SPDEs, criticality, and renormalisation

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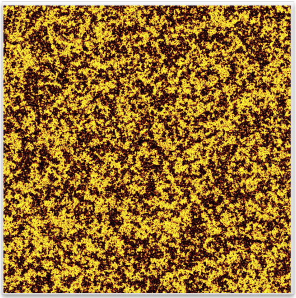
Potsdam, 06.11.2013

An interesting model from Physics I

Ising model



Spin configurations: $\sigma(k) \in \{-1, 1\} \qquad k \in \Lambda,$

Energy: $H(\sigma) = -\frac{1}{2} \sum_{k \sim l} \sigma(k)\, \sigma(l),$

Inverse temperature: $\beta,$

Gibbs measure: $\mu_\beta(\sigma) \propto \exp\big(-\beta H(\sigma)\big).$

An equation that should describe critical Ising I

Kac-model: (e.g. Presutti, Lebowitz, ... 90s): Spins interact with spins in a whole neighbourhood.

Interaction kernel: $\kappa_\gamma(k) = \gamma^n \kappa(\gamma k)$,

Energy: $H_\gamma(\sigma) = -\frac{1}{2}\sum_{k,l} \kappa_\gamma(k-l)\sigma(k)\,\sigma(l)$,

Coarse grained field: $h_\gamma(k) = \sum_l \kappa_\gamma(k-l)\sigma(l)$.

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Evolution equation:

$$h_\gamma(t,k) = h_\gamma(0,k) + \int_0^t \mathscr{L}_\gamma\, h_\gamma(s,k)\, ds + m_\gamma(s,k), \qquad (*)$$

where (for $\beta \approