturm-Liouville problem* $-(au')' = \lambda u$ *for multiplier function* $m(s) = s^{\alpha}$*,* $n = 10000$ *and different values for* α *using a finite difference method*

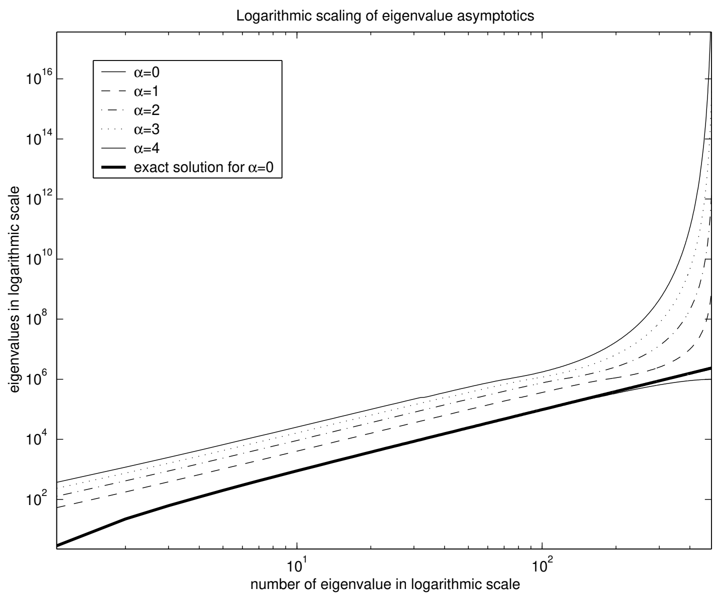

Figure 6.2: *Computed eigenvalues of Sturm-Liouville problem* $-(au')' = \lambda u$ *for* $n = 500$*, multiplier function* $m(s) = s^{\alpha}$ *and different values for* α *and exact eigenvalues for* $\alpha = 0$ *in logarithmic scales*

α	n	h	ε	COEFFICIENT OF n^2
0	500	0.002	0.001	9.8444
0	1000	0.001	0.0001	9.8584
0	10000	0.0001	0.00001	9.8686
1	500	0.002	0.001	39.4254
1	1000	0.001	0.0001	39.4657
1	10000	0.0001	0.00001	39.4790
2	500	0.002	0.001	88.5877
2	1000	0.001	0.0001	88.7683
2	10000	0.0001	0.00001	88.8278
3	500	0.002	0.001	157.2007
3	1000	0.001	0.0001	157.7383
3	10000	0.0001	0.00001	157.9156

Table 6.1: *Results for eigenvalue asymptotics of finite difference method for $m(s) = s^\alpha$ for equidistant meshes and several values of α, $n = \frac{1}{h}$ and $\varepsilon := a_0$*

α	n	ε	COEFFICIENT OF n^2
0	500	0.0066	63.5644
0	1000	4.3171e-05	24.6671
0	10000	4.5173e-05	9.8700
1	500	0.0066	38.5485
1	1000	4.3171e-05	38.5486
1	10000	4.5173e-05	39.4700
2	500	0.0066	83.9955
2	1000	4.3171e-05	83.9955
2	10000	4.5173e-05	88.7815

Table 6.2: *Results for eigenvalue asymptotics of finite difference method for $m(s) = s^\alpha$ for non-equidistant meshes and several values of α, $n = \frac{1}{h}$ and $\varepsilon := a_0$*

that the exact eigenvalues for $\alpha = 0$ are indeed approximated from below, i.e. (6.13) holds. For $\alpha > 0$ we did not plot the last $2-4$ eigenvalues, since they were complex conjugated (the reason for this was already mentioned in the previous analytic part). We also observe that the plots in logarithmic scaling have the same gradient (except for large numbers of eigenvalues), with shifts in direction of the y-axis for different values of α. The gradient for small computed eigenvalues is 2. The computed eigenvalues of large magnitude are no good approximations to the true eigenvalues, since the error increases, the larger the computed eigenvalues get. We could see this by computing the approximate eigenvalues for larger n, i.e. $n = 1000$ and plotting the first 500 ones in the same figure 6.2. We also refer to the next section, where we apply Galerkin's method to the integral equations and therefore do not have the difficulties with the approximation of the left hand boundary condition (6.8).

The plots suggest that $\lambda_n(A) = f(\alpha)O(n^2)$, where f is a monotone increasing function depending on α, as we have seen from figure 6.2. Hence, we compute the first 10 eigenvalues of smallest magnitude for several values of n, ε and α. Then, we apply curve fitting (with a polynomial of order 2) to those values in order to get the coefficients for n^2 and summarize the computed values for equidistant meshes in table 6.1, the ones for non-equidistant meshes in table 6.2. We observe, that the coefficients for n^2 converge for $n \to \infty$.

We are going to give a short comparison between finite difference method with equidistant and with non-equidistant meshes. From tables 6.1 and 6.2, especially for $\alpha = 0$ we observe, that the equidistant method yields better approximations. This is due to the fact that the equidistant finite difference method is convergent of order h^2, whilst the non-equidistant method only converges with order h. Furthermore, we are going to compare the condition numbers in both cases. For convenience we are only going to consider the multiplier function $m(s) = s$. In table 6.3 we summarize condition numbers $\kappa(A_h)$ of the matrix A_h from (6.10). From this table, we observe, that the

n	$\kappa(A_h)$ for equidistant meshes	$\kappa(A_h)$ for non-equidistant meshes
500	9.784e+09	7.051e+05
1000	3.321e+12	6.673e+06
5000	9.785e+13	5.976e+13
10000	3.321e+16	2.928e+22

Table 6.3: *Comparison of condition numbers of matrix A_h arising in the finite difference method for integral equation with multiplier function $m(s) = s$*

condition number of A_h starts off to be smaller for non-equidistant than for equidistant meshes for relatively small values of n. This is due to the fact that we are not close enough to zero yet in the case of non-equidistant meshes. But the growth rate of the condition numbers is much faster for non-equidistant meshes than for the equidistant case. This is due to the fact that there are much more grid points close to zero, i.e. for the entries of A_h in (6.10) we have $a(\tau) \to \infty$ in the case of non-equidistant meshes.

For non-equidistant meshes, the value of n cannot be enlarged any further, since serious problems with the large condition number of A_h occur.

Hence, both the tables and the plots suggest, that the eigenvalues have the asymptotics

$$\lambda_n^{\text{approx}}(A) \sim f(\alpha)n^2.$$

Then we apply curve fitting with the help of regression. We are going to find the best fitting second order polynomial to the first ten computed eigenvalues. The coefficients for the best fitting polynomial are summarized in table 6.4 for several values of α. We take a fixed value of n for each α, since we have already observed that the coefficients converge in the previous tables. We know the eigenvalues of the trivial operator (5.10) with $a(\tau) \equiv 1$ are given by (5.11),

$$\lambda_n = \frac{\pi^2(2n-1)^2}{4} = \pi^2 n^2 - \pi^2 n + \frac{\pi^2}{4}, \quad n \in \mathbb{N}. \tag{6.16}$$

Therefore we are going to compare the computed coefficients of the assumed

$$\lambda_n^{\text{approx}}(A) = an^2 + bn + c,$$

with the ones from the eigenvalues of the trivial operator. From table 6.4 we observe a

α	n	a	b	c	$\frac{a}{\pi^2}$	$\frac{b}{\pi^2}$	$\frac{c}{\pi^2}$
0	10000	9.8686	-9.8685	2.4670	0.9999	-0.9999	0.2500
0.5	10000	22.2069	3.6970	-0.2692	2.2500	0.3746	-0.0273
1	10000	39.4790	9.8598	-0.5988	4.0001	0.9990	-0.0607
1.5	10000	61.6860	18.4887	-0.9894	6.2501	1.8733	-0.1002
2	10000	88.8278	29.5840	-1.4418	9.0001	2.9975	-0.1461
2.5	10000	120.9044	43.1461	-1.9563	12.2502	4.3716	-0.1982
3	10000	157.9156	59.1753	-2.5334	16.0002	5.9957	-0.2567
3.5	10000	199.8614	77.6721	-3.1736	20.2502	7.8698	-0.3216
4	10000	246.7418	98.6369	-3.8776	25.0002	9.9940	-0.3929
4.5	10000	298.5566	122.0703	-4.6460	30.2501	12.3683	-0.4707
5	10000	355.3058	147.9728	-5.4795	36.0000	14.9928	-0.5552

Table 6.4: *Coefficients of the best fitting second order polynomial* $\lambda_n^{approx} = an^2 + bn + c$ *for eigenvalue asymptotics for multiplier function* $m(s) = s^\alpha$ *with* $n = 10000$ *and several values of* α

certain regularity for the coefficients a and b. We therefore conjecture

$$\lambda_n^{\text{approx}}(A) = (\alpha+1)^2\pi^2 n^2 + \frac{1}{2}(\alpha^2+\alpha)\pi^2 n + O(1) \tag{6.17}$$

for the eigenvalues of the Sturm-Liouville problem (6.1) with $a(\tau) = \dfrac{1}{m^2(\tau)}$ and $m(\tau) = \tau^\alpha$. Hence, the singular values of the integral operator have the asymptotics

$$\sigma_n^{\text{approx}}(B) \sim \frac{1}{(\alpha+1)\pi n}. \tag{6.18}$$

Therefore we conjecture (from numerical experiments), that the multiplier function only influences the coefficient for n^{-1} (i.e. the constants $\underline{C}$ and $\overline{C}$ in (3.14)), but the degree of ill-posedness does not change.

6.1.4 Multiplier functions of the form $m(s) = e^{-\frac{1}{s^\alpha}}$

Now we consider multiplier functions of type $m(s) = e^{-\frac{1}{s^\alpha}}$, i.e.

$$a(s) = e^{\frac{2}{s^\alpha}}.$$

Since the function $m(s) = e^{-\frac{1}{s^\alpha}}$ converges to zero much faster than the function $m(s) = s^\alpha$, serious problems with the condition number of A_h occur already for small sizes n of the matrix A_h. Figure 6.3 shows plots for the first 20 eigenvalues $\lambda_1, \ldots, \lambda_{20}$ and

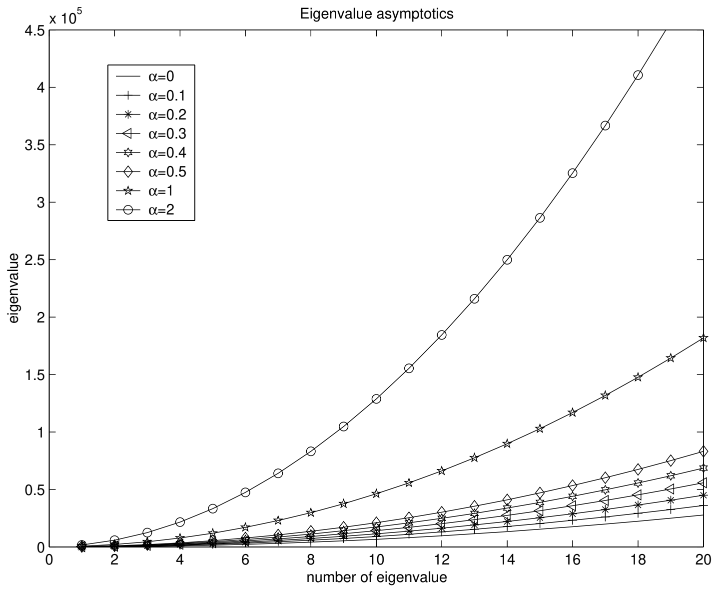

Figure 6.3: *First* 30 *computed eigenvalues of Sturm-Liouville problem* $-(au')' = \lambda u$ *for multiplier function* $m(s) = e^{-\frac{1}{s^\alpha}}$, $n = 10000$ *and different values for* α *using a finite difference method*

$n = 10000$ for different values of α. Again, we are going to consider the logarithmic scale plot for all computed eigenvalues for $n = 100$ and several values of α, since the previous plots suggest that λ_n is proportional to n^2. The results are shown in figure 6.4. In this plot, we observe that the exact eigenvalues for $\alpha = 0$ are indeed approximated from below, i.e. again (6.13) holds. For $\alpha > 0$ we did not plot the last 20 eigenvalues, since they are complex conjugated (which was already mentioned in the previous analytic part). We also observe that the plots in logarithmic scaling have the same gradient for the eigenvalues with small magnitude, with shifts in direction of the y-axis for different values of α. The gradient for small computed eigenvalues is 2. The computed eigenvalues of large magnitude are inaccurate, since the error increases, the larger the computed eigenvalues get. By computing the approximate eigenvalues for larger n, i.e. $n = 2000$ and plotting the first 100 ones in the same figure 6.4, we could see, that the large eigenvalues are no good approximations to the true ones. We also refer to the next

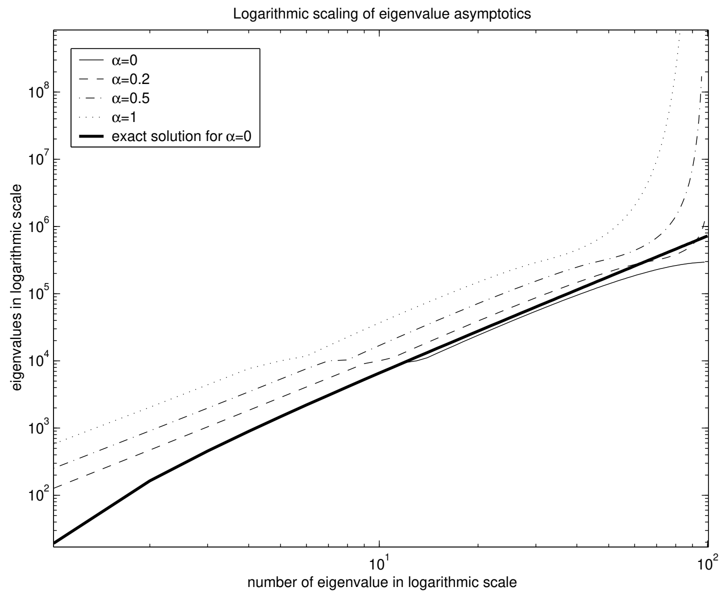

Figure 6.4: *Computed eigenvalues of Sturm-Liouville problem $-(au')' = \lambda u$ for $n = 100$, multiplier function $m(s) = e^{-\frac{1}{s^{\alpha}}}$ and different values for α and exact eigenvalues for $\alpha = 0$ in logarithmic scales*

sections, where we apply Ritz-Galerkin methods to the integral equations and therefore do not have the difficulties of approximating the left hand boundary condition (6.8).

The plots suggest that $\lambda_n^{\text{approx}}(A) = f(\alpha)O(n^2)$, where f is a monotone increasing function depending on α, as we have seen from figure 6.4. Hence, as in the previous section, we compute the first 10 eigenvalues of smallest magnitude for several values of n, ε and α. Then, we apply curve fitting (with a polynomial of degree 2) to those values in order to get the coefficients for n^2 and summarize the computed values, this time the results for equidistant and non-equidistant meshes in the same table (see table 6.5). We observe that the coefficients for n^2 converge for $n \to \infty$. We cannot increase n, since the matrix A_h becomes badly scaled and the results may get inaccurate. For comparisons between equidistant and non-equidistant methods we refer to section 6.1.3. The table confirms that the eigenvalues have the asymptotics

$$\lambda_n^{\text{approx}}(A) \sim f(\alpha)n^2. \tag{6.19}$$

Hence, we find the best fitting second order polynomial to the first ten computed eigenvalues. The coefficients for the best fitting polynomial are summarized in table 6.6 for several small values of α. Again, we take a fixed value of n for each α, since we have observed in table 6.5 that the coefficients converge. We find a certain regularity in the coefficients of a and conjecture

$$\lambda_n^{\text{approx}}(A) = g(\alpha^2)\pi^2 n^2 \tag{6.20}$$

α	n	h	ε	COEFFICIENT OF n^2
0	500	0.002	0.001	72.7407
0	1000	0.001	0.0001	72.8440
0	5000	0.0002	0.0001	72.9121
0	10000	0.0001	0.00001	72.9197
0.1	500	0.002	0.001	89.4280
0.1	1000	0.001	0.0001	89.8596
0.1	5000	0.0002	0.0001	89.8534
0.1	10000	0.0001	0.00001	89.8648
0.3	500	0.002	0.001	138.2995
0.3	1000	0.001	0.0001	138.3802
0.3	5000	0.0002	0.0001	138.4059
0.3	10000	0.0001	0.00001	138.4067
0.5	500	0.002	0.001	205.2736
0.5	1000	0.001	0.0001	205.4490
0.5	5000	0.0002	0.0001	205.5051
0.5	10000	0.0001	0.00001	205.5068
1	300	0.0033	0.003	445.6933
1	2000	not equidistant	0.1352	448.3335
1	2500	not equidistant	0.0820	448.3331

Table 6.5: *Results for eigenvalue asymptotics of finite difference method for* $m(s) = e^{-\frac{1}{s^\alpha}}$ *for equidistant and non-equidistant meshes and several values of* $\alpha, n = \frac{1}{h}$ *and* $\varepsilon := a_0$

α	n	a	b	c	$\frac{a}{\pi^2}$	$\frac{b}{\pi^2}$	$\frac{c}{\pi^2}$
0	10000	72.9197	-72.9191	18.2291	7.3883	-7.3883	1.8470
0.1	10000	89.8648	5.3561	-1.0132	9.1052	0.5427	-0.1027
0.2	10000	111.7152	13.4608	-2.9461	11.3191	1.3639	-0.2985
0.3	10000	138.4047	23.4205	-5.2019	14.0235	2.3730	-0.5271
0.4	10000	169.7267	34.9170	-7.5728	17.1969	3.5378	-0.7673
0.5	10000	205.5068	47.9005	-10.0372	20.8222	4.8533	-1.0170
1	2500	448.3331	136.9218	-25.7772	45.4256	13.8731	-2.6118

Table 6.6: *Coefficients of the best fitting second order polynomial* $\lambda_n = an^2 + bn + c$ *for eigenvalue asymptotics for multiplier function* $m(s) = e^{-\frac{1}{s^\alpha}}$ *and several values of* α

for the eigenvalues of the Sturm-Liouville problem (6.1) with $a(\tau) = \dfrac{1}{m^2(\tau)}$ and $m(\tau) =$

$e^{-\frac{1}{\tau\alpha}}$. In order to find an expression for $f(\alpha)$ in (6.20) (as for multiplier functions $m(s) = s^{\alpha}$) we are going to consider the values $\dfrac{a}{\pi^2}$ in table 6.6 and two different approaches: If we assume $g(\alpha) = (l\alpha + e)^2$, we get $l \approx 4.3$. But if we assume $g(\alpha) = (e^{m\alpha+1})^2$ we get $m \approx 1$. The second attempt is definitely stronger then the first one, hence we can certainly always state

$$\lambda_n^{\text{approx}}(A) \leq (e^{\alpha+1})^2 \pi^2 n^2, \tag{6.21}$$

but it is hard to find a more exact approximation as for multiplier function $m(s) = s^{\alpha}$, since we would need more values for α. Hence, the singular values of the integral operator satisfy

$$\sigma_n^{\text{approx}}(B) \geq \frac{1}{(l\alpha + e)\pi n}, \quad l > 1, \tag{6.22}$$

where $l \approx 4.3$ has been computed from the values in the sixth column of table 6.6 for $\alpha \leq 1$.

We can conjecture from our numerical experiments that this type of multiplier function also only influences the coefficient for n^{-1} but not the degree of ill-posedness.

In section 6.5 we conjecture how the multiplier functions $m(s)$ influence the coefficient exactly.

6.2 The Galerkin method for the Fredholm integral equation of the first kind

6.2.1 Introduction

We consider the the original Fredholm integral equation of the first kind (4.14)

$$[K(x)](s) = \int_0^1 k(s,t)x(t)dt = g(s), \quad (0 \leq s \leq 1), \tag{6.23}$$

or in operator notation

$$Kx = g,$$

with kernel function $k(s,t)$ given by (4.15)

$$k(s,t) = \begin{cases} m(s), & (0 \leq t \leq s \leq 1) \\ 0, & (0 \leq s \leq t \leq 1). \end{cases} \tag{6.24}$$

From the theory of Hilbert-Schmidt operators in section 3.1 (see definition 3.4), we know, that every square integrable kernel has a singular value expansion (SVE) of the form

$$k(s,t) = \sum_{j=1}^{\infty} \sigma_j u_j(t) v_j(s), \quad 0 \leq s \leq 1, \quad 0 \leq t \leq 1, \tag{6.25}$$

in which σ_j are the singular values and $\{u_j, v_j\}$ are the left and right singular functions of the kernel. The norm of the kernel is given by

$$\|K\|^2 = \int_0^1 \int_0^1 (k(s,t))^2 dtds. \tag{6.26}$$

The algebraic singular value decomposition (SVD) of a real square matrix $A \in \mathbb{R}^{n,n}$ is given by (see, for example [26])

$$A = U\Sigma V^T = \sum_{j=1}^{n} s_j \mathbf{u}_j \mathbf{v}_j^T, \tag{6.27}$$

where

$$\Sigma = \text{diag}(s_1, s_2, \ldots, s_n) \in \mathbb{R}^{n,n},$$

$$U = [\mathbf{u}_1, \mathbf{u}_1, \ldots, \mathbf{u}_n] \in \mathbb{R}^{n,n},$$

$$V = [\mathbf{v}_1, \mathbf{v}_1, \ldots, \mathbf{v}_n] \in \mathbb{R}^{n,n},$$

with U and V being orthogonal. The singular values of A are the scalars s_j, the vectors $\{\mathbf{u}_j\}$ and $\{\mathbf{v}_j\}$ are the left and right singular vectors of A. The left and right singular functions are orthonormal, and both the sets $\{u_j\}$ and $\{v_j\}$ are complete. Hence, with the theory of Hilbert-Schmidt operators (see lemma 3.2) we can write down the norm of $\|K\|$ in terms of the singular values as

$$\|K\|_{HS}^2 := \sum_{j=1}^{\infty} \sigma_j^2 < \infty. \tag{6.28}$$

In the following we mean $\|K\| := \|K\|_{HS}$. The matrix norm of A, that corresponds to the norm of K is the *Frobenius norm* $\|A\|_F$ (see, for example [77]), which is defined by

$$\|A\|_F^2 := \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 = \sum_{j=1}^{n} s_j^2. \tag{6.29}$$

We will summarize some relationships between the SVE for operators in the infinite dimensional space and the SVD for real square matrices in the finite dimensional space in table 6.7. This table is taken from a paper by Hansen [36]. As we have seen in the previous sections, finding the SVE is hard and can only be determined analytically for some special cases (see section 3.3.2). However, we will show, how the SVE can be computed numerically by discretizing (6.23) and calculating the SVD of the obtained matrix. We are going to examine the relationship between the SVE and the SVD, i.e. the approximation properties of the algorithm that provides the singular values numerically. Especially Hansen ([37] and [36]) was dealing with the relationship between the SVE and the SVD. Another related work is [15]. Two papers considering mainly the condition number of the matrix obtained are [2] and [84].

SVE		SVD	
$\sigma_1 \geq \sigma_2 \geq \sigma_3 \ldots$ with $\sigma_n \to 0, \quad n \to \infty$		$s_1 \geq s_2 \geq \ldots s_n \geq 0$	
$\|K\|^2 = \sum_{j=1}^{\infty} \sigma_j^2$		$\|A\|_F^2 = \sum_{j=1}^{n} s_j^2$	
$\langle u_i, u_j \rangle = \delta_{ij}$ $\langle v_i, v_j \rangle = \delta_{ij}.$	$i, j = 1, 2, \ldots$	$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \delta_{ij}$ $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}.$	$i, j = 1, 2, \ldots, n$
$\int_0^1 k(s,t) v_j(s) ds = \sigma_j u_j(t)$ $\int_0^1 \overline{k(s,t)} u_j(t) dt = \sigma_j v_j(s).$	$j = 1, 2, \ldots$	$A\mathbf{v}_j = s_j \mathbf{u}_j$ $A^T \mathbf{u}_j = s_j \mathbf{v}_j.$	$j = 1, 2, \ldots, n$

Table 6.7: *Some properties of the SVE and the SVD*

6.2.2 The algorithm and approximation properties

We are going to use the so called *Galerkin method* (see [6], [84] or [35]) for discretizing the integral equation (6.23). Hansen [36] introduces a generalization to the Galerkin method, the so-called *moment method* with orthonormal basis functions. It is a universal expansion method for discretizing the integral equation (6.23).

Let $\{\Psi_j\}$ and $\{\Phi_j\}$ be two orthonormal sets of basis functions , the left basis functions $\Psi_1, \ldots \Psi_n$ in the interval $I_s = [0,1]$ and the right basis functions $\Phi_1, \ldots \Phi_n$ in the interval $I_t = [0,1]$. Then we approximate the solution x of operator (6.23) by

$$x(t) = \sum_{j=1}^{n} x_j \Phi_j(t). \tag{6.30}$$

Hence,

$$g(s) = \sum_{j=1}^{n} x_j \int_0^1 k(s,t) \Phi_j(t) dt \quad (0 \leq s \leq 1). \tag{6.31}$$

Multiplying (6.31) by $\Psi_i(s)$ and integrating with respect to s yields

$$\int_0^1 g(s) \Psi_i(s) ds = \sum_{j=1}^{n} x_j \int_0^1 \int_0^1 k(s,t) \Phi_j(t) \Psi_i(s) dt ds, \tag{6.32}$$

or, in an easier notation

$$g_i = \sum_{j=1}^{n} a_{ij} x_j, \tag{6.33}$$

where

$$\begin{aligned} g_i &= \int_0^1 g(s) \Psi_i(s) ds \\ a_{ij} &= \int_0^1 \int_0^1 k(s,t) \Phi_j(t) \Psi_i(s) dt ds \\ &= \langle K\Phi_j, \Psi_i \rangle_{L^2(0,1)}, \quad i, j = 1, \ldots, n \end{aligned}$$

in the normal inner product in $L^2(0,1)$. Therefore the equation (6.23) in operator form

$$g = Kx$$

becomes

$$\mathbf{g} = A\mathbf{x},$$

in matrix form, where $\mathbf{g}$ and $\mathbf{x}$ are vectors in $\mathbb{R}^n$ and $A \in \mathbb{R}^{n,n}$ is a matrix. By considering this obtained matrix $A \in \mathbb{R}^{n,n}$

$$A = (a_{ij}), \quad i, j = 1, \ldots n \tag{6.34}$$

we may derive decay rates of its singular values. In the case of Hermitian kernels $k(s,t)$, the Galerkin equations above are precisely the equations obtained from the Rayleigh-Ritz method described below (section 6.3).

We could use the approach introduced by Hansen ([36]). He suggests to compute the SVD of the matrix A numerically. Then he shows that the computed algebraic singular values $(s_1, ..., s_n)$ are indeed good approximations to the first n true singular values $\sigma_i, \quad i = 1, \ldots, n$ of the infinite dimensional problem (6.23). We may also compute approximations to the singular functions u_j and v_j of K using

$$\tilde{u}_j(s) = \sum_{i=1}^{n} u_{ij} \Psi_i(s) \quad \tilde{v}_j(t) = \sum_{i=1}^{n} v_{ij} \Phi_i(t), \quad j = 1, \ldots, n, \tag{6.35}$$

where u_{ij} and v_{ij} are the elements of U and V in (6.27).

Another approach is the one by Wing et. al. using condition numbers (see [84], [2] or [48]). The condition of the matrix $A \in \mathbb{R}^{n,n}$ arising in the discretization process using the Galerkin method (see (6.34)) is very important. If A is well-conditioned, the vector $\mathbf{x}$ can be found quite accurately, even if there is some error in $\mathbf{g}$. An ill-conditioned A may lead to unsatisfactory solutions $\mathbf{x}$. The matrix condition number $\kappa(A)$ of a square matrix A is defined as

$$\kappa(A) = \|A\| \cdot \|A^{-1}\|,$$

where $\|\cdot\|$ is any valid matrix norm (see [26]). If we use the usual Euclidean norm $\|\cdot\|_2$ of vectors and the associated matrix norm, then the condition number is the ratio of the largest singular value of matrix A to the smallest, i.e.

$$\kappa(A) = \frac{s_1(A)}{s_n(A)}.$$

Since we know from section 3.2, inequality (3.14), that the singular values σ_j from the infinite dimensional problem satisfy

$$\sigma_j \sim j^{-\nu}, \quad \text{as} \quad j \to \infty$$

where $\nu > 0$ is the degree of ill-posedness there is a correspondence between the computed singular values $s_n(A)$ of A to the growth rate of the condition number (see [48]),

$$\kappa(A) \sim \frac{1}{s_n(A)} \sim n^{\nu},$$

with the condition number given as above.

We will consider Hansen's ([36]) algorithm more detailed. First, we summarize the algorithm:

Algorithm 6.2 (Galerkin method).

1. *Choose* $\{\Psi_j\}$ *and* $\{\Phi_j\}$*, two orthonormal sets of basis functions in the intervals* $I_s = (0,1)$ *and* $I_t = (0,1)$*, respectively.*
2. *Determine the matrix* $A \in \mathbb{R}^{n,n}$ *with*
$$a_{ij} = \langle K\Phi_j, \Psi_i \rangle_{L^2(0,1)} \quad i,j = 1,\ldots,n.$$
3. *Compute the singular value decomposition of this matrix*
$$A\mathbf{v} = s^{(n)}\mathbf{u}.$$

Optionally, we could also compute approximations to the singular functions u_j and v_j of K using (6.35). Note, when $I_s = I_t$ and K is symmetric, the algorithm is identical to the Galerkin method for computing eigensolutions of K. Then the above procedure is often called the *Rayleigh-Ritz procedure* for which we refer to [10] or [68]. We will examine this special case in the next section.

We are now going to examine the approximation properties of the algorithm given above. The singular values $s_i^{(n)}$ of A are approximations to the singular values of K. Hansen (see [36] or [37]) has proved the following results, which we summarize in lemma 6.1 and proposition 6.2. The analysis relies on the following definition of the singular values σ_i.

Definition 6.4. *The singular values* σ_i *of the real square integrable kernel* K *are the stationary values of the functional*
$$F[\Phi,\Psi] := \frac{\langle K\Phi, \Psi \rangle}{\|\Phi\|\|\Psi\|}, \tag{6.36}$$
where $\dfrac{\Psi}{\|\Psi\|}$ *and* $\dfrac{\Phi}{\|\Phi\|}$ *are the corresponding left and right singular functions.*

Notice that this definition is similar to the characterization of the algebraic singular values of a real matrix, see [26]. Furthermore, there is a similarity to the Rayleigh quotient R for a symmetric kernel K, which is given by (see [10])
$$R[\Phi] := \frac{\langle K\Phi, \Phi \rangle}{\|\Phi\|^2}.$$
We will come back to this fact later. Now, the basic idea of the algorithm is to approximate the kernel K by a degenerate kernel $\tilde{K}_n$ whose exact SVE can be determined with the help of the algebraic SVD. We can formulate this connection explicitly in the following lemma 6.1 by Hansen, which we want to prove for a better understanding.

Lemma 6.1 (Hansen 1987). *The singular values s_i and functions $\tilde{u}_i$ and $\tilde{v}_i$ computed by the above algorithm are the exact singular values and functions of the degenerate kernel*

$$\tilde{K}_n(s,t) \equiv \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\Psi_i(s)\Phi_j(t), \tag{6.37}$$

and they also satisfy

$$\langle K\tilde{v}_i, \tilde{u}_i\rangle = s_i. \tag{6.38}$$

Proof. From definition (6.4) it follows that the singular values of $\tilde{K}_n$ are the stationary values of the functional

$$\tilde{F}_n[\Phi,\Psi] = \langle \tilde{K}_n\Phi, \Psi\rangle$$

which are also the stationary values of the functional $F[\Phi,\Psi]$ with the restrictions $\Phi \in \operatorname{span}\{\Phi_1,\ldots\Phi_n\}$ and $\Psi \in \operatorname{span}\{\Psi_1,\ldots\Psi_n\}$. Then, writing

$$\Phi(t) = \sum_{i=1}^{n} x_i\Phi_i(t) \quad \text{and} \quad \Psi(s) = \sum_{i=1}^{n} y_i\Psi_i(s),$$

the latter problem leads to the algebraic problem of computing the stationary values of

$$G[\mathbf{x},\mathbf{y}] = \frac{\mathbf{y}^T A\mathbf{x}}{\|\mathbf{x}\|\|\mathbf{y}\|}, \tag{6.39}$$

with A given by (6.34), and these stationary values are exactly the singular values of A. For the proof of (6.38) we refer to [10]. □

Again, note the similarity of (6.39) to the algebraic Rayleigh quotient of a symmetric matrix A

$$R[\mathbf{x}] = \frac{\mathbf{x}^T A\mathbf{x}}{\|\mathbf{x}\|^2}.$$

Also notice that if the basis functions are not orthonormal, then the computational problem involved in the Galerkin approach leads to the computation of the stationary values of

$$G_{S,T}[\mathbf{x},\mathbf{y}] = \frac{\mathbf{y}^T A\mathbf{x}}{(\mathbf{x}^T S\mathbf{x}\mathbf{y}^T T\mathbf{y})^{\frac{1}{2}}}, \tag{6.40}$$

with symmetric matrices S and T whose elements are given by

$$s_{ij} = \langle\Psi_i,\Psi_j\rangle \quad \text{and} \quad t_{ij} = \langle\Phi_i,\Phi_j\rangle \quad i,j = 1,\ldots,n.$$

Note that for orthonormal basis functions $S = T = I$. If S and T are well conditioned, then a change of variables, via computing the Cholesky factors of S and T, leads to the desired results. If the basis functions are sufficiently linearly independent and S and T symmetric positive definite, then there exist unique lower triangular matrices G and H

such that $S = GG^T$ and $T = HH^T$. Substituting $\bar{\mathbf{x}} = G^T\mathbf{x}$ and $\bar{\mathbf{y}} = H^T\mathbf{y}$ into (6.40) leads to

$$G[\bar{\mathbf{x}}, \bar{\mathbf{y}}] = \frac{\bar{\mathbf{y}}^T \bar{A} \bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\| \|\bar{\mathbf{y}}\|},$$

where $\bar{A} = H^T A G^{-T}$ and we can apply the same analytical tools as before with changed variables. Finally we are going to state approximation properties for the above algorithm.

Proposition 6.2 (Hansen 1987). *Let $\|K\|$ denote the norm of K, defined by*

$$\|K\|^2 := \int_0^1 \int_0^1 |k(s,t)|^2 dt ds = \sum_{i=1}^{\infty} \sigma_i^2.$$

Moreover, let A be the matrix (6.34) in the above algorithm, let $\tilde{K}_n$ be the corresponding kernel given by (6.37) and let $\|A\|_F$ denote the Frobenius norm of A. Then

$$\delta_n^2 := \|K - \tilde{K}_n\|^2 = \|K\|^2 - \|A\|_F^2. \tag{6.41}$$

holds and the algebraic singular values $s_i^{(n)}$, where n is the number of basis functions, are increasingly better approximations to the true singular values σ_i,

$$s_i^{(n)} \leq s_i^{(n+1)} \leq \sigma_i, \quad i = 1, \ldots, n. \tag{6.42}$$

Furthermore, the errors of the approximate singular values $s_i^{(n)}$ are bounded by

$$0 \leq \sigma_i - s_i^{(n)} \leq \delta_n, \quad i = 1, \ldots, n, \tag{6.43}$$

where δ_n is given in (6.41). The sum of squares of the errors of the approximate singular values $s_i^{(n)}$ is bounded by

$$\sum_{i=1}^{n} [\sigma_i - s_i^{(n)}]^2 \leq \delta_n^2. \tag{6.44}$$

Moreover, the true singular values σ_i of K are bounded in terms of the computed singular values $s_i^{(n)}$ of A by

$$s_i^{(n)} \leq \sigma_i \leq [(s_i^{(n)})^2 + \delta_n^2]^{\frac{1}{2}}, \quad i = 1, \ldots, n. \tag{6.45}$$

The singular functions $\tilde{u}_i$ and $\tilde{v}_i$ are orthonormal and the errors of the singular functions corresponding to distinct singular values are bounded by

$$\max\{\|u_i - \tilde{u}_i\|, \|v_i - \tilde{v}_i\|\} \leq \left(\frac{2\delta_n}{\sigma_i - \sigma_{i+1}} \right)^{\frac{1}{2}}, \quad i = 1, \ldots, n. \tag{6.46}$$

Proof. We state the proof from [36]. First, we show (6.41). From definition (6.26) it follows, that

$$\begin{aligned}
\delta_n^2 &= \|K\|^2 + \|\tilde{K}_n\|^2 - 2\int_0^1 \int_0^1 K(s,t)\tilde{K}_n(s,t)dsdt \\
&= \|K\|^2 + \|\tilde{K}_n\|^2 - 2\|\tilde{K}_n\|^2 = \|K\|^2 - \|\tilde{K}_n\|^2 \\
&= \|K\|^2 - \|A\|_F^2,
\end{aligned}$$

where we have used lemma 6.1, equation (6.28) and the equation

$$\int_0^1 \int_0^1 K(s,t)\tilde{K}_n(s,t)dsdt = \int_0^1 \int_0^1 |\tilde{K}_n(s,t)|^2 dsdt = \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2,$$

which holds for orthonormal left and right basis functions $\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t)$ and $\Psi_1(s), \Psi_2(s), \ldots, \Psi_n(s)$. Equation (6.42) follows immediately from the facts that all the basis functions $\Phi_1, \ldots, \Phi_n$ and $\Psi_1, \ldots, \Psi_n$ are orthogonal and that the singular values $s_i^{(n)}$ and $s_i^{(n+1)}$ are the stationary values of the functional $F[\Phi, \Psi]$ where Φ and Ψ are restricted to n-dimensional and $n+1$-dimensional subspaces, respectively. Obviously, increasing the size of the subspaces yields increasingly better approximations to the singular values σ_i.

The first inequality in (6.43) follows directly from (6.42). In order to prove the right inequality, notice that by Schwarz's inequality we may derive

$$\begin{aligned}
F[\Phi, \Psi] - \tilde{F}_n[\Phi, \Psi] &= [\Psi, (K\Phi - \tilde{K}_n\Phi)] \\
&\leq \|\Psi\| \|K - \tilde{K}_n\| \|\Phi\| = \|K - \tilde{K}_n\|
\end{aligned}$$

since $\|\Psi\| = \|\Phi\| = 1$. Using lemma 6.1 we get that the singular values of $\tilde{K}_n$ are the stationary values of the functional $\tilde{F}_n[\Phi, \Psi] = \langle \tilde{K}_n\Phi, \Psi \rangle$ we get

$$F[\Phi, \Psi] \leq s_i^{(n)} + \|K - \tilde{K}_n\|, \quad i = 1, \ldots, n.$$

Finally σ_i are the stationary values of $F[\Phi, \Psi]$ and we obtain

$$\sigma_i \leq s_i^{(n)} + \|K - \tilde{K}_n\|, \quad i = 1, \ldots, n,$$

and equation (6.41) leads to (6.43). This is the same argument as used in [10, page 121]. The result (6.44) follows from the relation

$$\begin{aligned}
\sum_{i=1}^n [\sigma_i - s_i^{(n)}]^2 &= \sum_{i=1}^n \sigma_i^2 + \sum_{i=1}^n [s_i^{(n)}]^2 - 2\sum_{i=1}^n \sigma_i s_i^{(n)} \\
&\leq \sum_{i=1}^n \sigma_i^2 + \|A\|_F^2 - 2\sum_{i=1}^n [s_i^{(n)}]^2 \\
&\leq \|K\|^2 - \|A\|_F^2 = \delta_n^2.
\end{aligned}$$

The left inequality in (6.45) us just equation (6.42), which we have shown already and which also leads to $[s_i^{(n)}]^2 \leq \sigma_i^2$, $i = 1,\dots,n$. Hence

$$\sum_{l=1}^{n}[s_l^{(n)}]^2 - [s_i^{(n)}]^2 \leq \sum_{l=1}^{n}\sigma_l^2 - \sigma_i^2, \quad i = 1,\dots,n,$$

and therefore

$$\begin{aligned} \sigma_i^2 - [s_i^{(n)}]^2 &\leq \sum_{l=1}^{n}\sigma_l^2 - \sum_{l=1}^{n}[s_l^{(n)}]^2 \\ &\leq \sum_{l=1}^{\infty}\sigma_l^2 - \sum_{l=1}^{n}[s_l^{(n)}]^2 = \|K\|^2 - \|A\|_F^2 = \delta_n^2. \end{aligned}$$

Finally the orthonormality of $\tilde{u}_i$ and $\tilde{v}_i$ follows immediately from definition (6.35), the orthogonality of U and V and the orthonormality of the basis functions. For the result (6.46) we refer to the appendix of [36]. □

This proposition implies that if $\delta_n \to 0$ as $n \to \infty$, then the approximate singular values $s_i^{(n)}$ converge uniformly in n to the true singular values σ_i, and the corresponding approximate singular functions $\tilde{u}_i$ and $\tilde{v}_i$ converge in the mean to the true singular functions. Furthermore, the Galerkin method always gives lower bounds for the first n eigenvalues of the operator K.

Finally, we want to make some remarks on the convergence of the singular values (and singular functions respectively). When

$$\Phi_1(t), \Phi_2(t), \dots, \Phi_n(t) \quad \text{and} \quad \Psi_1(t), \Psi_2(t), \dots, \Psi_n(t)$$

are part of complete orthonormal sets $\{\Phi\}$ and $\{\Psi\}$, the convergence of the singular values is immediate.

The norm $\|K\|$ of the kernel can often be evaluated rather precisely, and so the quantity δ_n from (6.41) is not only of theoretical interest but also of practical importance.

6.2.3 Multiplier functions of the form $m(s) = s^\alpha$

We are now going to apply the Galerkin method described above to the problem (4.1) with special multiplier functions $m(s) = s^\alpha$ for several values of α. The orthonormal basis functions $\{\Psi_j\}$ and $\{\Phi_j\}$ are simply chosen to give piecewise constant approximations to the singular functions. The intervals $I_s = I_t = [0,1]$ are divided into n subintervals $[s_i, s_{i+1}]$ and $[t_j, t_{j+1}]$ of the same length h_s and h_t, respectively. Since our intervals I_s and I_t are the same, we can chose the same subintervals and interval lengths. The basis functions (that are also the same) are then given by

$$\Phi_j(t) = \begin{cases} \sqrt{t_j - t_{j-1}}, & t \in [t_{j-1}, t_j] \\ 0, & \text{else} \end{cases}, \quad j = 1,\dots,n \tag{6.47}$$

$$\Psi_i(s) = \begin{cases} \sqrt{s_i - s_{i-1}}, & s \in [s_{i-1}, s_i] \\ 0, & \text{else} \end{cases}, \quad i = 1,\dots,n. \tag{6.48}$$

With $h := h_s = h_t = (s_i - s_{i-1}) = (t_j - t_{j-1})$ and by taking the same subintervals, such that $s_i = t_i$, $\forall i$ we get the following entries for the matrix A from (6.34):

$$a_{ij} = \begin{cases} \dfrac{1}{h}\displaystyle\int_{s_i}^{s_{i+1}} m(s)(s-s_i)ds, & \text{if } i = j, \\ \displaystyle\int_{s_i}^{s_{i+1}} m(s)ds, & \text{if } j < i, \\ 0, & \text{else} \end{cases} \tag{6.49}$$

Hence, A becomes a lower triangular matrix. This is due to the fact that the kernel $k(s,t)$ is only defined on the lower triangle. For $m(s) = s^\alpha$ we can integrate these functions analytically and determine the entries of A quite easily and exactly. We are then going to compute the SVD of this matrix A for several values of $n = \frac{1}{h}$ and α. We are interested in the singular value asymptotics. Therefore, we compute the first 15 singular values $s_1, \ldots, s_{15}$ of largest magnitude for a fixed value of $n = 1000$ and plot them in figure 6.5. Since the plots suggest that the computed singular values s_n are

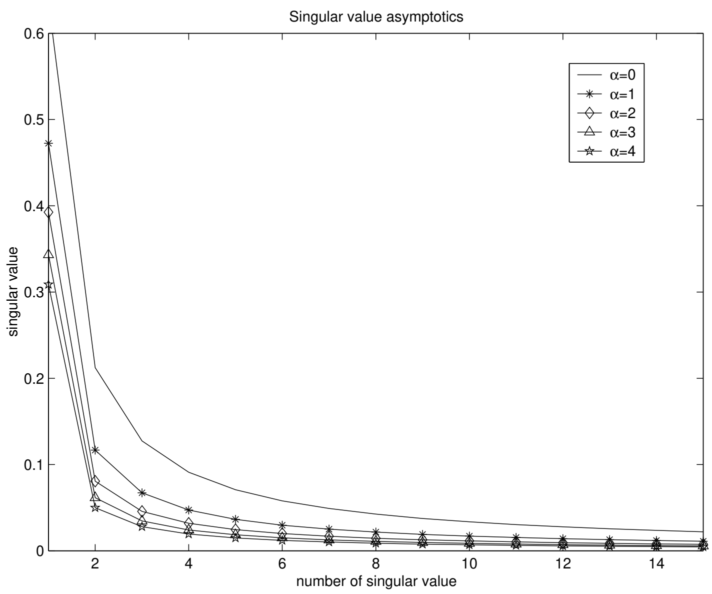

Figure 6.5: *First* 15 *computed singular values of integral equation* $Bv = \sigma u$ *for multiplier function* $m(s) = s^\alpha$*,* $n = 1000$ *and different values for* α *using a Galerkin method*

proportional to $\dfrac{1}{n}$, we are going to consider the logarithmic scale plot for all computed singular values for $n = 100$ and several values of α. The results are shown in figure 6.6. There we observe that the plots in logarithmic scaling have the same gradients, with shifts in direction of the y-axis for different values of α. Using the Galerkin method for the integral equation, we do not have any problems with boundary conditions such as in with the finite difference method. Moreover, proposition 6.2 even supplies error bounds for the computed singular values.

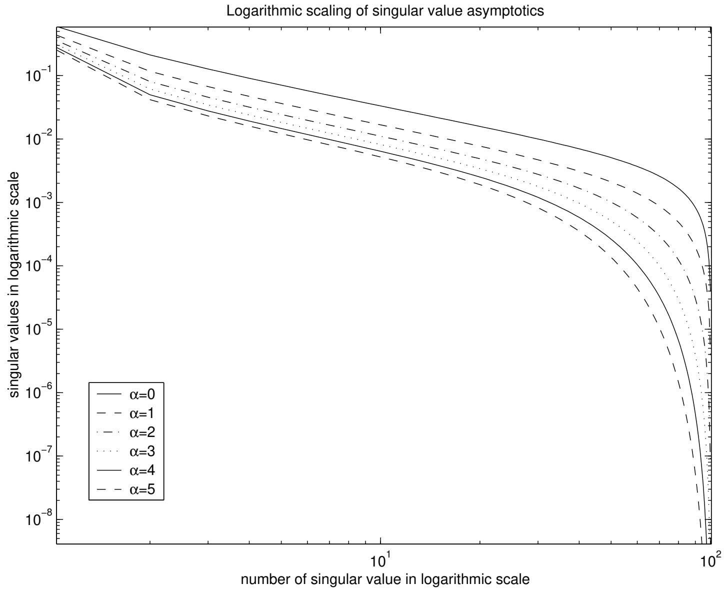

Figure 6.6: *Computed singular values of integral equation $Bv = \sigma u$ for $n = 100$, multiplier function $m(s) = s^{\alpha}$ and different values for α in logarithmic scales*

In table 6.8 we have summarized the error δ_n as given in (6.41) for several n and α. The error is computed by $\delta_n = \sqrt{\|K\|^2 - \|A\|_F^2}$. We can determine $\|K\|$ by

$$\begin{aligned} \|K\|^2 &= \int_0^1 \int_0^1 |k(s,t)|^2 dsdt \\ &= \int_0^1 m(s)^2 sds, \end{aligned}$$

and therefore, for our special multiplier function $m(s) = s^{\alpha}$

$$\|K\| = \frac{1}{\sqrt{2\alpha + 2}}. \tag{6.50}$$

The value of $\|A\|_F$ can be computed numerically. From table 6.8 we see, that $\delta_n \to 0$ as $n \to \infty$ and therefore, we may follow, that the algebraic singular values converge to the true ones.

The gradient that we can observe in figure 6.6 for the computed singular values of largest magnitude is -1. The computed singular values of small magnitude are no good approximations to the true singular values, since the error increases, the smaller the computed singular values get. We will illustrate this effect in figure 6.7. For a better clearness we only consider the special integral equation with multiplier function $m(s) = s^{\alpha}$ for $\alpha = 1$, but we could use any value of α. From figure 6.7 and with the help of proposition 6.2 we follow, that the true singular values lie between the thick solid and the dotted line. By computing the first 100 singular values for $n = 1000$

α	n	h	δ_n
0	50	0.02	0.07071
0	100	0.01	0.05000
0	500	0.002	0.02236
0	1000	0.001	0.01581
1	50	0.02	0.04123
1	100	0.01	0.02901
1	500	0.002	0.01292
1	1000	0.001	0.00913
2	50	0.02	0.03240
2	100	0.01	0.02264
2	500	0.002	0.01002
2	1000	0.001	0.00708
3	50	0.02	0.02794
3	100	0.01	0.01933
3	500	0.002	0.00849
3	1000	0.001	0.00599

Table 6.8: *Results for the errors of the Galerkin method applied to integral equations $Bv = \sigma u$ with multiplier functions $m(s) = s^\alpha$ for different values of n and α*

(dashed line) this statement is confirmed. From the dashed line we also see that the smallest computed singular values (i.e. those that are close to the matrix size n) are very inaccurate. Therefore we only consider the largest singular values and the plots suggest, that their asymptotics are

$$\sigma_n^{\text{approx}}(B) = g(\alpha)O(n^{-1}),$$

where g is a monotone decreasing function depending on α. Therefore the reciprocal value

$$\frac{1}{\sigma_n^{\text{approx}}(B)}$$

would yield a linear function in n. Knowing that, we are going to determine the values $\frac{1}{s_i^{(n)}}$, $i = 1, \dots, 10$ for the first ten computed singular values. Then we apply curve fitting to those values, assuming that

$$\frac{1}{\sigma_n^{\text{approx}}(B)} = dn + f,$$

and summarize the results for different values of α and constant $n = 1000$ in table 6.9. From this table we observe a regularity for the coefficients d and f. Both coefficients

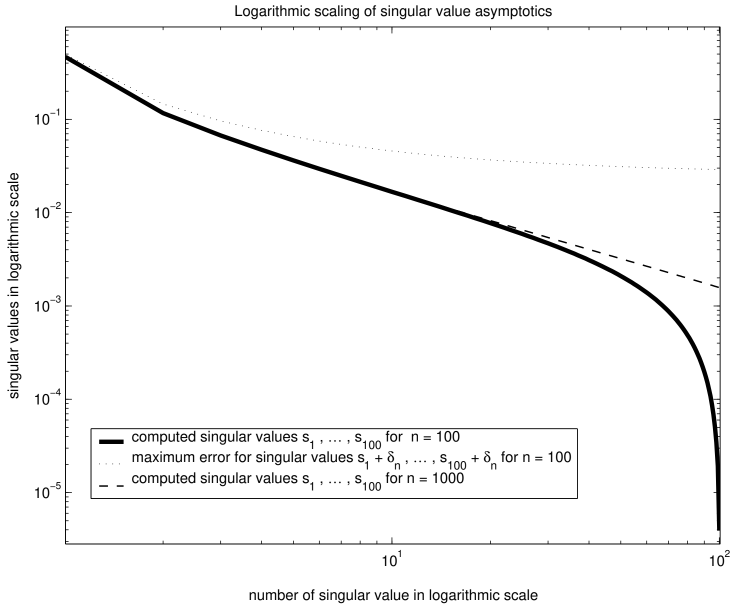

Figure 6.7: *Computed singular values of integral equation $Bv = \sigma u$ for $n = 100$, multiplier function $m(s) = s^{\alpha}$ and $\alpha = 1$ together with error bound*

α	n	d	f	$\frac{d}{\pi}$	$\frac{f}{\pi}$
0	1000	3.1418	-1.5714	1.0001	-0.5002
0.5	1000	4.7196	-2.8081	1.5023	-0.8938
1	1000	6.3002	-4.0655	2.0054	-1.2941
1.5	1000	7.8826	-5.3362	2.5091	-1.6986
2	1000	9.4666	-6.6166	3.0133	-2.1061
2.5	1000	11.0517	-7.9046	3.5179	-2.5161
3	1000	12.6378	-9.1990	4.0228	-2.9281
3.5	1000	14.2253	-10.4990	4.5280	-3.3419
4	1000	15.8135	-11.8040	5.0336	-3.7573

Table 6.9: *Coefficients of the best fitting first order polynomial $\frac{1}{\sigma_n} = dn + f$ for singular value asymptotics for multiplier function $m(s) = s^{\alpha}$ with $n = 1000$ and several values of α*

are linear in α, we therefore conjecture

$$\frac{1}{\sigma_n^{\text{approx}}(B)} = (\alpha+1)\pi n + O(\alpha) \tag{6.51}$$

for the singular values of the integral operator (4.1) with multiplier function $m(s) = s^{\alpha}$. Hence, the computed approximate singular values of this integral operator have the asymptotics

$$\sigma_n^{\text{approx}}(B) \sim \frac{1}{(\alpha+1)\pi n},$$

something we have already observed in section 6.1.3. To this end the multiplier function only influences the coefficient for n^{-1} (i.e. the constants $\underline{C}$ and $\overline{C}$ in (3.14)), but the degree of ill-posedness does not change. This is the same observation as we made in section 6.1.3.

6.2.4 Multiplier functions of the form $m(s) = e^{-\frac{1}{s^\alpha}}$

Now, we are going to apply the Galerkin method to the problem (4.1) with special multiplier functions $m(s) = e^{-\frac{1}{s^\alpha}}$ for different values of α. The orthonormal basis functions $\{\Psi_j\}$ and $\{\Phi_j\}$ are chosen as in (6.47) and (6.48). Then we get the following entries for the lower triangular matrix A from (6.34):

$$a_{ij} = \begin{cases} \frac{1}{h}\int_{s_i}^{s_{i+1}} m(s)(s-s_i)ds, & \text{if } i = j, \\ \int_{s_i}^{s_{i+1}} m(s)ds, & \text{if } j < i, \\ 0, & \text{else} \end{cases} \tag{6.52}$$

This time, these functions cannot be integrated analytically and so we determine the entries of A using numerical integration. Then we compute the SVD of this matrix A for several values of $n = \frac{1}{h}$ and α. As above we compute the first 10 singular values of largest magnitude and determine the error $\delta_n = \sqrt{\|K\|^2 - \|A\|_F^2}$ for different n and α as given in (6.41). We have to compute $\|K\|$, given by

$$\|K\|^2 = \int_0^1 m(s)^2 s ds, \tag{6.53}$$

as well as $\|A\|_F$ numerically. The results are summarized in table 6.10. Since each entry of the matrix has to be computed by a numerical integration, the construction of the matrix takes a very long time. Hence, we keep n, the size of the matrices restricted to $n = 200$. We see, that $\delta_n \to 0$ as $n \to \infty$ and therefore, we follow, that the algebraic singular values converge to the true ones. We are interested in the singular value asymptotics. We compute the first 10 singular values $s_1, \ldots, s_{10}$ of largest magnitude for a fixed value of $n = 200$ and plot them in figure 6.8. Since the plots suggest that the computed singular values s_n are proportional to $\frac{1}{n}$, we are going to consider the logarithmic scale plot for all computed singular values for $n = 100$ and several values of α again. The results are shown in figure 6.9. We observe that the plots in logarithmic scaling have the same gradients, with shifts in direction of the y-axis for different values of α. Using the Galerkin method for the integral equation, we do not have any difficulties with boundary conditions such as in with the finite difference method. Moreover, proposition 6.2 supplies error bounds for the computed singular values.

α	n	h	δ_n
0	50	0.02	0.02601
0	100	0.01	0.01839
0	200	0.005	0.01301
0.1	50	0.02	0.02362
0.1	100	0.01	0.01669
0.1	200	0.005	0.01179
0.3	50	0.02	0.01994
0.3	100	0.01	0.01407
0.3	200	0.005	0.00994
0.5	50	0.02	0.01747
0.5	100	0.01	0.01231
0.5	200	0.005	0.00869
1	50	0.02	0.01390
1	100	0.01	0.00976
1	200	0.005	0.00687

Table 6.10: *Results for the errors of the Galerkin method applied to integral equations $Bv = \sigma u$ with multiplier functions $m(s) = e^{-\frac{1}{s^\alpha}}$ for different values of n and α*

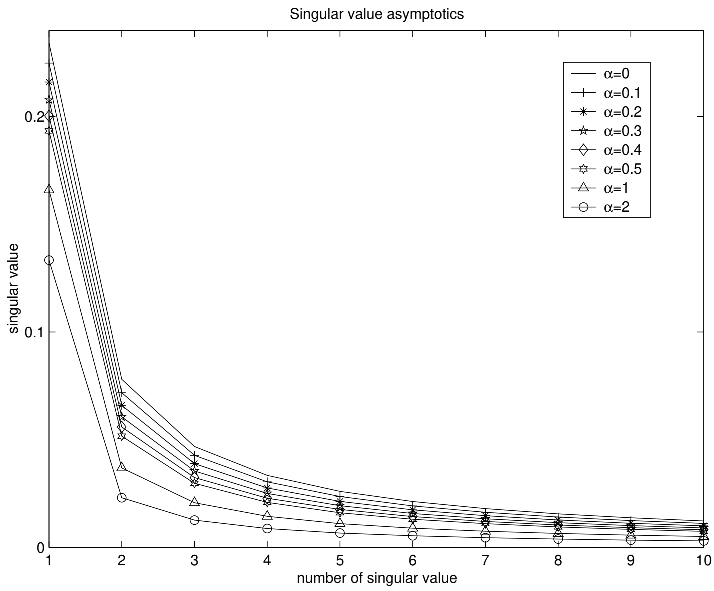

Figure 6.8: *First* 10 *computed singular values of integral equation $Bv = \sigma u$ for multiplier function $m(s) = e^{-\frac{1}{s^\alpha}}$, $n = 200$ and different values for α using a Galerkin method*

The gradient that we observe in figure 6.9 for the computed singular values of largest magnitude is -1. The computed singular values of small magnitude are no good approximations to the true singular values, since the error increases, the smaller the computed singular values get. We refer to the discussion in section 6.2.3. Again, the plots suggest, that the singular values have the asymptotics

$$\sigma_n^{\text{approx}}(B) = g(\alpha)O(n^{-1}).$$

Then the reciprocal value

$$\frac{1}{\sigma_n^{\text{approx}}(B)}$$

yields a linear function in n. By the same examination as in the previous section, i.e. assuming that

$$\frac{1}{\sigma_n^{\text{approx}}(B)} = dn + f,$$

we obtain the values in table 6.11. We observe that the coefficients d are linear in α and therefore conjecture

$$\frac{1}{\sigma_n^{\text{approx}}(B)} = h(\alpha)\pi n + O(\alpha) \tag{6.54}$$

for some function $h(\alpha)$ and for the singular values of the integral operator (4.1) with multiplier function $m(s) = e^{-\frac{1}{s^\alpha}}$. We may also compare the values of d in table 6.11

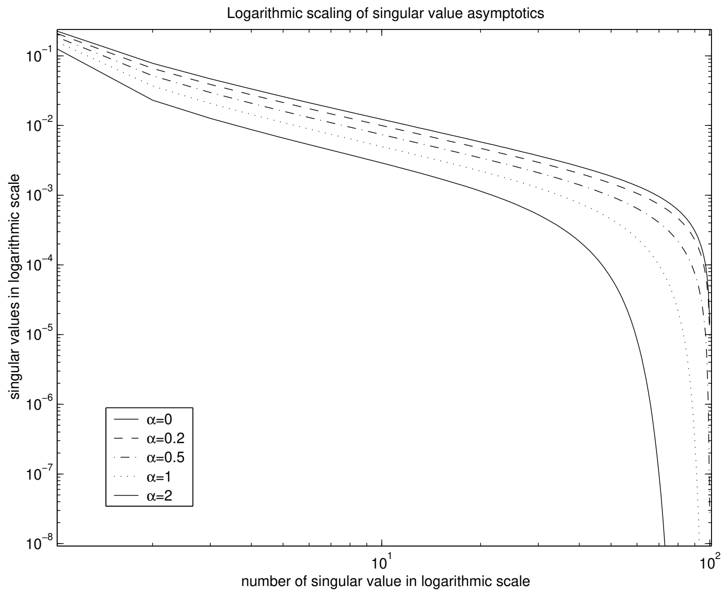

Figure 6.9: *Computed singular values of integral equation $Bv = \sigma u$ for $n = 100$, multiplier function $m(s) = e^{-\frac{1}{s^\alpha}}$ and different values for α in logarithmic scales*

α	n	d	f	$\frac{d}{\pi}$	$\frac{f}{\pi}$
0	200	8.5555	-4.3129	2.7233	-1.3728
0.1	200	9.4918	-5.0727	3.0213	-1.6147
0.2	200	10.5682	-5.9964	3.3639	-1.9087
0.3	200	11.7514	-7.0389	3.7406	-2.2406
0.4	200	13.0094	-8.1580	4.1410	-2.5968
0.5	200	14.3184	-9.3250	4.5577	-2.9682
1	200	21.2426	-15.5019	6.7617	-4.9344

Table 6.11: *Coefficients of the best fitting first order polynomial* $\frac{1}{\sigma_n} = dn + f$ *for singular value asymptotics for multiplier function* $m(s) = e^{-\frac{1}{s^\alpha}}$ *with* $n = 200$ *and several values of* α

with the coefficients a in table 6.6 and we will see that $a \approx d^2$. Hence, the computed singular values of this integral operator have the asymptotics

$$\sigma_n^{\text{approx}}(B) \sim \frac{1}{(l\alpha + e)\pi n}, \quad l > 1$$

as we have already stated in section 6.1.4. The value of l can be computed by examining the coefficients d. Then we get $l \approx 4.3$. As in section 6.1.4 we can certainly always state

$$\sigma_n^{\text{approx}}(B) \geq \frac{1}{(e^{\alpha+1})\pi n}. \tag{6.55}$$

Hence, the multiplier function only seems to influence the coefficient for n^{-1}, but the degree of ill-posedness remains the same. For more details on the influence on the constant we refer to section 6.5.

6.3 The Rayleigh-Ritz method for symmetric kernels and the generalized eigenvalue problem

6.3.1 Introduction

The Rayleigh-Ritz method is a special case of Galerkin's method, although the method by Galerkin can be applied to non-Hermitian kernels. However, the Rayleigh-Ritz method for a Hermitian kernel is of special interest because we can guarantee one-sided bounds on the true eigenvalues.

We consider the self-adjoint integral operator B^*B from (4.26) in section 4.2.3

$$[B^*Bx](\tau) = \int_0^1 K(t,\tau)x(t)dt. \tag{6.56}$$

with kernel $K \in L^2(0,1)$ from (4.28), given by

$$K(t,\tau) = \int_{\max(t,\tau)}^{1} m^2(s)ds, \tag{6.57}$$

knowing that we have to solve the eigenvalue problem (in operator notation)

$$B^* B u = \lambda u, \tag{6.58}$$

in order to obtain the singular values $\sigma = \sqrt{\lambda}$, where the singular value problem is given by

$$Bu = \sigma v \quad \text{or} \quad B^* v = \sigma u.$$

For convenience we set $K := B^* B$ for the self-adjoint integral operator. The norm of K is given by

$$\|K\|^2 = \int_0^1 \int_0^1 (K(t,\tau))^2 dt d\tau,$$

as in the previous section. The algebraic eigenvalue decomposition of a real square matrix $A \in \mathbb{R}^{n,n}$ is given by (see for example [13])

$$A = Q\Lambda Q^T = \sum_{j=1}^{n} \lambda_j \mathbf{q}_j \mathbf{q}_j^T, \tag{6.59}$$

where Λ is the diagonal matrix of eigenvalues

$$\Lambda = \operatorname{diag}(l_1, l_2, \ldots, l_n) \in \mathbb{R}^{n,n},$$

and Q is the orthogonal matrix whose columns are eigenvectors,

$$Q = [\mathbf{q}_1, \mathbf{q}_1, \ldots, \mathbf{q}_n] \in \mathbb{R}^{n,n}.$$

Note that the singular value decomposition of a matrix $\mathcal{C}$, given by $\mathcal{C} = U\Sigma V^T$ is closely related to the eigendecomposition of the symmetric matrices $\mathcal{C}^*\mathcal{C}$ and $\mathcal{C}\mathcal{C}^*$ given by

$$\mathcal{C}^*\mathcal{C} = V\Sigma^T \Sigma V^T.$$

Hence, the algebraic singular values are the roots of the algebraic eigenvalues of $\mathcal{C}^*\mathcal{C}$ or $\mathcal{C}\mathcal{C}^*$, something we have already observed for the infinite dimensional case. Therefore, for self-adjoint operators the singular values are equal to the eigenvalues of the integral operator and for symmetric matrices the algebraic eigenvalues are equal to the algebraic singular values. As in section 6.2 with the theory of Hilbert-Schmidt operators we can write

$$\|K\| = \sqrt{\sum_{j=1}^{\infty} \lambda_j^2} < \infty.$$

The matrix norm of A corresponding to the norm of K is again the Frobenius norm, given by

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2} = \sqrt{\sum_{j=1}^{n} l_j^2}.$$

Relationships between the eigenvalue expansion for operators in the infinite dimensional space and and the eigenvalue decomposition for real square matrices are similar to the ones in table 6.7. The main difference is, that the left and right singular functions (or singular vectors respectively) are equal. Then we may use the same results as Hansen [36] for the special case of symmetric kernels. This special case of symmetric (or Hermitian) kernels was investigated by Cochran in [10] (Note, that Cochran uses λ^{-1} instead of λ) and in [68].

At this stage we are going to state an a posteriori error bound on the local error, taken from [6].

Proposition 6.3. *Suppose that the kernel $K(t,\tau)$ in the eigenvalue problem*

$$\lambda x(\tau) = \int_0^1 K(t,\tau)x(t)dt$$

is Hermitian and square integrable for $0 \le t,\tau \le 1$. If there is a non-zero number $\tilde{\lambda}$ and a non-null square integrable function $\tilde{x}(\tau)$ such that

$$\int_0^1 K(t,\tau)\tilde{x}(t)dt - \tilde{\lambda}\tilde{x}(\tau) = \eta(\tau),$$

then there is an eigenvalue λ of $K(t,\tau)$, such that

$$|\tilde{\lambda} - \lambda| \le \frac{\|\eta(\tau)\|_2}{\|\tilde{x}(\tau)\|_2}.$$

This proposition holds only for Hermitian kernels, and $\tilde{\lambda}$ and $\tilde{x}(\tau)$ may have been produced by any means whatever. For a proof we refer to [6]. The result may be used to state approximation properties of already computed eigenvalues.

6.3.2 The algorithm and approximation properties

We are going to use the same procedure as in section 6.2, but this time we call it *Rayleigh-Ritz* procedure as in [10] and [68], since the Rayleigh quotient plays an important role. The Rayleigh-Ritz method was also investigated by Baker [6] who states, that in the case of Hermitian kernels $K(t,\tau)$ the Galerkin equations from the previous section are precisely the equations obtained from the Rayleigh-Ritz method.

Let $\{\Phi_j\}$ be a linearly independent set of basis functions in the interval $I = [0,1]$. Notice that this time we need only one set of basis functions, since for an eigenvalue expansion we have just one set of eigenfunctions, whereas for a singular value expansion we have left and right singular functions (see previous section). The set $\{\Phi_j\}$ also does not necessary have to be orthonormal. Then we approximate the eigenfunction u of operator equation (6.58) by

$$u(t) = \sum_{j=1}^{n} u_j \Phi_j(t). \tag{6.60}$$

Hence, equation (6.58) becomes

$$\sum_{j=1}^{n} u_j \int_0^1 K(t,\tau)\Phi_j(t)dt = l^{(n)} \sum_{j=1}^{n} u_j \Phi_j(\tau) \quad (0 \leq \tau \leq 1), \tag{6.61}$$

where $l^{(n)} \approx \lambda$ is the approximate eigenvalue if we choose an n-dimensional subspace. Multiplying (6.61) by $\Phi_i(\tau)$ and integrating with respect to τ yields

$$\sum_{j=1}^{n} u_j \int_0^1 \int_0^1 K(t,\tau)\Phi_j(t)\Phi_i(\tau)dtd\tau = l^{(n)} \sum_{j=1}^{n} u_j \int_0^1 \Phi_j(\tau)\Phi_i(\tau)d\tau, \tag{6.62}$$

or, in an easier notation

$$\sum_{j=1}^{n} a_{ij}u_j = l^{(n)} \sum_{j=1}^{n} c_{ij}u_j, \tag{6.63}$$

where

$$\begin{aligned} a_{ij} &= \int_0^1 \int_0^1 K(t,\tau)\Phi_j(t)\Phi_i(\tau)dtd\tau \\ &= \langle K\Phi_j, \Phi_i \rangle_{L^2(0,1)}, \quad i,j = 1,\ldots,n \\ c_{ij} &= \int_0^1 \Phi_j(\tau)\Phi_i(\tau)d\tau \\ &= \langle \Phi_j, \Phi_i \rangle_{L^2(0,1)}, \quad i,j = 1,\ldots,n \end{aligned}$$

in the normal inner product in $L^2(0,1)$. Therefore the equation (6.58) in operator form

$$B^*Bu := Ku = \lambda u$$

becomes

$$A\mathbf{u} = l^{(n)}C\mathbf{u},$$

where $\mathbf{u}$ is a vector in $\mathbb{R}^n$ and $A \in \mathbb{R}^{n,n}$ and $C \in \mathbb{R}^{n,n}$ are matrices. Hence, the eigenproblem (6.58) in $L^2(0,1)$ becomes a *generalized eigenvalue problem* in $\mathbb{R}^n$, which is given by

$$[\langle B\Phi_j, B\Phi_i \rangle_{L^2(0,1)}]\mathbf{u} = l^{(n)}[\langle \Phi_j, \Phi_i \rangle_{L^2(0,1)}]\mathbf{u}, \quad i,j = 1,\ldots,n, \tag{6.64}$$

using the definitions of A and C and knowing that $\langle B^*Bx, y \rangle = \langle Bx, By \rangle$. Notice that for orthonormal sets of basis functions $\{\Phi_j\}$ we have $c_{ij} = \delta_{ij}$ and therefore $C = I$ holds. Then (6.64) becomes a standard eigenvalue problem. Also note that if we would have used non-orthonormal basis functions for the singular value problem in the previous section 6.2 we would have got a *generalized singular value problem.*

By solving the generalized eigenproblem (6.64) for A and C we will get approximations to the true eigenvalues of $K = B^*B$. This was shown in Cochran [10] and is actually a special case of the results obtained by Hansen [36]. First we will summarize the algorithm:

Algorithm 6.3 (Rayleigh-Ritz method).

1. *Choose $\{\Phi_j\}$, a linearly independent (or even orthonormal set) of basis functions in the intervals $I = (0,1)$.*
2. *Determine the matrices $A \in \mathbb{R}^{n,n}$ and $C \in \mathbb{R}^{n,n}$ with*
$$a_{ij} = \langle K\Phi_j, \Phi_i \rangle_{L^2(0,1)} = \langle B\Phi_j, B\Phi_i \rangle_{L^2(0,1)} \quad i,j = 1,\ldots,n \tag{6.65}$$
and
$$c_{ij} = \langle \Phi_j, \Phi_i \rangle_{L^2(0,1)} \quad i,j = 1,\ldots,n \tag{6.66}$$
3. *Compute the eigenvalue decomposition of the (generalized) eigenvalue problem*
$$A\mathbf{u} = l^{(n)} C\mathbf{u}.$$

We are now going to examine the approximation properties of the algorithm given above. The generalized eigenvalues $l_i^{(n)}$ of (A,C) are approximations to the eigenvalues of K. again, we can use proposition 6.2. Obviously this proposition also holds for symmetric kernels K. The functional $F[\Phi,\Psi]$ from definition 6.4 becomes the *Rayleigh quotient* (see [10])
$$R[\Phi] := \frac{\langle K\Phi, \Phi \rangle}{\|\Phi\|^2},$$
and the eigenvalues λ_i of the real square integrable symmetric kernel K are the stationary values of this functional. The corresponding orthonormal left and right eigenfunctions are equal and given by $\dfrac{\Phi}{\|\Phi\|}$. Then the algebraic singular values s_i of lemma 6.1 correspond to the algebraic eigenvalues l_i for the symmetric problem and the left and right singular functions $\tilde{u}_i$ and $\tilde{v}_i$ are equal and correspond to the eigenfunctions. Furthermore the functional G in (6.39) becomes the algebraic Rayleigh quotient (see [26]) of a symmetric matrix A
$$R[\mathbf{x}] = \frac{\mathbf{x}^T A\mathbf{x}}{\|\mathbf{x}\|^2}.$$
Again, if the basis functions are not orthonormal (the case that we actually want to consider in this section) then the algorithm leads to the computation of the stationary values of
$$R_C[\mathbf{x}] = \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T C\mathbf{x}}. \tag{6.67}$$

with a symmetric matrix C whose elements are given by

$$c_{ij} = \langle \Phi_j, \Phi_i \rangle_{L^2(0,1)} \quad i,j = 1,\dots,n.$$

We will see, that $C \in \mathbb{R}^{n,n}$ is a symmetric positive definite matrix and therefore we can do a Cholesky factorization of this matrix. There exists a unique lower triangular matrix $G \in \mathbb{R}^{n,n}$ such that $C = GG^T$. Substituting $\bar{\mathbf{x}} = G^T \mathbf{x}$ into (6.67) leads to

$$R[\bar{\mathbf{x}}] = \frac{\bar{\mathbf{x}}^T \bar{A} \bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|}. \tag{6.68}$$

Hence, we obtain the Rayleigh quotient for changed variables. The analysis stays the same as for unchanged variables. We are going to summarize the approximation results for the Rayleigh-Ritz method for Hermitian kernels in the following propositions taken from [6]

Proposition 6.4 (Cochran 1972,Baker 1977). *Suppose that $K(t,\tau)$, $0 \le t,\tau \le 1$ is Hermitian, positive definite and square integrable. If*

$$c_{ij} = \langle \Phi_j, \Phi_i \rangle_{L^2(0,1)} = \delta_{ij}, \quad i,j = 1,\dots,n$$

and

$$a_{ij} = \langle K\Phi_j, \Phi_i \rangle_{L^2(0,1)}, i,j = 1,\dots,n$$

is the matrix used in the Rayleigh-Ritz method, then the algebraic eigenvalues $l_i^{(n)}$ satisfying

$$A\mathbf{u} = l^{(n)}\mathbf{u},$$

where n is the number of basis functions, are increasingly better approximations to the true eigenvalues λ_i,

$$l_i^{(n)} \le l_i^{(n+1)} \le \lambda_i, \quad i = 1,\dots,n, \tag{6.69}$$

and the algebraic eigenvalues $l_i^{(n)}$ exist. Furthermore, if

$$c_{ij} = \langle \Phi_j, \Phi_i \rangle_{L^2(0,1)} \neq \delta_{ij}, i,j = 1,\dots,n$$

and $\{\Phi_i\}_{i=1}^n$ are square integrable on $[0,1]$. If $l_i^{(n)}$ is the ith positive eigenvalue satisfying

$$(A - l^{(n)}C)\mathbf{u} = 0,$$

then inequality (6.69) holds, too.

Proof. We will state the proof by Baker, see [6, page 316]. If $K(t,\tau)$ is Hermitian and positive definite then its real eigenvalues are

$$\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n \ge \dots \to 0, \quad n \to \infty.$$

If λ_i exists, it may be characterized by the relation

$$\begin{aligned}
\lambda_i = \lambda_i(K) &= \sup_{S_i} \inf_{\substack{\Phi(t)\in S_i \\ \|\Phi(t)\|_2=1}} \int_0^1 \int_0^1 K(t,\tau)\Phi(t)\Phi(\tau)dtd\tau && (6.70)\\
&= \sup_{S_i} \inf_{\substack{\Phi(t)\in S_i \\ \|\Phi(t)\|_2=1}} \langle K\Phi,\Phi\rangle_{L^2(0,1)}, && (6.71)
\end{aligned}$$

where S_i denotes some linear subspace, of square integrable functions, with dimension i. Notice the link to the *Minimum-maximum principle by Poincaré and Fischer* from lemma 4.1, where we stated this result for the singular values. Note that, since $K = K^*$, $\langle K\Phi,\Phi\rangle_{L^2(0,1)} = \langle \Phi, K\Phi\rangle_{L^2(0,1)}$. Equation (6.71) provides a natural extension to a similar characterization of the eigenvalues of a matrix $A = A^* \in \mathbb{R}^{n,n}$. If we suppose that the eigenvalues of A are $l_1^{(n)} \geq l_2^{(n)} \geq \ldots \geq l_n^{(n)}$ then

$$l_i^{(n)} = l_i^{(n)}(A) = \sup_{V_i\subseteq\mathbb{C}^n} \inf_{\substack{x\in V_i \\ x^*x=1}} x^*Ax, \tag{6.72}$$

where V_i denotes an i-dimensional subspace of the space $\mathbb{C}^n$ of complex n-vectors. This follows immediately from the Courant-Fischer Minimax Theorem, for which we refer to [26]. Now, we are given square integrable functions $\Phi_1, \Phi_2, \ldots$ such that

$$c_{ij} = \langle \Phi_j, \Phi_i\rangle_{L^2(0,1)} = \delta_{ij}, \quad i, j = 1, \ldots, n.$$

The functions $\Phi_1, \Phi_2, \ldots, \Phi_n$ form a basis of the linear space S_n comprising functions of the form

$$\Phi(t) = \sum_{i=1}^n a_i\Phi_i(t),$$

where

$$\|\Phi(t)\|_2 = \sum_{i=1}^n |a_i|^2 = a^*a$$

with $a = [a_1, a_2, \ldots, a_n]^T$. The correspondence $\Phi(t) \leftrightarrow a$ establishes an isomorphism between S_n and $\mathbb{C}^n$.

Suppose that S_i is an i-dimensional subspace of S_n. In view of (6.71)

$$\lambda_i(K) \geq \sup_{S_i\subseteq S_n} \inf_{\substack{\Phi(t)\in S_i \\ \|\Phi(t)\|_2=1}} \langle K\Phi,\Phi\rangle_{L^2(0,1)}, \tag{6.73}$$

In this expression $langleK\Phi,\Phi\rangle_{L^2(0,1)}$ has the form

$$\langle \sum_{j=1}^n a_jK\Phi_j, \sum_{i=1}^n a_i\Phi_i\rangle_{L^2(0,1)} = \sum_{i=1}^n\sum_{j=1}^n \bar{a}_ia_j\langle K\Phi_j,\Phi_i\rangle_{L^2(0,1)},$$

since $\Phi(t) \in S_i \subseteq S_n$. This equation is of the form

$$a^*Aa, \quad \text{where} \quad A = A^* = [\langle K\Phi_j,\Phi_i\rangle_{L^2(0,1)}] = \langle \Phi_j, K\Phi_i\rangle_{L^2(0,1)}.$$

Hence the isomorphism $\Phi(t) \leftrightarrow a$, where $a_i = \langle \Phi, Phi_i \rangle_{L^2(0,1)}$ maps each subspace S_i into an i-dimensional subspace V_i of$\mathbb{C}^n$, and as noted above, $\|\Phi(t)\|_2 = a^*a$. Consequently we may rewrite (6.73)

$$\lambda_i(K) \geq \sup_{V_i \subseteq \mathbb{C}^n} \inf_{\substack{a \in V_i \\ a^*a=1}} a^*Aa. \tag{6.74}$$

From (6.72) it follows that

$$\lambda_i(K) \geq l_i^{(n)}(A).$$

Finally, since $A = [\langle K\Phi_j, \Phi_i \rangle_{L^2(0,1)}]$ is the matrix constructed in the Rayleigh-Ritz method so that $l_i^{(n)}(A) = l_i^{(n)}$ and we obtain the result (6.69).

So far we have assumed that the functions $\Phi_i(t)$ were orthonormal. In practice this may not be the case and the Rayleigh-Ritz method then entails the solution of the eigenvalue problem

$$(A - l^{(n)}C)\mathbf{u} = 0,$$

where a_{ij} and c_{ij} are given as in the proposition and $\Phi_1, \Phi_2, \ldots, \Phi_n$ are linearly independent. Then the above consideration may be modified to cover this case. We require a slightly modified form of (6.71)

$$\lambda_i(K) = \sup_{S_i} \inf_{\substack{\Phi(t) \in S_i \\ \|\Phi(t)\|_2 \neq 0}} \frac{\langle K\Phi, \Phi \rangle_{L^2(0,1)}}{\langle \Phi, \Phi \rangle_{L^2(0,1)}}. \tag{6.75}$$

The equivalence to (6.71) is established by setting $\Psi(t) = \frac{\Phi(t)}{\|\Phi(t)\|_2}$ in (6.75). Now suppose

$$\Phi(t) = \sum_{i=1}^{n} a_i \Phi_i(t),$$

where $\Phi_1, \Phi_2, \ldots, \Phi_n$ are linearly independent. Then

$$\langle \Phi, \Phi \rangle_{L^2(0,1)} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{a}_j \langle \Phi_i, \Phi_j \rangle_{L^2(0,1)} = a^*Ca,$$

where $c_{ij} = \langle \Phi_i, \Phi_j \rangle_{L^2(0,1)}$ and $C = C^*$ is positive definite. When S_n is defined as the spaces spanned by $\Phi_1, \Phi_2, \ldots, \Phi_n$, then

$$\begin{aligned}
\lambda_i(K) &\geq \sup_{S_i \subseteq S_n} \inf_{\substack{\Phi(t) \in S_i \\ \|\Phi(t)\|_2 \neq 0}} \frac{\langle K\Phi, \Phi \rangle_{L^2(0,1)}}{\langle \Phi, \Phi \rangle_{L^2(0,1)}} \\
&= \sup_{V_i \subseteq \mathbb{C}^n} \inf_{\substack{a \in V_i \\ a^*Ca \neq 0}} \frac{a^*Aa}{a^*Ca} \\
&= \sup_{V_i \subseteq \mathbb{C}^n} \inf_{\substack{a \in V_i \\ a \neq 0}} \frac{a^*Aa}{a^*Ca},
\end{aligned}$$

where V_i is an i-dimensional subspace of $\mathbb{C}^n$. Hence the last equation characterizes the ith positive eigenvalue $l_i^{(n)}$ and we have extended the result (6.69) to generalized eigenvalue problems. □

Notice that Baker [6] has stated the results not only for positive definite kernels, but we only need them for the case of positive definite kernel functions.As in proposition 6.2 for the Galerkin method where we got error bounds on the computed singular values, we get error bounds on the computed eigenvalues by using the Rayleigh-Ritz method.

Proposition 6.5 (Cochran 1972,Baker 1977). *Under the conditions of proposition 6.4, the errors of the approximate singular values $l_i^{(n)}$ are bounded by*

$$0 \leq \lambda_i - l_i^{(n)} \leq \|K - \tilde{K}_n\|, \quad i = 1, \ldots, n, \tag{6.76}$$

where K and $\tilde{K}_n$ are the integral operators with Hermitian kernels $K(t,\tau)$ and $\tilde{K}_n(t,\tau)$, where

$$\tilde{K}_n(t,\tau) = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle K\Phi_j, \Phi_i \rangle_{L^2(0,1)} \Phi_i(t) \Phi_j(\tau). \tag{6.77}$$

Furthermore, we obtain

$$0 \leq \lambda_i - l_i^{(n)} \leq \left(\int_0^1 \int_0^1 K(t,\tau)^2 dt d\tau - \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}}, \quad i = 1, \ldots, n. \tag{6.78}$$

Note the similarity of equation (6.77) to (6.37) and of equation (6.78) to (6.43), where we defined the norm of K to be

$$\|K\|^2 := \int_0^1 \int_0^1 K(t,\tau)^2 dt d\tau$$

and the Frobenius norm of A as

$$\|A\|_F^2 := \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2,$$

and then we got

$$0 \leq \lambda_i - l_i^{(n)} \leq \sqrt{\|K\|^2 - \|A\|_F^2} =: \delta_n, \quad i = 1, \ldots, n. \tag{6.79}$$

Proof. In order to show proposition 6.5 we state the proof given in [6].

Suppose that the functions $\Phi_1, \Phi_2, \ldots, \Phi_n$ are orthonormal. Then the non-zero eigenvalues $l_i^{(n)}$ of the matrix $A = [\langle K\Phi_j, \Phi_i \rangle_{L^2(0,1)}]$ are precisely the non-zero eigenvalues of the Hermitian degenerate kernel

$$\tilde{K}_n(t,\tau) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \Phi_i(t) \Phi_j(\tau).$$

Notice that the kernel may be written in the form

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\Phi_i(t)\right)\Phi_j(\tau)$$

so that its non-zero eigenvalues are those of the matrix with the ij-th element

$$a_{ij} = \in_0^1 \Phi_i(\tau)\sum_{k=1}^{n} a_{kj}\Phi_k(\tau)d\tau.$$

Comparing the eigenvalues λ_i and $l_i^{(n)}$ we obtain (see Baker [6])

$$|\lambda_i - l_i^{(n)}| \leq \rho(K-\tilde{K}_n) \leq \|K-\tilde{K}_n\|_2.$$

Together with the conditions of proposition 6.4 we obtain the first result (6.76)

$$0 \leq \lambda_i - l_i^{(n)} \leq \|K-\tilde{K}_n\|, \quad i=1,\ldots,n.$$

Inequality (6.78) may be shown in the same way as the error bound (6.41) in proposition 6.2 and by applying the definition of $\|K\|$ and $\|A\|_F$. □

We may also obtain results for the approximate eigenvalues using the Rayleigh-Ritz method that correspond to inequalities (6.44) and (6.45) for approximate singular values obtained by Galerkin's method. For those results we refer also to [6] and [10]. Hence, propositions 6.4 and 6.5 imply that if $\delta_n \to 0$ as $n \to \infty$, then the approximate eigenvalues $l_i^{(n)}$ converge uniformly in n to the true eigenvalues λ_i, and the corresponding approximate eigenfunctions $\tilde{u}_i$ converges in the mean to the true eigenfunctions. Furthermore, the Rayleigh-Ritz method always gives lower bounds for the first n eigenvalues of the operator K.

Finally, we want to make some remarks on the convergence of the eigenvalues (and eigenfunctions respectively). When $\Phi_1(t),\Phi_2(t),\ldots,\Phi_n(t)$ are part of a complete orthonormal set $\{\Phi\}$, the convergence of the eigenvalues is immediate. To be precise, if $K(t,\tau)$ is square integrable and the functions $\{\Phi\}$ are complete, then

$$\lim_{n\to\infty}\int_0^1\int_0^1 |K(t,\tau)-\tilde{K}_n(t,\tau)|^2 dt d\tau = 0,$$

where

$$\tilde{K}_n(t,\tau) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\Phi_i(t)\Phi_j(\tau)$$

and $a_{ij} = \langle K\Phi_j,\Phi_i\rangle_{L^2(0,1)}$, $i,j=1,\ldots,n$. For a proof we refer to Courant and Hilbert [12, page 43].

Then we finally get the following proposition from [6].

Proposition 6.6. *If the conditions of proposition 6.4 are satisfied and* $\{\Phi_1(t), \Phi_2(t), \ldots\}$ *is a complete orthonormal system in* $L^2(0,1)$*, then*

$$\lim_{n\to\infty} |\lambda_i - l_i^{(n)}| = 0.$$

Note the similarity of inequality (6.43) in proposition 6.2 and of inequality (6.76) in proposition 6.5 to the inequality in the following proposition by Parlett [70]:

Proposition 6.7. *Let* $\mathcal{A}$ *be a symmetric n-by-n matrix and* Q *be any orthonormal n-by-m matrix. Associated with it are the submatrix* $\mathcal{H}(= Q^*\mathcal{A}Q)$ *and* $R(= \mathcal{A}Q - Q\mathcal{H})$*, the residual matrix. There are m eigenvalues of* $\mathcal{A}$*,* $\{\phi_{j'}, j = 1, \ldots, m\}$ *which can be put in one-to-one correspondence with the eigenvalues* θ_j *of* $\mathcal{H}$ *in such a way that*

$$|\theta_j - \phi_{j'}| \le \|R\|, \quad j = 1, \ldots, m.$$

For a proof, see [70]. Proposition 6.7 states, how eigenvalues of a symmetric n-by-n matrix $\mathcal{A}$ are approximated by the eigenvalues of $\mathcal{H}$, where $\mathcal{H}$ is the restriction of $\mathcal{A}$ to range(Q), i.e. to an m-dimensional subspace of the original n-dimensional space, where $m < n$. In proposition 6.2 we have got a more general result: The eigenvalues of an infinite dimensional operator are approximated by eigenvalues of a finite dimensional operator in a finite dimensional subspace. In proposition 6.2 we have stated this result for singular values, but the same holds for the eigenvalues of symmetric operators which we have stated in 6.4 and 6.5.

6.3.3 Multiplier functions of the form $m(s) = s^\alpha$

We are going to apply the Rayleigh-Ritz method to operator B^*B from (4.26) in section 4.2.3,

$$[B^*Bx](\tau) = \int_0^1 K(t,\tau)x(t)dt, \tag{6.80}$$

with kernel $K \in L^2(0,1)$ from (4.28), given by

$$K(t,\tau) = \int_{\max(t,\tau)}^1 m^2(s)ds, \tag{6.81}$$

in order to compute its eigenvalues. We consider multiplier function $m(s) = s^\alpha$ for several values of α. The basis functions $\{\Phi_j\}$ are chosen to be piecewise linear interpolations. The interval $I_s = I_t = I = [0,1]$ is divided into n subintervals $[t_j, t_{j+1}]$ of the same length h. On each interval I, Φ_j is a piecewise linear function given by

$$\Phi_j(t) = \begin{cases} \dfrac{t - t_{j-1}}{h}, & t \in [t_{j-1}, t_j] \\ \dfrac{t_{j+1} - t}{h}, & t \in [t_j, t_{j+1}] \\ 0, & \text{else} \end{cases}, \quad j = 1, \ldots, n-1. \tag{6.82}$$

For an illustration of (6.82), see for example [53, page 71]. Then the entries of matrix C from (6.66) are given by

$$c_{ij} = \begin{cases} \frac{2}{3}h, & \text{if } i = j, \\ \frac{1}{6}h, & \text{if } |i-j| = 1, \\ 0, & \text{else} \end{cases} . \tag{6.83}$$

Furthermore we get

$$[B\Phi_j](s) = \frac{m(s)}{h} \cdot \begin{cases} h^2, & \text{if } t_{j-1} \leq s, t_j \leq s, t_{j+1} \leq s, \\ h^2 - \frac{(t_{j+1}-s)^2}{2}, & \text{if } t_{j-1} \leq s, t_j \leq s, t_{j+1} > s, \\ \frac{(s-t_{j-1})^2}{2}, & \text{if } t_{j-1} \leq s, t_j > s, t_{j+1} > s, \\ 0, & \text{if } t_{j-1} > s, t_j > s, t_{j+1} > s \end{cases} . \tag{6.84}$$

for $[B\Phi_j](s)$ in (6.65). Then we may calculate the entries of matrix A using (6.65) and (6.84) and obtain

$$a_{ij} = \int_{t_{j+1}}^{1} m(s)^2 h^2 ds + \tilde{a}_{ij}, \tag{6.85}$$

where $\tilde{a}_{ij}$ is given by

$$\tilde{a}_{ij} = \begin{cases} \int_{t_{j-1}}^{t_j} \frac{m(s)^2}{h^2} \frac{(s-t_{j-1})^4}{4} ds \\ \qquad + \int_{t_j}^{t_{j+1}} \frac{m(s)^2}{h^2} \left(h^2 - \frac{(t_{j+1}-s)^2}{2}\right)^2 ds, & \text{if } i = j, \\ \int_{t_{j-1}}^{t_j} \frac{m(s)^2}{h^2} \frac{(s-t_{j-1})^2}{2} \left(h^2 - \frac{(t_j-s)^2}{2}\right) ds \\ \qquad + \int_{t_j}^{t_{j+1}} m(s)^2 \left(h^2 - \frac{(t_{j+1}-s)^2}{2}\right) ds, & \text{if } |i-j| = 1, \\ \int_{t_{j-1}}^{t_j} m(s)^2 \frac{(s-t_{j-1})^2}{2} ds \\ \qquad + \int_{t_j}^{t_{j+1}} m(s)^2 \left(h^2 - \frac{(t_{j+1}-s)^2}{2}\right) ds, & \text{if } |i-j| \geq 2, \end{cases} . \tag{6.86}$$

Obviously the matrix A is symmetric, since the operator $K := B^*B$ was self-adjoint. For $m(s) = s^\alpha$ we can integrate these functions analytically and determine the entries of A easily and even exactly. After setting up the matrices A and C we can determine the generalized eigenvalues of (A, C) by computing $l^{(n)}$ in

$$A\mathbf{u} = l^{(n)} C\mathbf{u}.$$

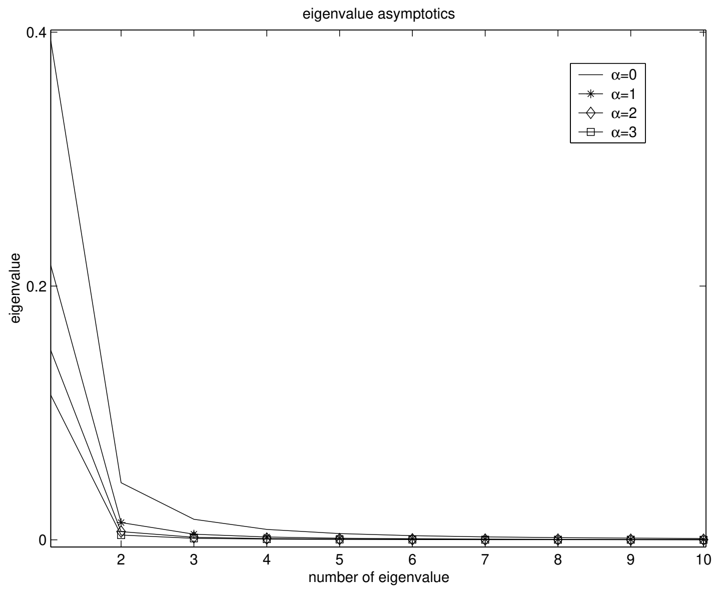

Figure 6.10: *First* 10 *computed eigenvalues of integral equation* $B^*Bu = \lambda u$ *for multiplier function* $m(s) = s^{\alpha}$, $n = 1000$ *and different values for* α *using a Rayleigh-Ritz method*

Since we are interested in the eigenvalue asymptotics, we compute the first 10 eigenvalues $l_1, \dots, l_{10}$ of largest magnitude for a fixed value of $n = 1000$ and plot them in figure 6.10. Since the plots suggest that the computed eigenvalues $l^{(n)}$ are proportional to $\frac{1}{n^x}$ for some x, we are going to consider the logarithmic scale plot for all computed singular values for $n = 100$ and several values of α. The results are shown in figure 6.11. In order to simplify a comparison, we take the same axis as in figure 6.6. In figure 6.11, we observe that the plots in logarithmic scaling have the same gradients, with shifts in direction of the y-axis for different values of α. If we compare the plots with the ones in figure 6.6 we observe that the gradient is larger. As the Galerkin method the Rayleigh-Ritz method for the integral equation does not yield any problems with boundary conditions such as with the finite difference method. Moreover, proposition 6.2 even supplies error bounds for the computed eigenvalues.

In table 6.12 we have summarized the error δ_n as given in (6.41) for several n and α. The error is computed by $\delta_n = \sqrt{\|K\|^2 - \|\bar{A}\|_F^2}$. We can determine $\|K\|$ by

$$\begin{aligned} \|K\|^2 &= \int_0^1 \int_0^1 |k(t,\tau)|^2 dt d\tau \\ &= \int_0^1 \int_0^1 \left(\int_{\max(t,\tau)}^1 m^2(s) ds \right)^2 dt d\tau. \end{aligned}$$

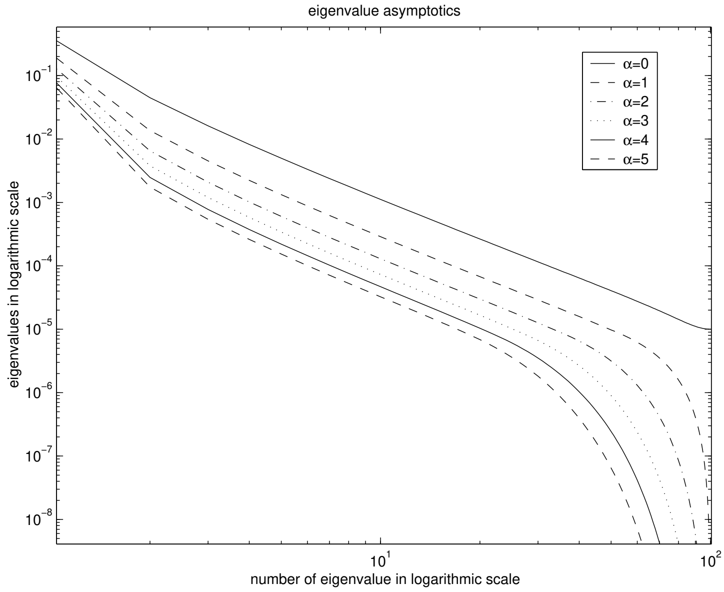

Figure 6.11: *Computed eigenvalues of integral equation $B^*Bu = \lambda u$ for $n = 100$, multiplier function $m(s) = s^{\alpha}$ and different values for α in logarithmic scales*

For the special multiplier function $m(s) = s^{\alpha}$ we may determine $\|K\|$ explicitly:

$$\|K\| = \frac{1}{\sqrt{2(\alpha+1)(2\alpha+3)}}. \tag{6.87}$$

The value of $\|\bar{A}\|_F$ can be computed numerically. Note that we have to determine the Frobenius norm of $\bar{A} = H^T A G^{-T}$ this time, since the basis functions are not orthonormal an we have to do a variable change via Cholesky factorization. From table 6.12 we see, that $\delta_n \to 0$ as $n \to \infty$ and therefore, we may follow, that the algebraic eigenvalues converge to the true ones. If we compare the results in table 6.12 to the ones in table 6.8, we notice that the error δ_n is smaller for each n when we use the Rayleigh-Ritz method. This is due to the use of piecewise linear functions as basis functions. They are better approximations to the true eigenfunctions than the piecewise constant basis functions which we used in section 6.2.

The gradient that we can observe in figure 6.11 for the computed eigenvalues of B^*B of largest magnitude is -2. Again, the computed eigenvalues of small magnitude are no good approximations to the true eigenvalues, since the error increases, the smaller the computed eigenvalues get. This effect is illustrated in figure 6.12. For a better overview we only consider the special integral equation with multiplier function $m(s) = s^{\alpha}$ for $\alpha = 1$. From figure 6.12 and with the help of proposition 6.2 we follow, that the true singular values lie between the thick solid and the dotted line. After determining the first 100 eigenvalues of largest magnitude for $n = 1000$ (dashed line) this statement is confirmed. The dashed line suggests that the smallest computed eigenvalues (i.e. those that are close to the matrix size n) are inaccurate. Hence, only the largest eigenvalues

α	n	h	δ_n
0	50	0.02	0.07973
0	100	0.01	0.05640
0	500	0.002	0.02523
0	1000	0.001	0.01784
1	50	0.02	0.03712
1	100	0.01	0.02619
1	500	0.002	0.01169
1	1000	0.001	0.00826
2	50	0.02	0.02427
2	100	0.01	0.01709
2	500	0.002	0.00761
2	1000	0.001	0.00538
3	50	0.02	0.01806
3	100	0.01	0.01270
3	500	0.002	0.00565
3	1000	0.001	0.00399

Table 6.12: *Results for the errors of the Rayleigh-Ritz method applied to integral equations $B^*Bu = \lambda u$ with multiplier functions $m(s) = s^\alpha$ for different values of n and α*

are considered and the plots in 6.11 suggest that their asymptotics are

$$\lambda_n^{\text{approx}}(B^*B) = p(\alpha)O(n^{-2}),$$

where p is a monotone decreasing function depending on α. Therefore the reciprocal value

$$\frac{1}{\lambda_n^{\text{approx}}(B^*B)}$$

would yield a quadratic function in n. Knowing that, we are going to determine the values $\frac{1}{l_i^{(n)}}$, $i = 1, \dots, 10$ for the first ten computed eigenvalues. As in the previous sections we apply curve fitting to those values, assuming that

$$\frac{1}{\lambda_n^{\text{approx}}(B^*B)} = an^2 + bn + c,$$

and summarize the results for different values of α and constant $n = 1000$ in table 6.13. From this table we observe a regularity for the coefficients a, b and c. All three coefficients are quadratic in α. Knowing that

$$\frac{1}{\lambda_n} = \frac{\pi^2(2n-1)^2}{4} = \pi^2 n^2 - \pi^2 n + \frac{\pi^2}{4}, \quad n \in \mathbb{N} \tag{6.88}$$

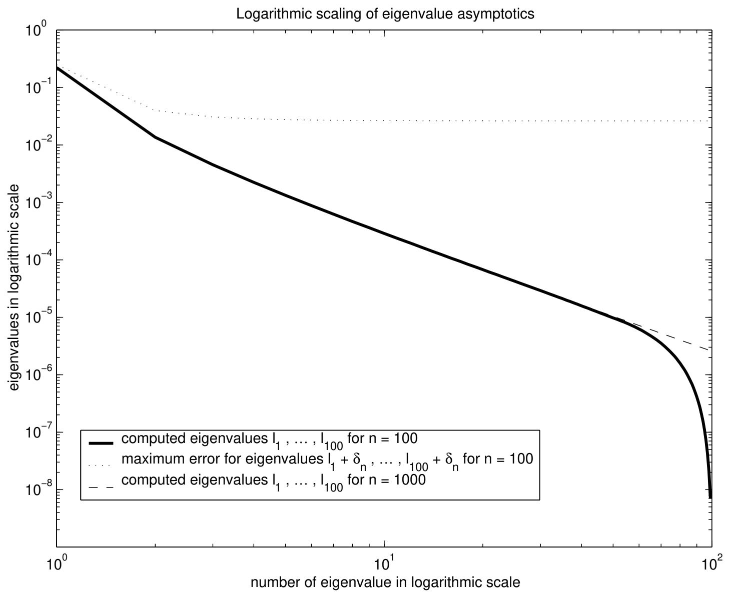

Figure 6.12: *Computed eigenvalues of integral equation $B^*Bu = \lambda u$ for $n = 100$, multiplier function $m(s) = s^\alpha$ and $\alpha = 1$ together with error bound*

α	n	a	b	c	$\frac{a}{\pi^2}$	$\frac{b}{\pi^2}$	$\frac{c}{\pi^2}$
0	1000	9.8753	-9.8753	2.4688	1.0006	-1.0006	0.2501
0.5	1000	22.2104	-25.9193	7.1664	2.2503	-2.6262	0.7261
1	1000	39.4841	-49.3940	14.3330	4.0006	-5.0047	1.4522
1.5	1000	61.6946	-80.2924	23.9628	6.2509	-8.1353	2.4279
2	1000	88.8416	-118.6141	36.0557	9.0015	-12.0181	3.6532
2.5	1000	120.9249	-164.3582	50.6112	12.2522	-16.6529	5.1280
3	1000	157.9446	-217.5254	67.6303	16.0031	-22.0399	6.8524
3.5	1000	199.9006	-287.1146	87.1116	20.2542	-28.1789	8.8262
4	1000	246.7929	-346.1268	109.0564	25.0059	-35.0700	11.0497

Table 6.13: *Coefficients of the best fitting second order polynomial $\frac{1}{\lambda_n} = an^2 + bn + c$ for eigenvalue asymptotics for multiplier function $m(s) = s^\alpha$ with $n = 1000$ and several values of α*

from (6.16) for the trivial problem without multiplier function we conjecture

$$\frac{1}{\lambda_n^{\text{approx}}(B^*B)} = \pi^2(q(\alpha)n^2 + r(\alpha)n + s(\alpha)) \tag{6.89}$$

for the eigenvalues of the integral operator B^*B with multiplier function $m(s) = s^\alpha$. We may determine the functions $q(\alpha)$, $r(\alpha)$ and $s(\alpha)$ which are quadratic functions in α

using the values in table 6.13 and applying polynomial curve fitting. Then we get

$$\begin{aligned} q(\alpha) &= (\alpha+1)^2, \\ r(\alpha) &= -\left(\frac{3}{2}\alpha^2+\frac{5}{2}\alpha+1\right), \\ s(\alpha) &= \frac{1}{2}\alpha^2+\frac{\sqrt{2}}{2}\alpha+\frac{1}{4}. \end{aligned}$$

Notice that this result does not yield a contradiction to the results for $\alpha = 0$ (i.e. no multiplier function), where we know the analytic solution which is given by (6.88).

Hence, we may conjecture that the eigenvalues of integral operator B^*B with multiplier function $m(s) = s^\alpha$ have the asymptotics

$$\lambda_n^{\text{approx}}(B^*B) \sim \frac{1}{(\alpha+1)^2\pi^2 n^2}.$$

Knowing that the singular values of B are given by the square roots of the positive eigenvalues of B^*B (or BB^*), we get

$$\sigma_n^{\text{approx}}(B) \sim \frac{1}{(\alpha+1)\pi n}.$$

something that we have already observed in section 6.1.3 and 6.2.3. Therefore, multiplier functions of the form $m(s) = s^\alpha$ only influence the coefficient for n^{-1}, in the same way as we have already stated in sections 6.1.3 and 6.2.3.

6.3.4 Multiplier functions of the form $m(s) = e^{-\frac{1}{s^\alpha}}$

Now, we are now going to apply the Rayleigh-Ritz method described above to the integral operator equation $B^*Bu = \lambda u$ from (6.56) with special multiplier functions $m(s) = e^{-\frac{1}{s^\alpha}}$ for several values of α. The linearly independent non-orthonormal basis functions $\{\Phi_j\}$ are chosen to be the same as in (6.82). Then the entries for the matrices A and C can be determined like the ones in (6.85) and (6.83). This time the entries of A have to be computed by numerical integration, since the integrals $m(s) = e^{-\frac{1}{s^\alpha}}$ cannot be determined analytically. After setting up the symmetric matrices A and C we can determine the generalized eigenvalues of (A,C)

$$A\mathbf{u} = l^{(n)}C\mathbf{u}$$

for several values of $n = \frac{1}{h}$ and α. As before we compute the first 10 eigenvalues of largest magnitude and determine the error $\delta_n = \sqrt{\|K\|^2 - \|\bar{A}\|_F^2}$ as given in (6.41). We have to compute $\|K\|$, given by

$$\|K\|^2 = \int_0^1 \int_0^1 \left(\int_{\max(t,\tau)}^1 m^2(s)ds \right)^2 dt d\tau. \tag{6.90}$$

α	n	h	δ_n
0	50	0.02	0.01079
0	100	0.01	0.00763
0	200	0.005	0.00540
0.1	50	0.02	0.00962
0.1	100	0.01	0.00680
0.1	200	0.005	0.00481
0.3	50	0.02	0.00776
0.3	100	0.01	0.00548
0.3	200	0.005	0.00388
0.5	50	0.02	0.00643
0.5	100	0.01	0.00454
0.5	200	0.005	0.00320
1	50	0.02	0.00443
1	100	0.01	0.00312
1	200	0.005	0.00220

Table 6.14: *Results for the errors of the Rayleigh-Ritz method applied to integral equations $B^*Bu = \lambda u$ with multiplier functions $m(s) = e^{-\frac{1}{s^\alpha}}$ for different values of n and α*

as well as $\|\bar{A}\|_F$ numerically. The results are summarized in table 6.14. Proposition 6.2 provides error bounds for the computed eigenvalues. We observe that $\delta_n \to 0$ as $n \to \infty$ and therefore we may follow that the algebraic eigenvalues converge to the true ones. Hence, we are interested in the eigenvalue asymptotics. We compute the first 10 eigenvalues $l_1, \dots, l_{10}$ of largest magnitude for a fixed value of $n = 200$ and plot them in figure 6.13. Since the plots suggest that the computed eigenvalues $l^{(n)}$ are proportional to $\frac{1}{n^x}$ for some x, we are going to consider the logarithmic scale plot for all computed eigenvalues and $n = 100$ and several values of α again. The results are shown in figure 6.14. In order to simplify a comparison, we take the same axis as in figure 6.9. As before, in figure 6.14 we observe that the plots in logarithmic scaling have the same gradients, with shifts in direction of the y-axis for different values of α.

The gradient that we observe in figure 6.14 for the computed singular values of largest magnitude is -2. The computed eigenvalues of small magnitude are no good approximations to the true eigenvalues, since the error increases, the smaller the computed eigenvalues become. We refer to the discussion in section 6.3.3. Again, the plots suggest, that the eigenvalues have the asymptotics

$$\lambda_n^{\text{approx}}(B^*B) = p(\alpha)O(n^{-2}),$$

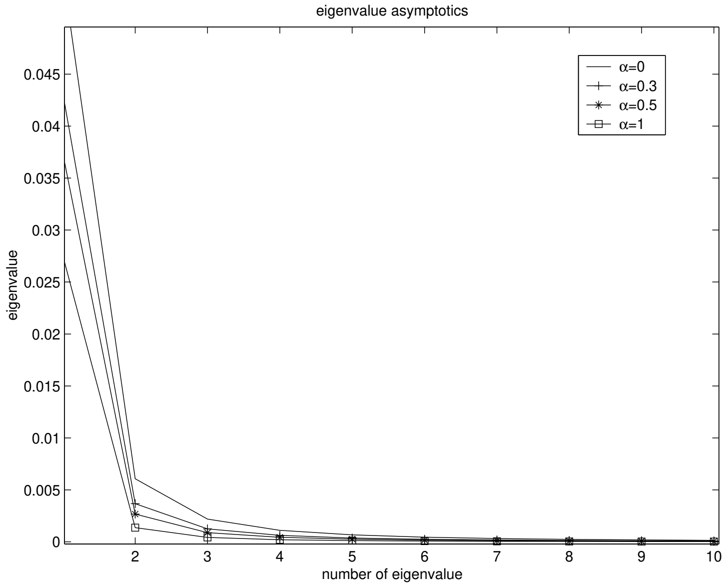

Figure 6.13: *First* 10 *computed eigenvalues of integral equation* $B^*Bu = \lambda u$ *for multiplier function* $m(s) = e^{-\frac{1}{s^\alpha}}$, $n = 200$ *and different values for* α *using a Rayleigh-Ritz method*

for some function p. Then the reciprocal value

$$\frac{1}{\lambda_n^{\text{approx}}(B^*B)}$$

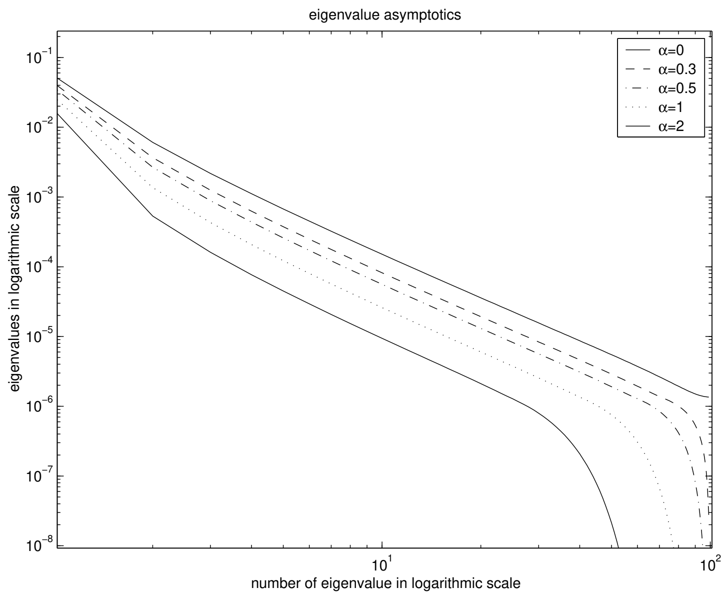

Figure 6.14: *Computed eigenvalues of integral equation* $B^*Bu = \lambda u$ *for* $n = 100$, *multiplier function* $m(s) = e^{-\frac{1}{s^\alpha}}$ *and different values for* α *in logarithmic scales*

yields a quadratic function in n. By the same examination as in the previous section, i.e. assuming that

$$\frac{1}{\lambda_n^{\text{approx}}(B^*B)} = an^2 + bn + c,$$

we obtain the values in table 6.15. From this table we observe a regularity for the

α	n	a	b	c	$\frac{a}{\pi^2}$	$\frac{b}{\pi^2}$	$\frac{c}{\pi^2}$
0	200	73.1389	-73.1439	18.2920	7.4105	-7.4110	1.8534
0.1	200	89.9191	-95.2912	25.2915	9.1107	-9.6550	2.5626
0.2	200	111.4622	-126.2863	36.7695	11.2935	-12.7955	3.7255
0.3	200	137.8438	-165.9935	52.3591	13.9665	-16.8187	5.3051
0.4	200	168.9138	-213.6950	71.3029	17.1145	-21.6518	7.2245
0.5	200	204.5018	-268.7980	93.0755	20.7204	-27.2350	9.4305
1	200	446.9897	-647.5496	239.2323	45.2895	-65.6105	24.2393

Table 6.15: *Coefficients of the best fitting second order polynomial $\frac{1}{\lambda_n} = an^2 + bn + c$ for eigenvalue asymptotics for multiplier function $m(s) = e^{-\frac{1}{s^\alpha}}$ with $n = 200$ and several values of α*

coefficients a, b and c. As before, we conjecture

$$\frac{1}{\lambda_n^{\text{approx}}(B^*B)} = \pi^2(q(\alpha)n^2 + r(\alpha)n + s(\alpha)) \tag{6.91}$$

for the eigenvalues of the integral operator B^*B with multiplier function $m(s) = e^{-\frac{1}{s^\alpha}}$ for some functions q, r and s depending on α. We will determine those functions by using the values in table 6.15 and by applying curve fitting. Then we get

$$\begin{aligned} q(\alpha) &\approx 22.65\alpha^2 + 15.28\alpha + 7.38, \\ r(\alpha) &\approx -(37.5\alpha^2 + 20.88\alpha + 7.27), \\ s(\alpha) &\approx 14.8\alpha^2 + 7.75\alpha + 1.72. \end{aligned}$$

Note that these are only approximate results, we do not get such convenient coefficients as for multiplier functions of the form $m(s) = s^\alpha$. The results do not yield a contradiction to the results for $\alpha = 0$ (i.e. multiplier function $e \approx 2.718$), where we know the analytic solution from (6.88) which is given by

$$\frac{1}{\lambda_n} = e^2\frac{\pi^2(2n-1)^2}{4} = e^2\pi^2n^2 - e^2\pi^2n + e^2\frac{\pi^2}{4}, \quad n \in \mathbb{N}. \tag{6.92}$$

Another attempt which is stronger than the above one and which we have already used in section 6.1.4 provides the inequality

$$\frac{1}{\lambda_n^{\text{approx}}(B^*B)} \leq (e^{\alpha+1})^2\pi^2n^2,$$

but it is hard to find a more exact dependence on α as for multiplier function $m(s) = s^{\alpha}$.

Hence, we may conjecture, that the eigenvalues of integral operator B^*B with multiplier function $m(s) = e^{-\frac{1}{s^{\alpha}}}$ have the asymptotics

$$\lambda_n^{\text{approx}}(B^*B) \sim \frac{1}{(22.65\alpha^2 + 15.28\alpha + 7.38)\pi^2 n^2}.$$

Knowing that the singular values of B are given by the square roots of the positive eigenvalues of B^*B (or BB^*), we get

$$\sigma_n^{\text{approx}}(B) \sim \frac{1}{\sqrt{22.65\alpha^2 + 15.28\alpha + 7.38}\pi n}.$$

Furthermore we may certainly always state

$$\sigma_n^{\text{approx}}(B) \geq \frac{1}{(e^{\alpha+1})\pi n}, \tag{6.93}$$

as in section 6.1.4 and 6.2.4. Therefore, multiplier functions of the form $m(s) = e^{-\frac{1}{s^{\alpha}}}$ only influence the coefficient for n^{-1} in the same way as we have stated already in sections 6.1.4 and 6.2.4, but the degree of ill-posedness remains the same.

6.4 Some further numerical approaches

So far we have seen three numerical approaches for determining the singular values of integral operator B from (4.1). In this section we are going to state some further methods, which are closely related to the ones described in sections 6.1, 6.2 and 6.3.

Besides the finite difference approach for the *Sturm-Liouville problem*

$$-(au')' = \lambda u$$

we could consider a finite element method for the generalized eigenproblem from (5.8)

$$[u, \eta]_A = \lambda \langle u, \eta \rangle_{L^2(0,1)}, \quad \forall \eta \in H_A. \tag{6.94}$$

We choose an n-dimensional subspace $H_{\hat{A}} := \mathcal{L}\{u_1, \dots, u_n\} \subset H_A$ and trial functions

$$u = \sum_{i=1}^{n} z_i u_i, \quad u \in H_{\hat{A}}.$$

Then (see for example [31], [24] or [5, page 42]), applying proposition 5.2 we may state that

$$\lambda_i(\hat{A}) \geq \lambda_i(A), \quad i = 1, \dots, n$$

holds, i.e. the n approximate eigenvalues

$$\lambda_1(\hat{A}) \leq \lambda_2(\hat{A}) \leq \dots \leq \lambda_n(\hat{A})$$

obtained by the generalized matrix eigenvalue decomposition of

$$\mathcal{A}\mathbf{z} = \hat{\lambda}\mathcal{B}\mathbf{z},$$

where $\mathcal{A} = \{[u_j, u_k]_A\}_{j,k=1}^n$ and $\mathcal{B} = \{\langle u_j, u_k\rangle_{L^2(0,1)}\}_{j,k=1}^n$ are always upper bounds for the first n exact eigenvalues of the *Sturm-Liouville problem* $-(au')' = \lambda u$, since the governing equation is self-adjoint and positive definite. This fact was shown in [25], for example. There we can also find error bounds for $|\lambda_i(A) - \lambda_i(\hat{A})|$. Note that this method also yields a Ritz-Galerkin method, but this time for the Sturm-Liouville problem (see, for example [66] or [76]). Using the definitions of $displaystyle\lambda = \frac{1}{\sigma^2}$ from chapter 5, where σ is the singular value of the integral operator (4.1) we get

$$\sigma_k^{\text{approx}}(B) \leq \sigma_k^{\text{exact}}(B), \quad k = 1, \ldots, n.$$

for the singular values of integral operator B. Together with the analytic result (see chapter 4)

$$\sigma_k^{\text{exact}}(B) \leq Ck^{-1}, \quad k \to \infty \tag{6.95}$$

we would get upper and lower bounds on the decay rate of the singular values of the integral operator (4.1). The limitation of this method lies in the approximation of the boundary condition at the left hand boundary, which we have already seen for the finite difference methods. We would have to choose linear independent functions u_i that satisfy the boundary condition (6.1).

With the Galerkin and Rayleigh-Ritz methods described in sections 6.2 and 6.3 that are directly applied to the *integral operators*, we do not have any difficulties with boundary conditions. Hence, the Galerkin method applied to the integral equation (4.1) which yields the singular value problem

$$Bu = \sigma v,$$

with integral operator B or the Rayleigh-Ritz method applied to the integral equation (4.26) which yields the eigenvalue problem

$$B^*Bu = \lambda u,$$

with integral operator B^*B, where $\sigma = \sqrt{\lambda}$, are more reliable than the methods for the Sturm-Liouville problem and we even obtain error bounds through proposition 6.2.

Note that we called the approach, where we computed the singular value decomposition, the Galerkin method and the approach, where we computed the eigenvalue decomposition, the Rayleigh-Ritz method. This was just for a better distinction. In literature we will find the Galerkin method just as a generalization of the Rayleigh-Ritz method. Actually both methods apply to eigenvalue problems, but Galerkin's method does not require any symmetry in the kernel $k(s,t)$ (therefore we were able to use it for the numerical computation of the singular value decomposition), whereas the Rayleigh-Ritz method applies only to Hermitian kernels (therefore we were able to use it for the computation of the eigenvalues of the symmetric kernel of B^*B). For more detailed explanation, see [6].

Besides the Galerkin and Rayleigh-Ritz methods for the integral operators B or B^*B that we have used we could have considered several variations of these approaches.

Instead of using orthonormal basis functions in algorithm 6.2 we could have used only linearly independent basis functions. Then we would have got a *generalized singular value problem* (see [26])

$$A\mathbf{u} = s^{(n)}C\mathbf{v},$$

where

$$a_{ij} = \langle K\Phi_j, \Psi_i\rangle_{L^2(0,1)} \quad \text{and} \quad c_{ij} = \langle \Phi_j, \Psi_i\rangle_{L^2(0,1)}, \quad i,j = 1,\ldots,n.$$

Furthermore, in algorithm 6.3 we could have used orthonormal basis functions in order to get a normal symmetric eigenvalue problem

$$A\mathbf{u} = l^{(n)}C\mathbf{u}, \quad C = I,$$

where

$$a_{ij} = \langle K\Phi_j, \Phi_i\rangle_{L^2(0,1)} \quad \text{and} \quad c_{ij} = \langle \Phi_j, \Phi_i\rangle_{L^2(0,1)} = \delta_{ij}, \quad i,j = 1,\ldots,n.$$

In addition, for both the algorithms 6.2 and 6.3 we could have used higher order basis functions instead of only piecewise constant (as in (6.47) and (6.48)) or piecewise linear approximations (as in (6.82)), in order to get even better numerical results.

Many more methods for eigenvalue problems for integral equations such as collocation methods, quadrature methods, least-squares methods etc., including convergence properties are summarized in [6].

6.5 Some further investigations on multiplier functions without a zero

In this section we are going to examine the influence of multiplier functions $\tilde{m}(s)$, which do not have a zero, i.e. $\tilde{m}(s) \neq 0 \quad \forall s \in [0,1]$. To this end we are going to consider three different functions $\tilde{m}(s)$ given by

$$\begin{aligned}
\tilde{m}_1(s) &= m(s) + \delta, \\
\tilde{m}_2(s) &= m(s+\delta), \\
\tilde{m}_3(s) &= \begin{cases} m(s), & (\delta \leq s \leq 1) \\ m(\delta), & (s < \delta). \end{cases}
\end{aligned}$$

Figure 6.15 shows the three types of functions for the special case $m(s) = s^2$ and $\delta = 0.1$. Notice that for $\delta \to 0$ we get $\tilde{m}_i(s) \to m(s) \quad \forall i$. We are going to choose small values for δ but $\delta \neq 0$ and modify the Galerkin method from section 6.2 such that

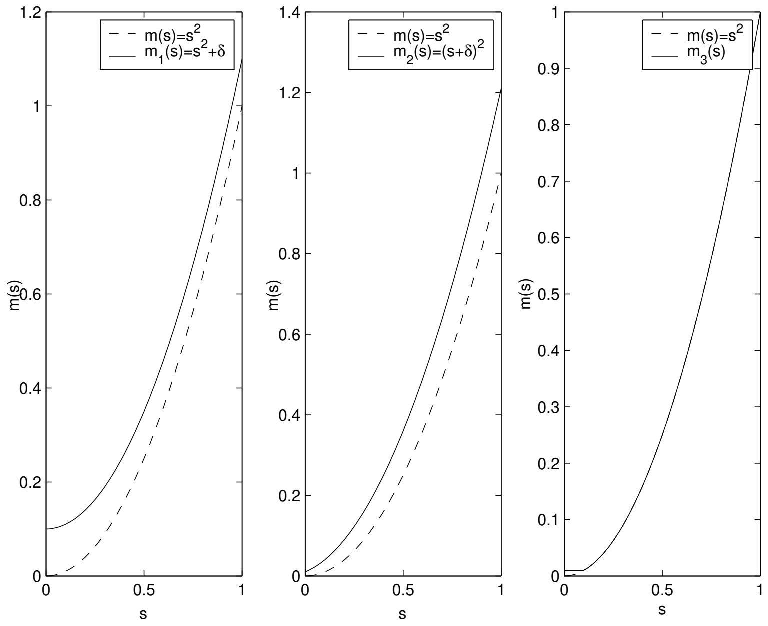

Figure 6.15: *Modified multiplier functions $\tilde{m}(s)$ for $m(s) = s^2$ and $\delta = 0.1$*

it may deal with the multiplier functions $\tilde{m}_i(s), \quad i = 1,2,3$ instead of $m(s)$. Before stating any numerical outcomes, we want to review some analytical results. In chapter 5 we obtained inequality (5.15) given by

$$\frac{2}{\pi(2n-1)\sqrt{a_1}} \leq \sigma_n(B) \leq \frac{2}{\pi(2n-1)\sqrt{a_0}}, \tag{6.96}$$

where a_0 and a_1 are determined from

$$a_1 \geq a(s) = \frac{1}{\tilde{m}^2(s)} \geq a_0. \tag{6.97}$$

Obviously we may compute a_0 and a_1 for the modified multiplier functions $\tilde{m}(s)$. For multiplier function $m(s) = s^\alpha$ we then get

$$\delta\sigma_n(J) \leq \sigma_n(B) \leq (1+\delta)\sigma_n(J),$$

for $\tilde{m}_1(s)$,

$$\delta^\alpha\sigma_n(J) \leq \sigma_n(B) \leq (1+\delta)^\alpha\sigma_n(J),$$

for $\tilde{m}_2(s)$ and

$$\delta^\alpha\sigma_n(J) \leq \sigma_n(B) \leq \sigma_n(J),$$

for $\tilde{m}_3(s)$. We may do the same calculation for the modified multiplier function $\tilde{m}(s)$ with $m(s) = e^{-\frac{1}{s^\alpha}}$. Then we get

$$\delta\sigma_n(J) \leq \sigma_n(B) \leq (e^{-1}+\delta)\sigma_n(J),$$

for $\tilde{m}_1(s)$,

$$e^{-\frac{1}{\delta^\alpha}}\sigma_n(J) \le \sigma_n(B) \le e^{-\frac{1}{(1+\delta)^\alpha}}\sigma_n(J),$$

for $\tilde{m}_2(s)$ and

$$e^{-\frac{1}{\delta^\alpha}}\sigma_n(J) \le \sigma_n(B) \le e^{-1}\sigma_n(J),$$

for $\tilde{m}_3(s)$. In all three cases for the two types of multiplier functions the degree of ill-posedness of operator B is $\nu = 1$. We are now going to calculate the singular value decomposition of operator B with multiplication operator $\tilde{m}$ and several values for δ. Without loss of generality and just for a better overview we only consider the special multiplier functions $m(s) = s^2$ and $m(s) = e^{-\frac{1}{s}}$. First we consider the operator B with $\tilde{m}_i(s)$ for the potential type function $m(s) = s^2$. In table 6.16 we summarize the quotient

$$Q = \frac{\sigma_n(B)}{\sigma_n(J)},$$

for several δ in order to compare these results to our previous ones. Firstly we observe

δ	$\tilde{m}_1(s)$	$\tilde{m}_2(s)$	$\tilde{m}_3(s)$
1	1.333	2.314	1
0.5	0.833	1.072	0.414
0.1	0.434	0.440	0.332
0.01	0.341	0.341	0.332
0.001	0.333	0.333	0.332

δ	$\tilde{m}_1(s)$	$\tilde{m}_2(s)$	$\tilde{m}_3(s)$
1	1.333	2.333	1
0.5	0.833	1.083	0.348
0.1	0.433	0.443	0.333
0.01	0.343	0.343	0.333
0.001	0.334	0.334	0.333

Table 6.16: *Left table: The quotient $Q = \sigma_n(B)/\sigma_n(J)$ for $\tilde{m}_i(s), i = 1,2,3,\ m(s) = s^2$, $n = 1000$ and several values of δ. Right table: The integral $\int_0^1 \tilde{m}_i(s)ds$ for $\tilde{m}_i(s), i = 1,2,3$, $m(s) = s^2$ and several values of δ*

that for $\delta \to 0$ the singular value asymptotics and the coefficient Q is the same as the one obtained in the previous section, i.e. (here with $\alpha = 2$)

$$Q = \frac{1}{\alpha+1}.$$

Since we observe a certain relation between Q and the integral $\int_0^1 \tilde{m}_i(s)ds$ we are going to determine this integral (which can be done analytically in this case) and summarize the values in the right table of 6.16. From those two tables we may conjecture that

$$\sigma_n(B) \sim \int_0^1 m(s)ds \cdot \sigma_n(J) \sim \int_0^1 m(s)ds \cdot \frac{2}{\pi(2n-1)},$$

for the integral operator $B = M \circ J$. Notice that this result does not yield a contradiction to our previous outcomes from sections 6.1-6.3.

Now we consider the operator B with $\tilde{m}_i(s)$ for the exponential type function $m(s) = e^{-\frac{1}{s}}$. In tables 6.17 we summarize the quotient

$$Q = \frac{\sigma_n(B)}{\sigma_n(J)},$$

in the left hand side table and the integral

$$I = \int_0^1 \tilde{m}_i(s)ds$$

in the right hand side table. We observe the same relationship between Q and the

δ	$\tilde{m}_1(s)$	$\tilde{m}_2(s)$	$\tilde{m}_3(s)$
1	1.144	0.502	0.367
0.5	0.645	0.350	0.196
0.1	0.244	0.186	0.148
0.01	0.156	0.152	0.148
0.001	0.149	0.148	0.148

δ	$\tilde{m}_1(s)$	$\tilde{m}_2(s)$	$\tilde{m}_3(s)$
1	1.148	0.504	0.368
0.5	0.648	0.353	0.197
0.1	0.248	0.187	0.148
0.01	0.158	0.152	0.148
0.001	0.149	0.149	0.148

Table 6.17: *Left table: The quotient $Q = \sigma_n(B)/\sigma_n(J)$ for $\tilde{m}_i(s)$, $i = 1,2,3$, $m(s) = e^{-\frac{1}{s}}$, $n = 200$ and several values of δ. Right table: The integral $\int_0^1 \tilde{m}_i(s)ds$ for $\tilde{m}_i(s)$, $i = 1,2,3$, $m(s) = e^{-\frac{1}{s}}$ and several values of δ*

integral $\int_0^1 \tilde{m}_i(s)ds$ and we may conjecture that

$$\sigma_n(B) \sim \int_0^1 m(s)ds \cdot \sigma_n(J) \sim \int_0^1 m(s)ds \cdot \frac{2}{\pi(2n-1)},$$

for the integral operator $B = M \circ J$. These results do not yield a contradictions to our previous outcomes from sections 6.1-6.3.

Hence, numerically we have found that

$$\sigma_n(B) \sim \int_0^1 m(s)ds \cdot \frac{2}{\pi(2n-1)}.$$

At this stage we might refer to the result obtained by Vu Kim Tuan and Gorenflo [78] again. From (4.12) with $r = 1$ we know

$$[s^{-\alpha}Jx](s) := s^{-\alpha} \int_0^s x(t)dt, \quad (0 \leq s \leq 1) \tag{6.98}$$

in $X = Y = L^2(0,1)$ has got the same singular value asymptotics as operator J, i.e. $\sigma_n(s^{-\alpha}J) \sim n^{-1}$, if $\frac{1}{2} > \alpha \geq 0$. We may conjecture that this result also holds for $1 > \alpha \geq 0$, since the integral $I = \int_0^1 m(s)ds$ exists. It does not hold for larger values of α, since the integral I does not exist. Notice that this was also conjectured by Vu Kim Tuan and Gorenflo [78], but has not been proved yet.

6.6 Summary of numerical results

We are going to summarize our results for the different numerical approaches, stating also approximation properties and limitations for the different methods. Finally we want to compare the results for the various multiplier functions.

Finite difference methods First, we consider the finite difference methods examined in section 6.1. We do not know how the the computed eigenvalues $\lambda_k^{\text{approx}}$ obtained by the numerical method approximate the true eigenvalues of the Sturm-Liouville problem

$$Au = \lambda u, \quad Au := -(a(\tau)u'(\tau))' \quad u(1) = u'(0) = 0$$

given by (5.5) and we have not found any error bounds. But we have found that the numerical approximation yields

$$\sigma_k^{\text{approx}}(B) \sim \frac{1}{(\alpha+1)\pi k} \quad \text{or} \quad \sigma_k^{\text{approx}}(B) \sim \frac{1}{(l\alpha+e)\pi k}$$

for the computed singular values for the different types of multiplier functions. In chapter 5 equation (5.12) yields

$$\sigma_k(B) \leq Ck^{-1}, \tag{6.99}$$

i.e. an upper bound on the singular values of integral operator B and therefore a lower bound, but no upper bound, on the degree of ill-posedness. Therefore the numerical result confirms the analysis and even provides a constant C.

The method is limited in the sense, that the left hand boundary condition cannot be approximated properly, i.e. we cannot set ε from (6.9) to zero. For values of ε that are very close to zero, occurring especially for multiplier functions $m(s) = e^{-\frac{1}{s^\alpha}}$, the condition number of the matrix A_h becomes very large and the results for the eigenvalues become inexact. Finally we want to compare the results for multiplier functions $m(s) = s^\alpha$ and $m(s) = e^{-\frac{1}{s^\alpha}}$. We know from (6.18) and (6.22), that the singular value asymptotics are

$$\sigma_n(B) \sim \frac{1}{(\alpha+1)\pi n}$$

for multiplier functions of the form $m(s) = s^\alpha$ and

$$\sigma_n(B) \sim \frac{1}{(l\alpha+e)\pi n}$$

with $l \approx 4.3$ for multiplier functions of the form $m(s) = e^{-\frac{1}{s^\alpha}}$. In figure 6.16 we see a comparison between the first computed eigenvalues for the two types of multiplier functions. We observe that the eigenvalues of the exponential type multiplier functions integrate into the ones of the potential type multiplier functions.

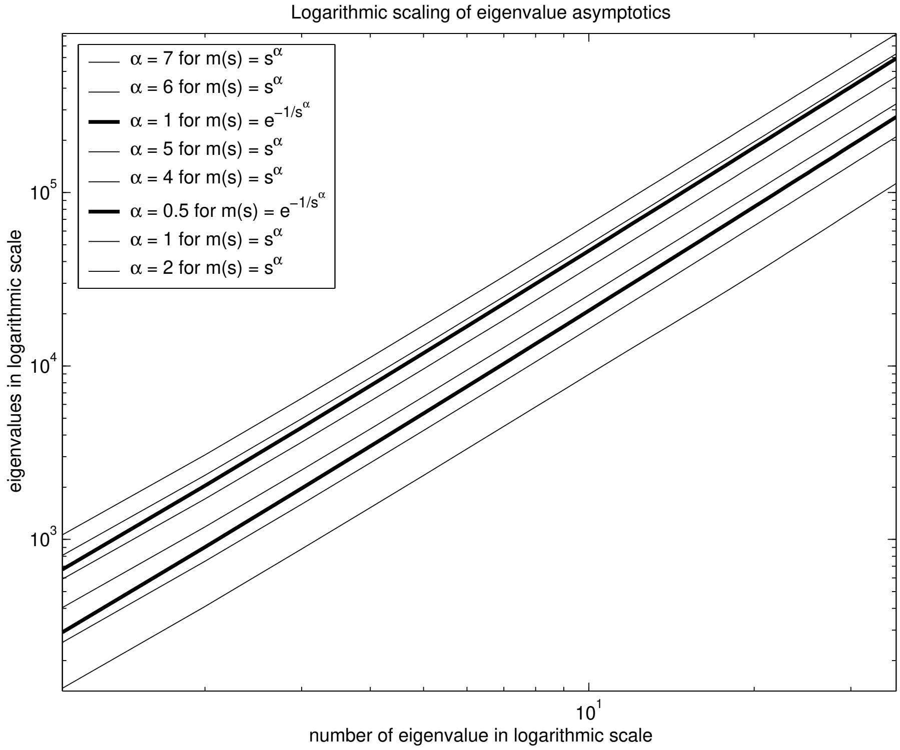

Figure 6.16: *Comparison between first* 80 *computed eigenvalues of Sturm-Liouville problem* $-(au')' = \lambda u$ *for multiplier functions* $m(s) = s^{\alpha}$ *and several values of* α*,* $m(s) = e^{-\frac{1}{\sqrt{s}}}$ *and* $m(s) = e^{-\frac{1}{s}}$

Hence, we can say, that the degree of ill-posedness of the integral operator for these two multiplier functions is the same, but the constants $\underline{C}$ and $\overline{C}$ in (3.14) are different. For example we could compare the ill-posedness of the integral equation with multiplier function $m(s) = e^{-\frac{1}{s}}$ with the one of an integral operator with multiplier function $m(s) = s^6$. Hence, we conjecture that the speed $m(s) \to 0$ as $s \to 0$ does not influence the degree of ill-posedness.

Galerkin method Considering the Galerkin approach, we see, that this numerical method provides both lower bounds (see (6.42)) and upper bounds (see (6.45)) on the true singular values of the integral operator. We even obtain error estimates (see (6.43)). Hence, from inequality (6.42), we know, that the singular values $\sigma_k^{\text{approx}}(B)$ obtained by the numerical method are always a *lower bound* to the true singular values of the singular value problem for the integral equation

$$Bu = \sigma v, \quad Bu(s) := \int_0^1 k(s,t)u(t)dt, \quad (0 \le s \le 1)$$

given by (4.14), i.e.

$$\sigma_k^{\text{approx}}(B) \le \sigma_k^{\text{exact}}(B), \quad k = 1, \dots, n. \tag{6.100}$$

Analytically (see chapter 4) we have found upper bounds on the singular values, i.e. (see (4.34))

$$\sigma_k^{\text{exact}}(B) \le Ck^{-1}, \quad k \to \infty. \tag{6.101}$$

Since the numerical approximation yields

$$\sigma_k^{\text{approx}}(B) \sim \frac{1}{(\alpha+1)\pi k} \quad \text{or} \quad \sigma_k^{\text{approx}}(B) \sim \frac{1}{(l\alpha+e)\pi k}$$

for multiplier functions $m(s) = s^{\alpha}$ and $m(s) = e^{-\frac{1}{s^{\alpha}}}$, respectively, we can combine analytical and numerical results in order to get

$$\underline{C}k^{-1} \sim \sigma_k^{\text{approx}}(B) \le \sigma_k^{\text{exact}}(B) \le \overline{C}k^{-1}, \quad k \to \infty, \tag{6.102}$$

where $\underline{C}$ is determined appropriately according to the multiplier function. Hence, with the help of analytical examinations *and* numerical results we have found that the degree of ill-posedness does not change for integral equations with specific multiplication operators. We have even found values for the constants $\underline{C}$ and $\overline{C}$. The Galerkin method is not as limited as the finite difference method, since we do not have any boundary conditions. The only problem, that occurs is, that the setting up of the matrix A can take some time. Especially for $m(s) = e^{-\frac{1}{s^{\alpha}}}$ the numerical computation of the integrals in (6.49), which has to be as accurate as possible, takes a long time. It is a lower tridiagonal matrix, since the kernel $k(s,t)$ of the integral operator B which we want to approximate vanishes in the upper triangle $\{(s,t) : 0 \le s < t \le 1\}$. Therefore the matrix A is not sparse, which slows down the computation of the singular value decomposition.

Concerning the comparison between the two kinds of multiplier functions we refer to figure 6.17. We observe the same results as discussed for the finite difference method, i.e. the singular values of the exponential type multiplier functions integrate into the ones of the potential type multiplier functions.

Rayleigh-Ritz method If we consider the Rayleigh-Ritz method for the calculation of the eigenvalues, we see that this approach, like the Galerkin method, provides both lower bounds (see (6.42)) and upper bounds (see (6.45)) on the true eigenvalues of the integral operator B^*B. In addition we obtain error estimates (see (6.43)). We therefore know from inequality (6.42), that the eigenvalues $\lambda_k^{\text{approx}}(B^*B)$ of operator B^*B obtained by the numerical method are always a *lower bound* to the true eigenvalues of the eigenvalue problem for the integral equation

$$B^*Bu = \lambda u, \quad Bu(s) := \int_0^1 k(s,t)u(t)dt, \quad (0 \le s \le 1)$$

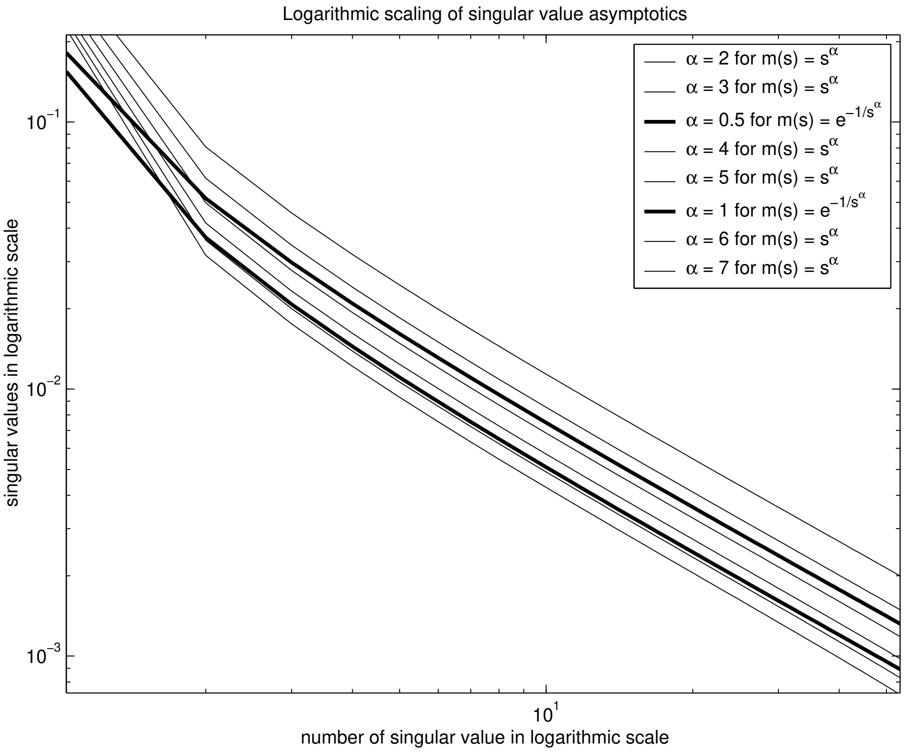

Figure 6.17: *Comparison between first* 80 *computed singular values of integral equation* $Bv = \sigma u$ *for multiplier functions* $m(s) = s^{\alpha}$ *and several values of* α, $m(s) = e^{-\frac{1}{\sqrt{s}}}$ *and* $m(s) = e^{-\frac{1}{s}}$

given by (4.14),i.e.

$$\lambda_k^{\text{approx}}(B^*B) \leq \lambda_k^{\text{exact}}(B^*B), \quad k = 1, \ldots, n. \tag{6.103}$$

Knowing that the singular values are the square roots of the eigenvalues of B^*B (or BB^*) and $\sigma_k > 0$, we get

$$\sigma_k^{\text{approx}}(B) \leq \sigma_k^{\text{exact}}(B), \quad k = 1, \ldots, n. \tag{6.104}$$

As stated above, we have found upper bounds on the singular values analytically (see chapter 4), i.e. (see (6.101))

$$\sigma_k^{\text{exact}}(B) \leq Ck^{-1}, \quad k \to \infty.$$

Since the numerical approximation using the Rayleigh-Ritz yields

$$\sigma_k^{\text{approx}}(B) \sim \frac{1}{(\alpha+1)\pi k} \quad \text{or} \quad \sigma_k^{\text{approx}}(B) \sim \frac{1}{\sqrt{22.65\alpha^2 + 15.28\alpha + 7.38}\pi k}$$

for multiplier functions $m(s) = s^{\alpha}$ and $m(s) = e^{-\frac{1}{s^{\alpha}}}$, respectively, we can combine analytical and numerical results in order to get

$$\underline{C}k^{-1} \sim \sigma_k^{\text{approx}}(B) \leq \sigma_k^{\text{exact}}(B) \leq \overline{C}k^{-1}, \quad k \to \infty, \tag{6.105}$$

where $\underline{C}$ is determined appropriately according to the multiplier function. By using analytical *and* numerical results we have therefore found that the degree of ill-posedness does not change for integral equations with multiplication operators of the form $m(s) = s^\alpha$ or $m(s) = e^{-\frac{1}{s^\alpha}}$ for some $\alpha > 0$. Furthermore we may determine values for the constants $\underline{C}$ and $\overline{C}$.

As the Galerkin method, the Rayleigh-Ritz method is not as limited as the finite difference methods, since there are no boundary conditions (that take the value infinity) which we would have to approximate. The only problem, that occurs is the setting up of the matrix A. This can take a long time, since the matrix is not sparse. Especially for $m(s) = e^{-\frac{1}{s^\alpha}}$ the numerical computation of the integrals in (6.85), which has to be as accurate as possible, takes a long time. The matrix A is symmetric, since the operator B^*B which we want to approximate is self-adjoint, and therefore not sparse, which slows down the computation of the generalized eigenvalue decomposition.

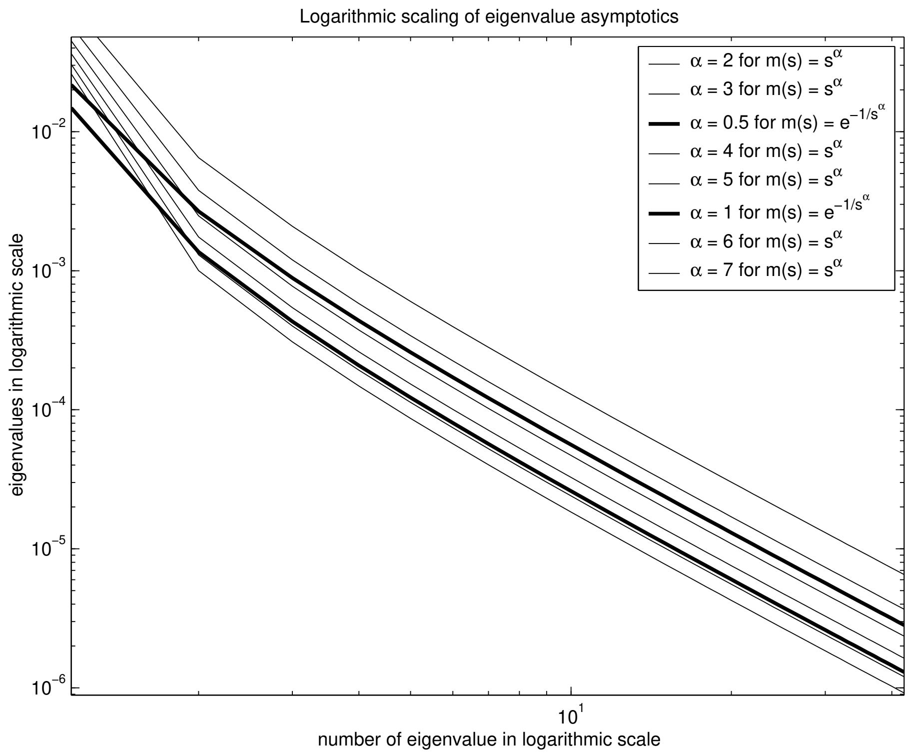

Figure 6.18: *Comparison between first* 80 *computed eigenvalues of integral equation* $B^*Bu = \lambda u$ *for multiplier functions* $m(s) = s^\alpha$ *and several values of* α, $m(s) = e^{-\frac{1}{\sqrt{s}}}$ *and* $m(s) = e^{-\frac{1}{s}}$

Note that we get slightly better approximations by using the Rayleigh-Ritz method with basis functions that are piecewise linear interpolations (see (6.82) than by using the Galerkin method with basis functions that are only piecewise constant approximations of the singular functions (see (6.47) and (6.48)). We could have used piecewise linear interpolations for Galerkin method as well and would have got better approximations.

We may also use higher order interpolations for the basis functions. But the better the approximations to the basis functions are, the more complicated becomes the determination of the matrix entries for either the singular value problem or the eigenvalue problem. Moreover, if we do not use orthonormal basis functions (as done in the second case for the Rayleigh-Ritz method), we obtain generalized singular value problems or generalized eigenvalue problems.

For the comparison between the two kinds of multiplier functions we refer to figure 6.18. We observe the same results as discussed for the finite difference method and the Galerkin method, i.e. the eigenvalues of the exponential type multiplier functions can be integrate into the ones of the potential type multiplier functions.

Hence the integral operator B with multiplier function $m(s) = e^{-\frac{1}{\sqrt{s}}}$ is observed to be as ill-posed as the integral operator with multiplier function $m(s) = s^{\alpha}$, where $3 < \alpha < 4$ and the integral operator B with multiplier function $m(s) = e^{-\frac{1}{s}}$ is observed to be as ill-posed as the integral operator with the potential type multiplier function $m(s) = s^{\alpha}$, where $5 < \alpha < 6$.

Hence, from figures (6.16)-(6.18) we observe that the rate of convergence $m(s) \to 0$ as $s \to 0$ does not influence the degree of ill-posedness.

Moreover, in section 6.5 we were able to conjecture that

$$\sigma_n(B) \sim \int_0^1 m(s)ds \cdot \sigma_n(J) \sim \int_0^1 m(s)ds \cdot \frac{2}{\pi(2n-1)},$$

holds for the integral operator $B = M \circ J$, if the integral $\int_0^1 m(s)ds$ exists. This outcome even confirms the results from sections 6.1-6.3.

Chapter 7

On the influence of multiplication operators on the Tikhonov regularization

In the previous chapter we have seen through several numerical approaches that the multiplication operator M in the composition $B = M \circ J$ does not influence the degree of ill-posedness of the linear integral operator J, which is given by $\nu = 1$.

In this chapter we want to consider the influence of this multiplication operator on the regularization opportunities of an inverse problem given by the linear integral operator

$$Bx = y, \quad x \in D(F) \subset X, y \in Y$$

with $B = M \circ J$ as well as on the chances of regularization of the inverse problem given by the nonlinear operator

$$F(x) = y, \quad x \in D(F) \subset X, y \in Y.$$

e Therefore, we are going to consider the regularization of the nonlinear problem itself as well as the regularization of the linearized problem.

To this end, we firstly state some results on local ill-posedness of nonlinear operators and the relationship between the ill-posedness of a nonlinear problem and the linearized problem and apply this to our special problem $F = N \circ J$ from chapter 2, equation (2.12).

Then we are going to consider Tikhonov regularization applied to the nonlinear problem as well as to the linear operator equation and examine the influence of the multiplication operator.

7.1 Local ill-posedness behavior of nonlinear operators

In chapter 2, definition 2.3 we have already seen the characterization of local ill-posedness of a nonlinear operator $F(x) = y$. We are now going to see some more definitions.

7.1.1 Some definitions and application to $F(x) = y$

Firstly we shall extend the definition of compactness from chapter 3 to nonlinear operators:

Definition 7.1. *A nonlinear operator $F : D(F) \subseteq X \Rightarrow Y$, mapping between Hilbert spaces X and Y, is called* ***compact*** *on a subset $S \subseteq D(F)$, if F is continuous in S and if each bounded subset $T \subseteq S$ in X maps into a relatively compact subset $F(T) = \{y \in Y : y = F(x), x \in T\}$ in Y.*

Another important characteristic of nonlinear operators is weak closedness (see [45]), which we define in the following

Definition 7.2. *A nonlinear operator $F : D(F) \subseteq X \to Y$, mapping between Hilbert spaces X and Y, is called* ***weakly closed*** *if for $x_n \subset D$ and weak convergence of the sequences $x_n \rightharpoonup x_0$ in X and $F(x_n) \rightharpoonup y_0$ in Y immediately implies the relationships $x_0 \in D$ and $F(x_0) = y_0$.*

We know that compact linear operators A with finite dimensional image space $R(A)$ always lead to ill-posed problems, since the stability condition of Hadamard is not satisfied. We may extend this result to nonlinear operators. From the definition of local well-posedness of a nonlinear inverse problem in chapter 2 we see that this corresponds to a local stability condition. Therefore, we may follow that also nonlinear compact operators lead to locally ill-posed problems. From the two definitions above we immediately get the following proposition taken from [45]:

Proposition 7.1. *Let the domain D of the weakly closed operator F, mapping between the infinite dimensional Hilbert spaces X and Y, satisfy the condition*

$$B_r(x_0) := \{x \in X : \|x - x_0\|_X \leq r\} \subseteq D(F)$$

for some $r > 0$. Furthermore, let F be compact on $B_r(x_0)$. Then the nonlinear operator $F(x) = y$ is locally ill-posed in x_0.

A proof of this proposition can be found in [45] or [46]. We are going to state some characteristics for the nonlinear operator F given by

$$F(x) = y, \quad x \in D(F) \subset L^2(0,1), \quad y \in L^2(0,1), \tag{7.1}$$

with F from (2.12)

$$[F(x)](t) = k(t, [J(x)](t)), \quad (0 \leq t \leq 1)$$

where the domain of the nonlinear operator $F = N \circ J$, as a composition of an outer Nemytskii operator $[N(z)](t) = k(t, z(t))$, where $k(t,s)$, $(t,s) \in [0,1] \times [0,\infty)$ is a sufficiently smooth kernel function and an inner integral operator J, is restricted to the domain

$$D(F) = \{x \in L^2(0,1) : x(t) \geq c \geq 0 \text{ a.e. in } [0,1]\}. \tag{7.2}$$

First, we need the following proposition on the continuity of Nemytskii operators (see [3, Theorem 2.2] for a proof).

Proposition 7.2. *Let $\Omega \in \mathbb{R}^n$ be bounded and suppose that the kernel function $k(t,s)$ satisfies the* Carathéodory condition*, i.e.*

(i) $s \to k(t,s)$ is continuous for almost every $t \in \Omega$,

(ii) $t \to k(t,s)$ is measurable for all $s \in \mathbb{R}$.

Furthermore let a growth condition

$$|k(t,s)| \leq a + b|s|^{\alpha}, \quad \alpha = \frac{p}{q},$$

with $p,q \geq 1$ and $a,b > 0$ be satisfied for the Nemytskii operator. Then the Nemytskii operator k is a continuous map from $L^p(\Omega)$ to $L^q\Omega$.

This proposition will be used in order to proof the following theorem on some characteristics of the nonlinear operator F.

Theorem 7.1. *Let the nonlinear operator $F : D(F) \subset L^2(0,1) \to L^2(0,1)$ possess a convex and weakly closed domain $D(F)$ as given by (7.2). Furthermore let the function $k(t,s)$ generating the Nemytskii operator be continuous for almost every $t \in I$, where $I = [0,1]$ and measurable for all $s \in [0,\infty)$ and let*

$$|k(t,s)| \leq a + b|s|, \quad a,b > 0$$

be valid. Then the nonlinear operator F is compact, continuous, weakly continuous and consequently weakly closed.

Proof. First we prove continuity and compactness. By applying proposition 7.2 with $\Omega = [0,1]$ and $p = q = 2$ to $k(s,t)$ we find that all assumptions of proposition 7.2 are satisfied and k maps continuously from $L^2(0,1)$ to $L^2(0,1)$.

The nonlinear operator $F = N \circ J$ is a composition of a compact linear operator $J : L^2(0,1) \to L^2(0,1)$ (see [57] for a proof of compactness) and $N : L^2(0,1) \to L^2(0,1)$, given by $[N(z)](t) = k(t,z(t))$, which is a continuous nonlinear operator, as we have just proved. Hence, the composite operator $F = N \circ J$ is compact and continuous, too. Weak continuity follows from continuity.

Finally we will prove weak closedness. Let the sequences $x_n \in D(F)$ and $F(x_n)$ be weak convergent, i.e.

$$x_n \rightharpoonup x_0 \quad \text{in} \quad X \quad \text{and} \quad F(x_n) \rightharpoonup y_0 \quad \text{in} \quad Y.$$

Knowing that $x_n \in D(F)$ and with the assumption that $D(F)$ is a weakly closed domain we have $x_0 \in D(F)$. By continuity of F we then get

$$x_n \rightharpoonup x_0 \quad \text{in} \quad D(F) \quad \Rightarrow \quad F(x_n) \rightharpoonup F(x_0) \quad \text{in} \quad Y.$$

Since the limit of a convergent sequence in Hilbert spaces is unique we get

$$F(x_0) = y_0,$$

which, together with $x_0 \in D(F)$, yields weak closedness. $\square$

Notice that the Carath*é*odory condition and the growth condition for $k(s,t)$ are certainly satisfied if $k(t,s)$ is continuous and uniformly bounded for all $(t,s) \in [0,1] \times [0,\infty)$ and we have continuity of N between spaces of power-integrable function. If we furthermore assume that the nonlinear operator F is injective, we may formulate the following lemma:

Lemma 7.1. *The nonlinear operator $F : D(F) \subset L^2(0,1) \to L^2(0,1)$ possessing a convex and weakly closed domain $D(F)$ as given by (7.2) is injective, compact, continuous, weakly continuous and consequently weakly closed and the inverse operator F^{-1} defined on $F(D(F))$ exists.*

For a proof and explanations to this lemma we refer to [38], where it was shown for the special case of a Nemytskii operator defined by the Black-Scholes function. This lemma, together with the results by Engl, Kunisch and Neubauer [18, Proposition A.3] or (7.1) provides the following proposition which implies the ill-posedness of operator F from (2.12).

Proposition 7.3. *For a given right-hand side $y_0 \in F(D(F))$ the operator equation (7.1) has a uniquely determined solution $x_0 \in D(F)$. For any ball $B_r(x_0)$ with centre x_0 and radius $r > 0$ there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset D(F) \cap B_r(x_0)$ satisfying*

$$x_n \rightharpoonup x_0, \quad \textit{but} \quad x_n \not\to x_0 \quad \textit{and} \quad F(x_n) \to y_0 \quad \textit{in} \quad L^2(0,1), n \to \infty.$$

Thus, equation (7.1) is locally ill-posed in the sense of definition 2.3 and F^{-1}, defined on $F(D(F) \cap B_r(x_0))$, is not continuous in x_0.

A proof can be found in [18]. Therefore, a regularization is required for the stable solution of (7.1). Regularization methods are going to be considered in section 7.2.4.

Before, we are going to examine the relationship between the ill-posedness of the nonlinear operator equations and its linearizations.

7.1.2 The relationship between the ill-posedness of a nonlinear problem and the linearized problem

As we have seen in the previous section the nonlinear problem (7.1) with $F = N \circ J$ given by (2.12) is locally ill-posed.

For linear problems

$$Ax = y, \quad A \in \mathcal{L}(X,Y), D(A) = X$$

the character of ill-posedness is global and depends only on properties of the operator A. Since we require the existence of Fr*é*chet derivatives $F'(x)$ (see definition 2.4) of F, we may compare the nonlinear operator (7.1) to its linearization

$$F'(x_0)x = y$$

at the point x_0. In chapter 2 we have introduced the linearized problem where the Fr*é*chet derivative is given by

$$[F'(x_0)(h)](t) = m(t)[J(h)](t), \quad (0 \le t \le 1, h \in X = L^2(0,1)),$$

i.e. a composition $F'(x_0) = M \circ J$ of the convolution operator J with a multiplication operator M.

In chapter 4 we have seen that this linearized operator $F'(x_0) = B = M \circ J$ is a compact bounded linear operator, if the multiplier function m satisfies $m \in L^\infty(0,1)$. This holds especially for continuous multiplier functions $m \in C[0,1]$. Furthermore we know from chapter 3 that compact linear operators B mapping into infinite dimensional image spaces $R(B)$ always lead to ill-posed problems.

But the connection between the ill-posedness of a nonlinear problem and its linearization is not as strong as we might think. Engl, Kunisch and Neubauer (see [18]) provided examples that show that ill-posedness of a nonlinear problem does not necessarily imply ill-posedness of its linearization. Conversely a well-posed nonlinear problem may have ill-posed linearizations.

But in the case of *compact* nonlinear operators we may state a relationship between the operator and its linearization. The compactness of the nonlinear operator transfers to the Fr*é*chet derivative (see [11]).

Lemma 7.2. *Let $F : D(F) \subseteq X \to Y$ be a compact operator on the open subset $S \subseteq D(F)$ which is Fréchet differentiable in an $x_0 \in S$. Then the Fréchet derivative $F'(x_0) \in \mathcal{L}(X,Y)$ is a compact operator, too.*

Several more interplays between the local ill-posedness of a nonlinear problem and its linearization using the Fr*é*chet derivative were considered by Hofmann [46].

The local degree of ill-posedness in x_0 is defined by the degree of ill-posedness of the corresponding linearized equation. In [49] several examinations were made on factors influencing the ill-posedness of nonlinear problems. From there we see that the correlation between the nonlinear operator and its linearization does not have to be that strict. As shown in [27] the local degree of ill-posedness for the autoconvolution operator may vary quite strongly depending on x_0.

For our special nonlinear equation $F(x) = y$ with $F = N \circ J$ as well as its linearization $F'(x_0) = M \circ J$ we have seen through the last chapters that they are both ill-posed and that the degree of ill-posedness is the same for the nonlinear and linearized problem. Through numerical methods we have found that the degree of ill-posedness of those operators is one and that the multiplication operator does not influence the degree of ill-posedness.

Here we finally want to note that we may force the local well-posedness of a nonlinear equation $F(x) = y$ by restricting the domain of feasible solutions to a set $\hat{D}$ given by

$$\hat{D} = \{x \in D : x - x_0 = F'(x_0)^* w,\, w \in Y, \|w\|_Y \le \tau\}, \tag{7.3}$$

where $F'(x_0)$ is the Fr*é*chet derivative in x_0, $F'(x_0)^*$ its adjoint operator and τ a sufficiently small positive constant. Elements in $\hat{D}$ satisfy a so-called source condition.

With this condition we may formulate the following proposition (see [50] or [45]):

Proposition 7.4. *Let the operator F be Fréchet differentiable with Lipschitz continuous derivative on $B_r(x_0)$, i.e.*

$$\|F(x) - F(x_0) - F'(x_0)(x - x_0)\|_Y \leq \frac{L}{2}\|x - x_0\|_X^2, \quad x \in B_r(x_0)$$

holds. Then the operator equation

$$F(x) = y, \quad x \in \hat{D} \subseteq X, y \in Y$$

with a restricted domain of feasible solutions $\hat{D}$ given by (7.3) is always locally well-posed in x_0. We may proof a stability estimate

$$\|x - x_0\|_X \leq \sqrt{\tau(1-\theta)}\sqrt{\|F(x) - F(x_0)\|_Y}, \quad \theta := \frac{L\tau}{2},$$

where $x \in \hat{D} \cap B_r(x_0)$.

For a proof we refer to [50] or [45].

Hence if we can choose approximate solutions to a problem that satisfy such source conditions we may overcome the instability of such a problem. We will return to the concept of source conditions in the next section, where we are going to consider regularization tools for our inverse problems with multiplication operators.

7.2 Tikhonov regularization of the ill-posed problem

7.2.1 Introduction and definitions

We want to find stable solutions to ill-posed problems by regularization methods. Hence we want to find

$$x_{mn} = A^{\dagger}y, \quad y \in R(A)$$

(see (3.12)) as stable and exact as possible, where $A \in \mathcal{L}(X,Y)$ is a compact linear operator and $A^{\dagger}$ its Moore-Penrose inverse. Since only estimated non-exact data $y_\delta \in Y$ satisfying $\|y_\delta - y\|_Y \leq \delta$ are given, we want the approximate solution to depend continuous on the data and have small errors. We may introduce the following definition of a linear regularizer from [32] or [17]:

Definition 7.3. *A family of bounded linear operators $\{R_\alpha\}$ with $R_\alpha \in \mathcal{L}(X,Y)$ where $\alpha > 0$ is the regularization parameter is called a linear regularizer for $A^{\dagger}$, if*

$$\lim_{\alpha \to 0} R_\alpha y = A^{\dagger}y \quad \forall y \in R(A). \tag{7.4}$$

From this definition we immediately get the following lemma, which holds especially for compact operators A (see [32] or [17], too).

Lemma 7.3. *If the image set of $A \in \mathcal{L}(X,Y)$ $R(A)$ is not closed then for the family of bounded linear operators $R_\alpha \in \mathcal{L}(X,Y)$ satisfying (7.4)*

$$\lim_{\alpha \to 0} \|R_\alpha\|_{\mathcal{L}(X,Y)} = \infty$$

holds.

If we apply the regularization operator R_α to the approximated given data y_δ we get regularized solutions

$$x_\alpha^\delta = R_\alpha y_\delta.$$

Therefore we may estimate the total regularization error

$$\|x_\alpha^\delta - x_{mn}\|_X \leq \|R_\alpha\|_{\mathcal{L}(X,Y)}\delta + \|R_\alpha y - A^\dagger y\|_X. \tag{7.5}$$

The stability component $\|R_\alpha\|_{\mathcal{L}(X,Y)}\delta$ expresses the influence of the data error and the approximation component $\|R_\alpha y - A^\dagger y\|_X$ is a measure for the distance of the regularized problem $x_\alpha = R_\alpha y$ to the original unstable problem $x_{mn} = A^\dagger y$. From previous considerations we know that the first component tends to infinity as $\alpha \to 0$ whilst the second component tends to zero. Hence we have to find an optimal α_{opt}, which minimizes $\|x_\alpha^\delta - x_{mn}\|_X$ by dealing with a compromise between accuracy and stability.

Finally we want to consider the question of of the convergence of a regularization, i.e. if for $\delta \to 0$ the regularization error tends to zero.

Definition 7.4. *A linear regularizer R_α for the operator $A^\dagger$ with $\alpha = \alpha(\delta, y_\delta)$ is called convergent if*

$$\lim_{\delta \to 0} \sup\{\|R_{\alpha(\delta,y_\delta)} y_\delta - x_{mn}\|_X : \|y_\delta - y\|_Y \leq \delta\} = 0$$

holds for all $y \in R(A)$.

From this definition we get the following proposition for linear regularizations (see [32] or [17]):

Proposition 7.5. *A linear regularizer R_α for the operator $A^\dagger$ is convergent according to definition 7.4 if the regularization parameter $\alpha = \alpha(\delta)$ is chosen such that*

$$\lim_{\delta \to 0} \alpha(\delta) = 0$$

and

$$\lim_{\delta \to 0} \|R_{\alpha(\delta)}\|_{\mathcal{L}(X,Y)}\delta = 0$$

are satisfied.

Hence, for a convergent regularization $\alpha \to 0$ if $\delta \to 0$ has to be satisfied. But since $\|R_\alpha\|_{\mathcal{L}(X,Y)} \to \infty$ for $\alpha \to 0$ the convergence must not be too fast.

For a compact operator one may construct such a linear convergent regularization via the singular value expansion and filter functions (see [45]).

Here we are only going to consider the method of Tikhonov regularization, which is now the standard method for the practical treatment of ill-posed problems. For further methods we refer to [45].

7.2.2 Tikhonov regularization of linear operators, convergence rates and application to $Bx = y$

We are now going to describe the method of Tikhonov regularization and apply it to $Bx = y$, where $B = M \circ J$. Thus, we have $X = Y = L^2(0,1)$. The generalized solution $x = B^\dagger y$ (see (3.12)) is a least squares solution and therefore satisfies the normal equations

$$B^*Bx = B^*y.$$

Now, the self-adjoint compact operator B^*B has nonnegative eigenvalues and therefore, for any positive number α the operator $B^*B + \alpha I$, where I is the identity operator in $L^2(0,1)$ has strictly positive eigenvalues. Then the operator $B^*B + \alpha I$ has a bounded inverse and the problem

$$(B^*B + \alpha I)x_\alpha = B^*y$$

is well-posed and

$$x_\alpha = (B^*B + \alpha I)^{-1}B^*y$$

is the Tikhonov approximation to $B^\dagger y$. For approximate data y_δ we then get the regularized solution

$$x_\alpha^\delta = (B^*B + \alpha I)^{-1}B^*y_\delta,$$

which is the same as the solution to the minimization problem for the Tikhonov functional

$$T_\alpha(x) := \|Bx - y_\delta\|^2_{L^2(0,1)} + \alpha\|x\|^2_{L^2(0,1)} \to \min! \quad \text{subject to} \quad x \in L^2(0,1).$$

Obviously we can define the family of linear regularizer R_α as it was introduced in the previous section to be

$$x_\alpha^\delta := R_\alpha y_\delta = (B^*B + \alpha I)^{-1}B^*y_\delta, \tag{7.6}$$

and we may state a proposition for a convergent linear Tikhonov regularization (see [45] for a proof):

Proposition 7.6. *The operator family $\{R_\alpha\}_{\alpha>0}$ for the Tikhonov regularization defined in (7.6) is a linear regularization for the operator $B^\dagger$ with $\|R_\alpha\|_{\mathcal{L}(X,Y)} \le \frac{1}{2\sqrt{\alpha}}$. If we choose $\alpha = \alpha(\delta)$ to satisfy the conditions*

$$\alpha(\delta) \to 0 \quad \textit{and} \quad \frac{\delta^2}{\alpha(\delta)} \to 0 \quad \textit{for} \quad \delta \to 0 \tag{7.7}$$

then the regularization is convergent.

Since R_α is a bounded linear operator the regularized solution x_α^δ depends continuously on the data y_δ. Now we want to investigate so called *convergence rates*, which state how fast the regularized solution x_α^δ converges to the solution $x_{mn} = B^\dagger y$. In section 7.1.2 we have already seen that ill-posed problems may be turned into well-posed ones by restricting the domain of the solution with the help of source conditions. For the Tikhonov regularization this approach of source conditions may lead to the achievement of certain convergence rates.

We may state and prove the following theorem for the convergence rate of the regularized solution to the minimum-norm solution $\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)}$:

Theorem 7.2. *Let c and C be positive constants. If a source condition*

$$x_{mn} = B^* w, \quad w \in Y \tag{7.8}$$

holds for the minimum-norm solution $x_{mn} = B^\dagger y$ and if we choose the regularization parameter to be $\alpha = \alpha(\delta) = c\delta$, then an estimate

$$\|x_\alpha^\delta - x_{mn}\|_X \le C\sqrt{\delta} \tag{7.9}$$

holds for the regularization error.

Proof. From (7.5) we get

$$\|x_\alpha^\delta - x_{mn}\|_X \le \|R_\alpha\|_{\mathcal{L}(X,Y)}\delta + \|R_\alpha y - B^\dagger y\|_X$$

Knowing that for Tikhonov regularization $\|R_\alpha\|_{\mathcal{L}(X,Y)} \le \frac{1}{2\sqrt{\alpha}}$ is valid and $x_\alpha = R_\alpha y$ as well as $x_{mn} = B^\dagger y$ hold by assumption, we get

$$\|x_\alpha^\delta - x_{mn}\|_X \le \frac{\delta}{2\sqrt{\alpha}} + \|x_\alpha - x_{mn}\|_X. \tag{7.10}$$

We are finally going to estimate $\|x_\alpha - x_{mn}\|_X$ by applying the exact source condition $x_{mn} = B^* w$

$$\begin{aligned}
\|x_\alpha - x_{mn}\|_X &= \|R_\alpha y - x_{mn}\|_X \\
&= \|(B^*B + \alpha I)^{-1}B^*Bx_{mn} - x_{mn}\|_X \\
&= \|-\alpha(B^*B + \alpha I)^{-1}x_{mn}\|_X \\
&= \|\alpha(B^*B + \alpha I)^{-1}B^*w\|_X \\
&= \|\alpha R_\alpha w\|_X \\
&\le \alpha\|R_\alpha\|_{\mathcal{L}(X,Y)}\|w\|_Y \le \frac{\sqrt{\alpha}}{2}\|w\|_Y.
\end{aligned}$$

Finally with $\alpha(\delta) = c\delta$ we obtain

$$\|x_\alpha^\delta - x_{mn}\|_X \le \frac{1 + c\|w\|_Y}{2\sqrt{c}}\sqrt{\delta},$$

i.e. inequality (7.9) with $C = \frac{1+c\|w\|_Y}{2\sqrt{c}}$. □

Notice that in [45] this convergence rate property was shown under the aspect of so-called discrepancy principle, i.e. the parameter $\alpha_{\text{dis}} = \alpha_{\text{dis}}(\delta, y_\delta)$ was chosen a posteriori to satisfy

$$\|Bx_{\alpha_{\text{dis}}} - y_\delta\|_Y = \delta.$$

Further convergence rate properties and other source conditions were introduced in [45].

We want to apply the principle of source conditions to our problem $Bx = y$ with $B = M \circ J$. Without multiplier function (i.e. $M = I$) we get $B^* = J^*$ and therefore the exact source condition $x_{mn} = B^* w$ becomes

$$x_{mn}(t) = \int_t^1 w(s)ds \quad (0 \le t \le 1, w \in L^2(0,1)).$$

This implies

$$x_{mn}(1) = 0 \quad \text{and} \quad x'_{mn} \in L^2(0,1), \tag{7.11}$$

i.e. $x_{mn} \in H^1(0,1)$, which is a certain smoothness requirement on the minimum norm solution x_{mn}.

With a multiplication operator we have $B^* = J^* \circ M$ and thus, the exact source condition becomes

$$x_{mn}(t) = \int_t^1 m(s)w(s)ds \quad (0 \le t \le 1, w \in L^2(0,1)),$$

implying

$$x_{mn}(1) = 0 \quad \text{and} \quad \frac{x'_{mn}}{m} \in L^2(0,1). \tag{7.12}$$

Since $m(s) \to 0$ as $s \to 0$ holds for our specific multiplier functions $\frac{1}{m} \notin L^\infty(0,1)$ and especially the last condition is hard to satisfy.

Therefore we introduce a so-called inexact source condition using a proposition first stated by Baumeister [7] with $X = Y = L^2(0,1)$:

Proposition 7.7 (Baumeister 1987). *Let X and Y be Hilbert spaces and B an injective linear bounded operator $B : X \to Y$ with $\overline{R(B)} \neq R(B)$, i.e. let the range be unclosed. Then*

$$Bx = y, \quad x \in X, y \in Y$$

is a linear ill-posed Operator equation. Consider the profile function

$$f(\alpha) = \|x_\alpha - x_{mn}\|_X$$

for Tikhonov regularized solutions $x_\alpha = R_\alpha y = (B^*B + \alpha I)^{-1} B^* y$ minimizing

$$T_\alpha(x) := \|Bx - y\|_Y^2 + \alpha \|x\|_X^2$$

over $L^2(0,1)$ for $y = Bx_{mn}$ corresponding to the regularization parameter α. Then we have

$$f(\alpha) \le \sqrt{\eta(R)^2 + \alpha R^2} \quad (\alpha > 0), \tag{7.13}$$

where the shift value

$$\eta(R) = \inf_{\|w\|_Y \le R} \|x_{mn} - B^* w\|_X$$

measures the violation w.r.t. the exact source condition $x_{mn} = B^* w$ for $x_{mn} \in X$. Furthermore

$$\lim_{R\to\infty} \eta(R) = 0$$

is valid, since $N(B) = \{0\}$ and $\overline{R(B^*)} = X$.

For a proof of this proposition we refer to [7].

Since we are especially interested in the influence of the multiplication operator, in a first step we assume that the exact source condition is satisfied for the operator without multiplier function, i.e. $x_{mn} = J^* w$ with $\|w\|_{L^2(0,1)} = R$. Then proposition (7.7) yields $\eta(R) = 0$ and

$$\|x_\alpha - x_{mn}\|_{L^2(0,1)} = R\sqrt{\alpha}.$$

For the operator with multiplier function $B = M \circ J$ only an inexact source condition is assumed and we are going to estimate the shift value

$$\eta(R) = \inf_{\|\tilde{w}\|_{L^2(0,1)} \le R} \|x_{mn} - B^* \tilde{w}\|_{L^2(0,1)}.$$

With the help of $x_{mn} = J^* w$ we find an upper bound on $\eta(R)$

$$\begin{aligned}
\eta(R)^2 &\le \|J^* w - B^* w\|^2_{L^2(0,1)} \\
&= \|J^* w - J^* \circ M w\|^2_{L^2(0,1)} \\
&= \|J^*(w - Mw)\|^2_{L^2(0,1)} \\
&\le \int_0^1 (1 - m(t))^2 (w(t))^2 dt
\end{aligned}$$

Hence, if

$$\eta(R) \le \left(\int_0^1 (1 - m(t))^2 (w(t))^2 dt \right)^{\frac{1}{2}} \tag{7.14}$$

is small, we get a small influence of this shift value on the convergence rate $\sqrt{\delta}$. As for the degree of ill-posedness (see chapter 6) we see that the integral of the multiplier function

$$\int_0^1 (1 - m(t))^2 dt$$

influences the regularization opportunities. If this integral is small, then $\eta(R)$ is small, too. Thus from proposition 7.7 we get

$$f(\alpha) \le \sqrt{\eta(R)^2 + \alpha R^2} \quad (\alpha > 0)$$

and together with (7.10) this yields

$$\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)} \le \frac{\delta}{2\sqrt{\alpha}} + \sqrt{\eta(R)^2 + \alpha R^2} \le \frac{\delta}{2\sqrt{\alpha}} + \eta(R) + R\sqrt{\alpha}, \qquad (7.15)$$

i.e. a small influence of the shift value $\eta(R)$ for $\delta \gg 0$ and $\alpha \sim \delta$. We also see that it is not important how fast the multiplier function $m(t)$ with $m(t) \to 0$ as $t \to 0$ approaches zero. Only the integral

$$\int_0^1 (1 - m(t))^2 dt$$

has got an influence on $\eta(R)$ and therefore on the regularization properties. If $w \in L^\infty(0,1)$, i.e. $w(t)$ is smooth enough without a peak at $t = 0$ and

$$\|w\|_{L^\infty(0,1)} < \infty$$

holds, then we may estimate (7.14) to get

$$\eta(R) \le \sqrt{\int_0^1 (1 - m(t))^2 dt}\,\|w\|_{L^\infty(0,1)}.$$

Therefore $\eta(R)$ is small if the above integral is small. To this end, let us consider the following multiplier function:

$$\begin{aligned} m(t) &= 1 \quad (\varepsilon \le t \le 1), && (7.16) \\ m(t) &\to 0 \quad \text{as} \quad t \to 0 && (7.17) \end{aligned}$$

A plot of this function can be seen in figure 7.1. Using this function and assuming that

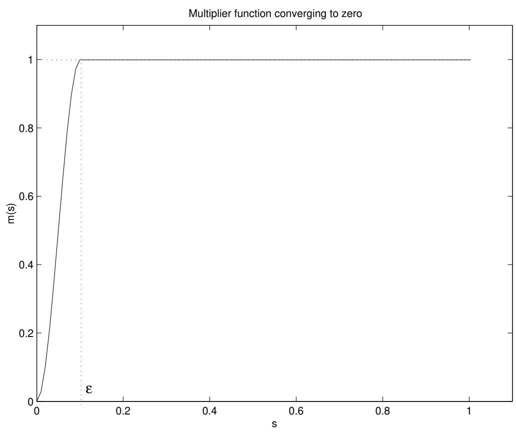

Figure 7.1: *Multiplier function with fast convergence $m(s) \to 0$ as $s \to 0$*

$w \in L^\infty(0,1)$, we get

$$\eta(R) \le \sqrt{\int_0^\varepsilon (1 - m(t))^2 dt}\,\|w\|_{L^\infty(0,1)},$$

If $\varepsilon > 0$ is small, then $\eta(R)$ is small, too. We find that not the velocity of $m(s) \to 0$ as $s \to 0$ is important, but only the area (it is the marked area in figure 7.1) influences $\eta(R)$ and therefore the regularization properties. We may find an upper bound for $\eta(R)$ by

$$\begin{aligned} \eta(R) &\leq \sqrt{\int_0^{\varepsilon} (1-m(t))^2 dt} \|w\|_{L^\infty(0,1)} \\ &\leq \sqrt{\int_0^{\varepsilon} dt} \|w\|_{L^\infty(0,1)} \\ &\leq \sqrt{\varepsilon} \|w\|_{L^\infty(0,1)} \end{aligned}$$

and substituting this result into (7.15) we get

$$\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)} \leq \frac{\delta}{2\sqrt{\alpha}} + \sqrt{\varepsilon} + \sqrt{\alpha} R.$$

If we choose the regularization parameter $\alpha = \alpha(\delta) = c\delta$ then we can estimate

$$\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)} \leq \frac{1}{2\sqrt{c}} \sqrt{\delta} + \sqrt{c}\sqrt{\delta} R + \sqrt{\varepsilon},$$

and for small ε

$$\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)} \leq C\sqrt{\delta}$$

for some constant $C > 0$ holds for the regularization error. Hence, we have observed that *not* the velocity $m(s) \to 0$ as $s \to 0$ influences the regularization properties, but only the integral $\int_0^1 (1-m(t))^2 dt$. This is the same result as for the degree of ill-posedness (see chapter 6), where we observed that the integral $\int_0^1 m(t)dt$ influences the constant C in (3.14).

In a further step we do not assume any exact source condition, since we have seen that these conditions are hard to satisfy. Especially the right hand boundary value has to be chosen to be exact (see (7.11) and (7.12)), which is not likely to happen and therefore unsatisfactory. Hence we will use Baumeister's lemma for inexaxt source conditions in order to find convergence rates for the Tikhonov regularization.

7.2.3 Inexact source conditions and the application of Baumeister's lemma to multiplication operators

The satisfaction of exact source conditions is somehow unnatural, therefore let the source condition not be satisfied exactly, i.e. we want to determine the shift value $\eta(R)$ from proposition 7.7 given by

$$\eta(R) = \inf_{\|w\|_{L^2(0,1)} \leq R} \|x_{mn} - A^* w\|_{L^2(0,1)}.$$

Notice that there are two cases:

If $\eta(R) = 0$ for $R \geq R_0$ then the classical source condition $x_{mn} = A^* w$ is satisfied with $\|w\|_{L^2(0,1)} = R_0$ and the convergence rate

$$\|x_\alpha - x_{mn}\|_{L^2(0,1)} \leq R_0 \sqrt{\alpha}$$

is valid. If $\eta(R) > 0$ for $R \in [0, \infty)$ then the source condition does not hold.
The prospects of reconstruction by regularization of the element $x_{mn} \in L^2(0,2)$ are not only influenced by the degree of ill-posedness of the operator equation $Ax = y$, which, for a compact operator A, can be computed via the singular value decomposition, as a measure for the strength of smoothness of the operator A. The opportunities of regularization are also influenced by the relative smoothness of the element x_{mn} with reference to A (see [65] for generalized source conditions with so-alled index functions).
For Tikhonov regularization $\eta(R)$ is a measure of this relative smoothness of x_{mn} with reference to A. In the case $\eta(R) > 0$ the decay rate of $\eta(R) \to 0$ for $R \to \infty$ firstly determines the smoothness of x_{mn} with respect to A, i.e. the faster $\eta(R) \to 0$, the better and faster is the reconstruction of x_{mn} with Tikhonov regularization. Furthermore it influences the convergence rate of the regularization error. By choosing $R = R(\alpha)$ such that

$$\eta(R(\alpha)) = \sqrt{\alpha} R(\alpha)$$

is satisfied, i.e. both terms in (7.13) have the same order of magnitude and we obtain the following property for regularization error:

$$\|x_\alpha - x_{mn}\|_{L^2(0,1)} \leq 2\sqrt{\alpha R^2(\alpha)} \leq 2\sqrt{\alpha} R =: \zeta(\alpha). \tag{7.18}$$

From (7.10) we get

$$\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)} \leq \frac{\delta}{2\sqrt{\alpha}} + \|x_\alpha - x_{mn}\|_{L^2(0,1)} \leq \frac{\delta}{2\sqrt{\alpha}} + \zeta(\alpha). \tag{7.19}$$

By equalizing

$$\frac{\delta}{2\sqrt{\alpha}} = \zeta(\alpha) \tag{7.20}$$

we may choose an optimal decay rate for the data error. Then, by setting $\alpha = g(\delta)$, determined from (7.20), we observe that the decay rate $\zeta(\alpha)$ of the regularized solution $\|x_\alpha - x_{mn}\|_{L^2(0,1)}$ influences the convergence rate of the Tikhonov regularization to the minimum-norm solution, i.e. the difference $\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)}$. The decay rate $\zeta(\alpha)$ may also transfer to convergence rate of the Tikhonov regularization to the minimum-norm solution, i.e.

$$\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)} = O(\zeta(\delta)), \quad \text{as} \quad \delta \to 0,$$

see [52], depending on the slowest rate of the two terms in the right hand side of (7.19). The resulting convergence rate is always less than $\sqrt{\delta}$, since the optimal convergence rate $\sqrt{\delta}$ can only be obtained for exact source conditions, i.e. $\eta(R) = 0$.

We want to determine the value of $\eta(R)$ in order to find convergence rates. For simplicity, we are going to consider the pure multiplication operator first, i.e.

$$(Ax)(t) = m(t)x(t), 0 \le t \le 1, x \in L^2(0,1), \tag{7.21}$$

whose adjoint operator is given by

$$(A^*x)(t) = m(t)x(t).$$

Thus, we want to optimize

$$\|x_{mn}(t) - m(t)w(t)\|_{L^2(0,1)} \to \min! \quad \text{subject to} \quad \|w\|_{L^2(0,1)} \le R. \tag{7.22}$$

Therefore we are going to use the theory of Lagrange multipliers (see [62]) and get the functional

$$L(w,\lambda) = \int_0^1 [x_{mn}(t) - m(t)w(t)]^2 dt + \lambda[\int_0^1 w^2(t)dt - R^2],$$

which has to be minimized. If the constraint is not active, i.e.

$$\left\| \frac{x_{mn}(t)}{m(t)} \right\|_{L^2(0,1)} > R$$

then the *exact* source condition may be satisfied, i.e. $x_{mn}(t) = m(t)w(t)$, if and only if

$$\frac{x_{mn}(t)}{m(t)} \in L^2(0,1).$$

If this condition is not satisfied, i.e. $\frac{x_{mn}(t)}{m(t)} \notin L^2(0,1)$, we proceed by applying Lagrange's technique. Using the Euler equations for constraint optimization

$$\nabla_w L(w,\lambda) = f_w = 0$$

has to be satisfied in order to achieve a minimum of L, where the function f is given by

$$f(w) = [x_{mn}(t) - m(t)w(t)]^2 + \lambda[w^2(t)dt - R^2].$$

Then we get

$$f_w = \lambda w(t) - [x_{mn}(t) - m(t)w(t)]m(t) = 0$$

and therefore

$$w_\lambda(t) = \frac{x_{mn}(t)m(t)}{\lambda + m^2(t)}, \quad \lambda > 0.$$

Using this value we may estimate the shift value $\eta(R)$:

$$\begin{aligned} \eta(R)^2 &= \int_0^1 [x_{mn} - m(t)w_\lambda(t)]^2 dt && (7.23)\\ &= \int_0^1 [x_{mn} - \frac{m^2(t)x_{mn}(t)}{\lambda + m^2(t)}]^2 dt && (7.24)\\ &= \int_0^1 \frac{\lambda^2}{(\lambda + m^2(t))^2} x_{mn}^2(t) dt \quad \lambda > 0. && (7.25) \end{aligned}$$

Knowing that $\int_0^1 w_\lambda^2(t)dt = R^2$ we get

$$R^2 = \int_0^1 \frac{m^2(t)}{(\lambda + m^2(t))^2} x_{mn}^2(t)dt. \tag{7.26}$$

Hence, via λ we may find an interdependence between $\eta(R)$ and R. Notice that it is not trivial to find this relationship, since the determination of λ is not easy. Then we can choose $R = R(\alpha)$ such that

$$\eta(R(\alpha)) = \sqrt{\alpha}R(\alpha)$$

is satisfied, i.e. both terms in (7.13) have the same order of magnitude and we obtain the following property for the convergence of the regularized solution to the minimum-norm solution:

$$\|x_\alpha - x_{mn}\|_{L^2(0,1)} \leq \sqrt{2\alpha R^2(\alpha)}.$$

Furthermore, from (7.10) we get

$$\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)} \leq \frac{\delta}{2\sqrt{\alpha}} + \|x_\alpha - x_{mn}\|_{L^2(0,1)}$$

for the regularization error of Tikhonov regularization and hence we obtain convergence rates for the Tikhonov regularization to the minimum-norm solution.
In a further step we should consider the integral multiplication operator $B = M \circ J$ instead of the simple multiplication operator $(Ax)(t) = m(t)x(t)$. By the same Lagrange approach as above, we get the functional

$$L(w, \lambda) = \|x_{mn} - B^* w\|_{L^2(0,1)}^2 + \lambda \|w\|_{L^2(0,1)}^2,$$

which has to be minimized. Using the Euler equations for constraint optimization we obtain a minimum for

$$w_\lambda = (BB^* + \lambda I)^{-1} B x_{mn}$$

and

$$R = \|(BB^* + \lambda I)^{-1} B x_{mn}\|_{L^2(0,1)}$$

and the shift value

$$\eta(R) = \|(B^*(BB^* + \lambda I)^{-1} B - I)x_{mn}\|_{L^2(0,1)}.$$

The further approach is as before and we get convergence rates for the Tikhonov regularization to the minimum-norm solution.
Both for the simple multiplication operators and for the integral multiplication operators we may summarize three different situations occurring for the decay rate of the shift value $\eta(R) \to 0$ for $R \to \infty$:

Logarithmic decay rate of $\eta(R)$

Let $\eta(R)$ satisfy a logarithmic decay rate, i.e.

$$\eta(R) \leq \frac{K}{(\ln R)^p}, \quad \text{with constants} \quad p > 0, K > 0.$$

Then, by the above considerations, we get

$$\|x_\alpha - x_{mn}\|_{L^2(0,1)} \leq \frac{K}{(\ln R)^p} + R\sqrt{\alpha}$$

for the regularization error. By setting $R = \sqrt[m]{\alpha}$, $m > 2$ we get

$$\|x_\alpha - x_{mn}\|_{L^2(0,1)} \leq \frac{m^p K}{(\ln \frac{1}{\alpha})^p} + \sqrt[l]{\alpha}, \quad l = \frac{m-2}{2m} > 0.$$

Since

$$\sqrt[l]{\alpha} \leq \frac{m^p K}{(\ln \frac{1}{\alpha})^p}$$

is valid, which can be proved for example by the rule of L'Hospital, we may follow

$$\|x_\alpha - x_{mn}\|_{L^2(0,1)} \leq C \frac{1}{(\ln \frac{1}{\alpha})^p} \tag{7.27}$$

for some positive constant C. From this regularization error we obtain

$$\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)} \leq \frac{C}{(\ln \frac{1}{\alpha})^p} + \frac{\delta}{2\sqrt{\alpha}},$$

for the convergence rate of the regularized solution to the minimum-norm solution. By setting $\alpha = c_0 \delta^s$, $0 < s < 2$ we get

$$\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)} = O\left(\frac{1}{(\ln \frac{1}{\delta})^p}\right), \quad \text{as} \quad \delta \to 0 \tag{7.28}$$

i.e. the decay rate of the regularization error (7.27) transfers to the decay rate of the regularized solution with data error. We obtain a very slow decay rate of the regularized solution to the minimum-norm solution for $\delta \to 0$.

Hölder type decay rate of $\eta(R)$

Now, let $\eta(R)$ satisfy a Hölder type decay rate, i.e.

$$\eta(R) \leq \frac{K}{R^{\frac{\mu}{1-\mu}}}, \quad \text{with} \quad 0 < \mu < 1, K > 0.$$

Notice that we can express any power R^s, $s > 0$. Then, by the above considerations, we get

$$\|x_\alpha - x_{mn}\|_{L^2(0,1)} \le \frac{K}{R^{\frac{\mu}{1-\mu}}} + R\sqrt{\alpha}.$$

By equalizing both terms $\frac{1}{R^{\frac{\mu}{1-\mu}}} = R\sqrt{\alpha}$ such that they are of the same order we obtain $R = R(\alpha) = \alpha^{\frac{\mu-1}{2}}$ and

$$\|x_\alpha - x_{mn}\|_{L^2(0,1)} \le C\alpha^{\frac{\mu}{2}} \tag{7.29}$$

for the regularization error. In order to find the convergence rate of Tikhonov regularized solution to the minimum-norm solution we have to equalize the two terms in the right-hand side of inequality

$$\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)} \le C\alpha^{\frac{\mu}{2}} + \frac{\delta}{2\sqrt{\alpha}}.$$

Hence, by choosing $\alpha = c_0\delta^{\frac{2}{1+\mu}}$ we get the convergence rate

$$\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)} = O(\delta^{\frac{\mu}{1+\mu}}), \quad \text{as} \quad \delta \to 0 \tag{7.30}$$

for the Tikhonov regularized solution. We observe that the larger μ, the larger is the decay rate of $\eta(R)$ and the faster is the convergence of x_α^δ to the minimum-norm solution x_{mn}. Notice that we get very close to the optimal convergence rate

$$\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)} = O(\sqrt{\delta}), \quad \text{as} \quad \delta \to 0$$

for $\mu \approx 1$, but the optimal convergence rate may only be obtained for exact source conditions, i.e. $\eta(R) = 0$.

We may give an example for Hölder type decay rates. To this end we are first going to state a proposition from Hofmann and Fleischer [48]:

Proposition 7.8. *Let there be a constant $K > 0$ such that the multiplier function $m(s)$ from (7.21) satisfies*

$$m(s) \ge Ks^\nu \quad (0 \le s \le 1).$$

If $\nu > \frac{1}{4}$ then for Tikhonov regularization

$$\|x_\alpha - x_{mn}\|_{L^2(0,1)} = O(\alpha^{\frac{1}{4\nu}}) \quad \textit{as} \quad \alpha \to 0 \tag{7.31}$$

holds for all $x_{mn} \in L^\infty(0,1)$.

For a proof we refer to [48]. Now let $x_{mn}(t) = 1$ and $m(t) = t$, such that the multiplication operator satisfies inequality (7.31) with $\nu = 1$. We can confirm this result by using Baumeister's Lemma 7.7. Equations (7.26) and (7.25) become

$$R^2 = \int_0^1 \frac{t^2}{(t^2+\lambda)^2}dt = -\frac{1}{2(1+\lambda)} + \frac{1}{2\sqrt{\lambda}}\arctan\left(\frac{1}{\sqrt{\lambda}}\right)$$

and

$$\eta^2(\lambda) = \int_0^1 \frac{\lambda^2}{(t^2+\lambda)^2} dt = \frac{\lambda}{2(1+\lambda)} + \frac{\sqrt{\lambda}}{2} \arctan\left(\frac{1}{\sqrt{\lambda}}\right).$$

By various estimates and calculations we then get that

$$\eta(R) \leq C\frac{1}{R}, \quad \text{for} \quad R \geq R_0 \quad \text{sufficiently large}$$

for some constant $C > 0$. Then, by proposition 7.7 we get

$$\|x_\alpha - x_{mn}\|_{L^2(0,1)} \leq \frac{1}{R} + \sqrt{\alpha}R$$

and by equalizing $\frac{1}{R} = \sqrt{\alpha}R$ we get

$$\|x_\alpha - x_{mn}\|_{L^2(0,1)} \leq 2\sqrt[4]{\alpha},$$

which is the optimal order of magnitude according to the proposition 7.8 stated above. It is a special case of Hölder type rates (7.29) with $\mu = \frac{1}{2}$.

Notice that for $x_{mn} = 1$ and multiplier functions $m(t) = t^\nu$ the exact source condition (i.e. $\eta(R) = 0$) is satisfied, if $t^{-\nu} \in L^2(0,1)$, i.e. $\nu < \frac{1}{2}$. Then we get the optimal convergence rate $\|x_\alpha - x_{mn}\|_{L^2(0,1)} = O(\sqrt{\alpha})$, as stated in section 7.2.2. For $\nu \geq \frac{1}{2}$ we only get inexact source conditions with the shift value $\eta(R) > 0$. The limit case $\nu = \frac{1}{2}$ is considered in the following paragraph.

Exponential decay rate of $\eta(R)$

Finally, let $\eta(R)$ satisfy an exponential decay rate, i.e.

$$\eta(R) \leq Ke^{-\frac{R^q}{2}}, \quad \text{with} \quad q > 0, K > 0.$$

Then for the regularization error

$$\|x_\alpha - x_{mn}\|_{L^2(0,1)} \leq Ke^{-\frac{R^q}{2}} + R\sqrt{\alpha}$$

holds and by the same considerations as above, we choose $R = \left(\ln\frac{1}{\alpha}\right)^{\frac{1}{q}}$ and obtain

$$\begin{aligned} \|x_\alpha - x_{mn}\|_{L^2(0,1)} &\leq K\sqrt{\alpha} + \left(\ln\frac{1}{\alpha}\right)^{\frac{1}{q}}\sqrt{\alpha} \\ &= O\left(\left(\ln\frac{1}{\alpha}\right)^{\frac{1}{q}}\sqrt{\alpha}\right), \quad \text{for} \quad \alpha \to 0. \end{aligned}$$

By applying the same procedure as above and by choosing $\alpha = c_0\delta$ we get the convergence rate

$$\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)} = O\left(\left(\ln\frac{1}{\delta}\right)^{\frac{1}{q}}\sqrt{\delta}\right), \quad \text{as} \quad \delta \to 0. \tag{7.32}$$

The decay rate of the regularization error transfers to the decay rate of the Tikhonov regularized solution to the minimum-norm solution. Notice that the convergence rate is slightly slower than the optimal rate $\sqrt{\delta}$. This optimal rate may only be obtained for with exact source conditions.

Finally, we give an example for exponential type decay rates. Let $x_{mn}(t) = 1$ and $m(t) = \sqrt{t}$. Equations (7.26) and (7.25) become

$$R^2 = \int_0^1 \frac{t}{(t+\lambda)^2} dt = \ln\left(1 + \frac{1}{\lambda}\right) - \frac{1}{1+\lambda}$$

and

$$\eta^2(\lambda) = \int_0^1 \frac{\lambda^2}{(t+\lambda)^2} dt = \frac{\lambda}{1+\lambda}.$$

By using several estimates again, we then get

$$\eta^2(R) \leq e^{-R^2} \quad \text{for} \quad R \geq R_0 \quad \text{sufficiently large}$$

and by setting $R = \sqrt{\ln\left(\frac{1}{\alpha}\right)}$ we get

$$\|x_\alpha - x_{mn}\|_{L^2(0,1)} \leq \sqrt{\alpha} + \sqrt{\alpha \ln\left(\frac{1}{\alpha}\right)} = O\left(\sqrt{\alpha \ln\left(\frac{1}{\alpha}\right)}\right), \quad \text{as} \quad \alpha \to 0,$$

which corresponds to the convergence rate in (7.32) for $q = 2$.

Finally, we may summarize that we can choose an arbitrary x_{mn} and we will be able to find a shift value $\eta(R)$ for all operators A by applying the proposition by Baumeister, stated in 7.7. Then we may find convergence rates for the regularization error

$$\|x_\alpha - x_{mn}\|_{L^2(0,1)} = O(\zeta(\alpha)) \quad \text{as} \quad \alpha \to 0,$$

defined in (7.18) and hence convergence rates for Tikhonov regularization to the minimum-norm solution

$$\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)} = O(\chi(\delta)) \quad \text{as} \quad \delta \to 0,$$

where $\chi(\delta)$ is defined by one of the functions in (7.28), (7.30) or (7.32). We have overcome the strict and somehow unnatural satisfaction of exact source conditions and we have observed that the convergence rates may get very close to the optimal convergence rate $\sqrt{\delta}$.

7.2.4 Tikhonov regularization of nonlinear operators, convergence rates and application to $F(x) = y$

Now the aim is to apply the method of Tikhonov regularization introduced in the previous section to nonlinear ill-posed problems, especially to (2.12).

We assume that $D \subseteq X$ is convex, F is a nonlinear continuously and weakly closed operator possessing a Fréchet derivative for all $x \in D$. Furthermore we have $X = Y = L^2(0,1)$ and we assume that there exists at least one solution $x_0 \in D$ with $F(x_0) = y$ for the exact right-hand side $y \in Y$. Hence we are looking for a so-called x^*-minimum-norm solution :

Definition 7.5. *Let $x^* \in X$ be a given reference element and $y \in Y$. An element $x_{mn} \in X$ is called a x^*-minimum-norm solution if*

$$F(x_{mn}) = y \tag{7.33}$$

and

$$\|x_{mn} - x^*\|_X = \min\{\|x - x^*\|_X : F(x) = y \quad x \in D\}.$$

In [18] this x^*-minimum-norm solution was generalized to a x^*-minimum-norm least square solution, i.e. (7.33) was replaced by

$$\|F(x_{mn}) - y\|_Y = \min\{\|F(x) - y\|_Y, x \in D(F)\}.$$

In general, an x^*-minimum-norm solution need not exist and even if it does, it need not be unique. Hence we assume that an x^*-minimum-norm solution exists. Engl et al. [18] investigated the applicability of the Tikhonov regularization to nonlinear inverse problems. Given some approximation $y_\delta \in Y$ of the right-hand side y with

$$\|y_\delta - y\|_Y \leq \delta$$

and some initial guess $x^* \in X$ they suggest to minimize the Tikhonov functional

$$T_\alpha^\delta(x) := \|F(x) - y_\delta\|_Y^2 + \alpha\|x - x^*\|_X^2 \tag{7.34}$$

where $\alpha > 0$ is a positive regularization parameter, in order to solve $F(x) = y$ approximately. Hence we get the regularized solutions $x_\alpha^\delta \in D$ according to Tikhonov by solving the regularized least-squares problem

$$T_\alpha^\delta(x) = \min!, \quad x \in D.$$

The existence of a solution to (7.34) is given by the following lemma (see [45] or [18] for a proof):

Lemma 7.4. *Let the conditions for the operator F formulated in this section hold. Then at least one minimizer $x_\alpha^\delta \in D$ of the Tikhonov functional, i.e. of the regularized least-squares problem (7.34) exists for all $y_\delta \in Y$ and $\alpha > 0$.*

By lemma 7.1 we know that our operator $F = N \circ J$ satisfies all the conditions for the previous lemma and thus a minimizer $x_\alpha^\delta \in D$ of the Tikhonov functional exists.

Under the same assumptions on F the solution $x_{\alpha(\delta)}^\delta$ obtained by the Tikhonov regularization converges to the x^*-minimum-norm solution x_{mn}, when the data quality δ tends to zero. We may extract this property from the following proposition (see [45] or [18, Theorem 2.3] for a proof):

Proposition 7.9. *Let $y \in R(A) \subset Y$ be perfect data and y_δ be disturbed data, where the error level is bounded by δ, i.e.*

$$\|y_\delta - y\|_Y \leq \delta.$$

Furthermore let $\alpha = \alpha(\delta)$ be a parameter choice rule such that

$$\alpha(\delta) \to 0 \quad \text{and} \quad \frac{\delta^2}{\alpha(\delta)} \to 0 \quad \text{as} \quad \delta \to 0.$$

Then under the assumptions on F stated above for every sequence $\{\delta_i\}_{i\in\mathbb{N}}$ with

$$\lim_{i\to\infty} \delta_i = 0 \quad \text{and} \quad \|y_{\delta_i} - y\|_Y \leq \delta_i$$

and $\alpha_i := \alpha(\delta_i)$ the sequence $x_{\alpha_i}^{\delta_i}$ of minimizers of (7.34) has a convergent subsequence. In addition the limit of every convergent subsequence is an x^*-minimum-norm solution. Furthermore, if the x^*-minimum-norm solution x_{mn} is unique, then

$$\lim_{\delta\to 0} x_{\alpha(\delta)}^{\delta} = x_{mn}.$$

Again, lemma 7.1 ensures that our operator $F = N \circ J$ satisfies the assumptions of this proposition and the solution x_α^δ obtained by Tikhonov regularization converges to x_{mn}.

Hence by lemma 7.4 and proposition 7.9 we find that for all regularization parameters $\alpha > 0$ and a fixed initial guess $x^* \in L^2(0,1)$ Tikhonov regularized solutions x_α^δ as minimizers of (7.34) exist for the inverse problem $F = N \circ J$ from (2.12) and stably depend on the data y_δ. Moreover, for

$$\alpha_i = \alpha(\delta_i) \to 0 \quad \text{and} \quad \frac{\delta_i^2}{\alpha(\delta_i)} \to 0 \quad \text{as} \quad \delta_i \to 0 \quad \text{for} \quad i \to \infty$$

any sequence $\{x_{\alpha_i}^{\delta_i}\}$ converges to x_{mn} in $L^2(0,1)$.

Since convergence stated in the previous proposition may be arbitrarily slow, we are interested in convergence rates which represent the asymptotic behaviour of the Tikhonov regularization for the error level δ. Therefore we state the following proposition which was first stated by Lukaschewitsch (see [63, Theorem 2.2.3] or [64, Theorem 4.1]).

Proposition 7.10. *Let X, Y denote Hilbert spaces and $F : D(F) \subset X \to Y$ be a continuous nonlinear operator between Hilbert spaces. Let $\delta > 0$ and $y_\delta \in Y$ be such that the data satisfy $\|y - y_\delta\|_Y \leq \delta$. Furthermore, let the following assumptions on F hold:*

1. *$D(F)$ is convex,*

2. *F is weakly sequentially closed,*

3. *an x^*-minimum-norm solution x_{mn} exists,*
4. *F is Fréchet differentiable in a ball $B_\rho(x_{mn})$,*
5. *the derivative $F'(x_{mn}) \in \mathcal{L}(X,Y)$ is Lipschitz continuous, i.e.*
$$\|F'(x) - F'(x_{mn})\|_{\mathcal{L}(X,Y)} \le L\|x - x_{mn}\|_X,$$
for all $x \in B_\rho(x_{mn})$, where L denotes the Lipschitz constant and
6. *let $\theta \in \mathbb{R}$ be positive, such that there exists an element $w_\theta \in Y$ satisfying*
$$\|x_{mn} - x^* - F'(x_{mn})^* w_\theta\|_X < \theta, \tag{7.35}$$
where $L\|w_\theta\|_Y \le c_0 < 1$ for a $c_0 > 0$.

Then, if $\rho > 2\|x_{mn} - x^*\|_X + \sqrt{\frac{\delta_0}{K}}$, where $\delta_0 > 0$ is such that $\delta < \delta_0$ for all error levels $\delta > 0$ of consideration, for the parameter choice $\alpha := K\delta$, where K denotes some positive constant, the asymptotic estimates

$$\|F(x_\alpha^\delta) - y_\delta\|_Y \le \left(1 + \frac{2Kc_0}{L}\right)\delta + \sqrt{2\rho K}\delta^{\frac{1}{2}}\theta^{\frac{1}{2}} \tag{7.36}$$

and

$$\|x_\alpha^\delta - x_{mn}\|_X \le \frac{1}{\sqrt{1-c_0}}\left(\left(\frac{1}{\sqrt{K}} + \frac{\sqrt{K}c_0}{L}\right)\delta^{\frac{1}{2}} + \sqrt{2\rho}\theta^{\frac{1}{2}}\right) \tag{7.37}$$

are valid.

A proof of this proposition can be found in [63]. This proposition is an extension of the theorem first stated by Engl, Kunisch and Neubauer (see [18, Theorem 2.4]), where the exact source condition
$$x_{mn} - x^* = F'(x_{mn})^* w,$$
i.e. $\theta = 0$ was used instead of the inexact source condition (7.35). By using the exact source condition we obviously get a convergence rate of

$$\|x_\alpha^\delta - x_{mn}\|_X \le O(\sqrt{\delta}).$$

from proposition 7.10. In lemma 7.1 we stated that assumptions (1)-(5) are satisfied for our nonlinear multiplication operator $F = N \circ J$. In [38] assumption (6) was examined. The source condition may be written as

$$(x_{mn} - x^*)(t) = \int_t^1 m(s)w(s)ds \quad (0 \le t \le 1, w \in L^2(0,1))$$

using $F'(x_0)^* = J^* \circ M^* = J^* \circ M$. Then the assumption (6) with exact source condition is satisfied if and only if

$$(x_{mn} - x^*)(1) = 0 \quad \text{and} \quad \frac{(x_{mn} - x^*)'}{m} \in L^2(0,1), \tag{7.38}$$

i.e. $x_{mn} - x^* \in H^1(0,1)$. Furthermore

$$\left\| \frac{(x_{mn} - x^*)'}{m} \right\|_{L^2(0,1)} < \frac{1}{L} \tag{7.39}$$

has to be satisfied in order to ensure a convergence rate of $\sqrt{\delta}$ for the exact source condition. These are quite strong conditions, since the initial guess has to be *exact* on the right-hand boundary (7.38) and (7.39) implies that the initial guess x^* has to be close to the solution x_{mn}. Furthermore condition (7.39) requires a certain smoothness of $x_{mn} - x^*$ and, since $m(s) \to 0$ as $s \to 0$ the condition (7.39) on the difference $x_{mn} - x^*$ is very rigorous with respect to small s.

We can partly overcome this restriction by considering inexact source conditions as examined in proposition 7.10. Firstly, we assume that the operator J, i.e. the integral operator *without* multiplier function satisfies the *exact source condition*, i.e.

$$x_{mn} - x^* = F'(x_{mn})^* w_\theta$$

holds for some $w_\theta \in L^2(0,1)$. Without multiplier function we get $F'(x_0)^* = J^*$ and therefore the exact source condition becomes

$$(x_{mn} - x^*)(t) = \int_t^1 w_\theta(s)ds \quad (0 \le t \le 1, w_\theta \in L^2(0,1)).$$

Again, this implies

$$(x_{mn} - x^*)(1) = 0 \quad \text{and} \quad (x_{mn} - x^*)' \in L^2(0,1), \tag{7.40}$$

i.e. with $x_{mn} - x^* \in H^1(0,1)$, a certain smoothness requirement on the difference $x_{mn} - x^*$ and

$$\|w_\theta\|_{L^2(0,1)} = \|(x_{mn} - x^*)'\|_{L^2(0,1)} < \frac{1}{L}, \tag{7.41}$$

where L is the Lipschitz constant for operator $J \in \mathcal{L}(L^2(0,1), L^2(0,1))$. Since we are only interested in the influence of the multiplication operator, this assumption of exact source conditions for the operator *without* multiplier function is justified. Assuming that an *inexact source condition* is satisfied for the operator with multiplier function $F'(x_{mn})^* = J^* \circ M^* = J^* \circ M$ we are going to estimate θ for some $\tilde{w}_\theta \in L^2(0,1)$:

$$\begin{aligned}
\|x_{mn} - x^* - F'(x_{mn})^* \tilde{w}_\theta\|^2_{L^2(0,1)} &= \|[J^*(w_\theta)] - [MJ^*(\tilde{w}_\theta)]\|^2_{L^2(0,1)} \\
&\le \|\int_t^1 w_\theta(s)ds - \int_t^1 m(s)w_\theta(s)ds\|^2_{L^2(0,1)} \\
&= \|\int_t^1 (1 - m(s))w_\theta(s)ds\|^2_{L^2(0,1)} \\
&\le \int_0^1 (1 - m(s))^2 ds \int_0^1 w_\theta^2(s)ds
\end{aligned}$$

Using (7.41) we get

$$\|x_{mn} - x^* - F'(x_{mn})^* \tilde{w}_\theta\|^2_{L^2(0,1)} \leq \frac{1}{L^2} \int_0^1 (1 - m(s))^2 ds,$$

and in order to achieve the inexact source condition (7.35)

$$\frac{1}{L^2} \int_0^1 (1 - m(s))^2 ds \leq \theta^2 \tag{7.42}$$

has to be satisfied. Since the integral $\int_0^1 (1 - m(s))^2 ds$ is bounded, the right-hand side of (7.42) is bounded, too. For our special multiplier functions $m(t) = t^\alpha$ or $m(t) = \exp\{-\frac{1}{t^\alpha}\}$, $\alpha > 0$

$$\int_0^1 (1 - m(s))^2 ds < 1$$

holds. For large L the value of θ may be chosen to be small. Thus, by considering (7.37) for $\delta \gg 0$ and small θ we get a low influence of multiplier functions. As for the consideration of the linearized case we may investigate the influence of the velocity of $m(s) \to 0$ as $s \to 0$. We can use the same function (7.16)-(7.17) which can be seen in figure 7.1. Then, by using the results of section 7.2.2 we may estimate the left-hand side of (7.42) to get

$$\frac{1}{L^2} \int_0^1 (1 - m(s))^2 ds = \frac{1}{L^2} \int_0^\varepsilon (1 - m(s))^2 ds \leq \frac{1}{L^2} \varepsilon$$

. Hence, we force (7.42) to satisfy

$$\frac{1}{L^2} \varepsilon \leq \theta^2.$$

By setting $\theta = \frac{\sqrt{\varepsilon}}{L}$ (7.37) we get

$$\|x_\alpha^\delta - x_{mn}\|_X \leq \frac{1}{\sqrt{1 - c_0}} \left(\left(\frac{1}{\sqrt{K}} + \frac{\sqrt{K} c_0}{L} \right) \delta^{\frac{1}{2}} + \sqrt{2\rho} \left(\frac{\sqrt{\varepsilon}}{L} \right)^{\frac{1}{2}} \right),$$

and therefore a small influence on the convergence rate for small $\varepsilon > 0$. Hence, only the integral $\int_0^1 (1 - m(s))^2 ds$ influences the regularization properties and not the decay rate of $m(s) \to 0$. This is the same result as observed for the Tikhonov regularization applied to the linearized problem. Therefore the convergence rates

$$\|F(x_\alpha^\delta) - y_\delta\|_{L^2(0,1)} \leq O(\delta)$$

and

$$\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)} \leq O(\sqrt{\delta})$$

are valid. Hence we still get the convergence rate of $\sqrt{\delta}$ for the Tikhonov regularization of operators with multiplier functions. We have overcome the restriction (7.39) which becomes even stronger as soon as $m(s)$ is close to zero. Therefore the influence of multiplication operators on the regularization properties is limited, however we still have to satisfy the rigorous conditions (7.40) and (7.41) which also have to be satisfied for the integral operator without multiplier function.

Chapter 8

Conclusions

The main objective of this diploma thesis was to investigate the influence of multiplication operators on the ill-posedness of inverse problems.

We have shown, how these operators arise via nonlinear problems in science, engineering and finance. Then we described how the degree of ill-posedness of those operators can be measured.

We provided several analytical tools in order to derive the degree of ill-posedness of an operator

$$F'(x_0) = M \circ J,$$

where J is a simple integral operator, which is ill-posed of degree $\nu = 1$. We have seen that it is hard to find analytical results for the singular value decomposition, i.e. the degree of ill-posedness, if the multiplication operator $m(t)$ has got a zero, for example at $t = 0$. Analytically we may always find lower bounds, but we are interested in upper bounds on the degree of ill-posedness.

By calculating the singular value decomposition numerically using several different approaches, such as finite difference methods for the corresponding Sturm-Liouville problem and Galerkin as well as Rayleigh-Ritz methods for the eigenvalue or singular value decomposition, we were able to find the degree of ill-posedness numerically. We have also provided approximation properties and even error estimates, such that the computations are reliable. Furthermore, we have found that the relationship

$$\sigma_n(F'(x_0)) = \int_0^1 m(s)ds\sigma_n(J) = \int_0^1 m(s)ds\frac{2}{\pi(2n-1)}.$$

is valid, i.e. only the integral $\int_0^1 m(s)ds$ influences the singular values, but the singular value asymptotics, which determine the degree of ill-posedness, remains the same.

Furthermore, via the consideration of the corresponding Sturm-Liouville problem

$$-(a(\tau)u'(\tau))' = \lambda u(\tau), \quad u(1) = u'(0) = 0$$

we could see that we may solve such problems with parameters $a(\tau)$ with satisfy $a(\tau) \to \infty$ as $\tau \to 0$.

The thesis is completed by an investigation of the influence of the multiplication operator on the properties of Tikhonov regularization. In the last chapter we have observed that this influence is limited, too. Again, the integral $\int_0^1 m(s)ds$ plays an important role. We have found convergence rates for $\|x_\alpha^\delta - x_{mn}\|_{L^2(0,1)}$, even with inexact source conditions, and we have observed that the convergence rate of the Tikhonov regularization to the minimum-norm solution are not influenced by the decay rate $m(s) \to 0$ but by the integral $\int_0^1 (1-m(s))^2 ds$. We may even get very close to the optimal convergence rate $\sqrt{\delta}$. This is a similar result to the one for the degree of ill-posedness.

Further work could involve the analytical verification of the numerical results such as it was done by Vu Kim Tuan and Gorenflo [78].

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Index

Thesen

zu der an der Fakultät für Mathematik der Technischen Universität Chemnitz eingereichten Diplomarbeit zum Thema

On the Influence of Multiplication Operators
on the Ill-Posedness of Inverse Problems

vorgelegt von Melina Freitag, MSc.

1. Diese Diplomarbeit beschäftigt sich mit dem Einfluss von Multiplikationsoperatoren auf die Inkorrektheit von inversen Problemen.

2. Bei der Modellierung physikalischer, technischer und auch finanzmathematischer Prozesse entstehen oftmals schlechtgestellte nichtlineare Probleme.

3. In dieser Arbeit werden solche nichtlineare Probleme betrachtet, deren Linearisierung über die Fr*é*chet-Ableitung sich als Komposition eines Multiplikationsoperators M und eines Integraloperators J mit $[J(x)](t) = \int_0^t x(\tau)d\tau$ darstellen lässt, also
$$F'(x_0) = M \circ J, \quad (0 \le t \le 1)$$
wobei $F'(x_0)$ zwischen den Hilberträumen $X = Y = L^2(0,1)$ abbildet.

4. Für $m \in L^\infty(0,1)$ überträgt sich die Kompaktheit von J auf den beschränkten linearen Operator $F'(x_0)$. $F'(x_0)$ stellt dann i.a. ein lineares schlecht gestelltes Problem dar. Für die Lösung inverser Aufgaben, insbesondere deren Regularisierung, ist es wichtig zu wissen, wie schlecht ein solches Problem gestellt ist.

5. Für die lokale Inkorrektheit des Problems mit kompakten Operator interessiert die Abklingrate der Singulärwerte von $F'(x_0)$, das Problem ist umso schlechter gestellt, je größer die Abklingrate der Singulärwerte ist.

6. Durch Multiplikationsoperatoren mit $0 < c \le |m(t)| \le C < \infty$ für alle t wird die Abklingrate der Singulärwerte und damit der Grad der Inkorrektheit nicht verändert.

7. Mit Hilfe verschiedener analytischer Betrachtungen wird der Grad der Inkorrektheit nach unten, jedoch nicht nach oben abgeschätzt.

8. Das Singulärwert- bzw. Eigenwertproblem wird diskretisiert und numerische Studien mittels Finite-Differenzen-Verfahren angewandt auf das zugehörige Sturm-Liouville-Eigenproblem sowie Galerkin- und Rayleigh-Ritz-Verfahren angewandt auf das Eigenwertproblem für die Integralgleichungen werden durchgeführt. Mehrere Tests liefern, zusammen mit Konvergenzsätzen, Approximationseigenschaften und Fehlerschätzern, obere Schranken für die Singulärwerte und damit deren Abklingrate.

9. Analytische und numerische Untersuchungen zeigen, dass der Grad der Inkorrektheit durch Multiplikationsoperatoren nicht beeinflusst wird. Sie liefern sogar die folgende Beziehung zwischen den Singulärwerten von J und $F'(x_0)$:

$$\sigma_n(F'(x_0)) = \int_0^1 m(s)ds\sigma_n(J) = \int_0^1 m(s)ds\frac{2}{\pi(2n-1)}.$$

10. Der Einfluss der Multiplikationsoperatoren auf die Tikhonov Regularisierung sowohl des linearen als auch des nichtlinearen Problems ist ebenfalls beschränkt. Die Konvergenzraten der Tikhonov Regularisierung zur Minimum-Norm-Lösung bleiben erhalten bzw. weichen nur leicht von der optimalen Konvergenzrate $\sqrt{\delta}$ ab.

11. Für die Tikhonov Regularisierung ist festzustellen, dass, ebenso wie für den Grad der Inkorrektheit, nicht die Geschwindigkeit der Konvergenz $m(s) \to 0$ wenn $s \to 0$ entscheidend ist, sondern dass Integral $\int_0^1 m(s)ds$.

12. Damit ist widerlegt, dass insbesondere Problem mit $m(s) = e^{-\frac{1}{s^\alpha}}$ stark schlecht gestellt sind. Sie sind ebenso schwach inkorrekt wie das Problem ohne Multiplikationsoperator.

Selbständigkeitserklärung

Ich versichere hiermit, dass ich die vorliegende Arbeit selbständig und nur unter der Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe. Die Arbeit hat in dieser oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegen.

Chemnitz, den 03. September 2004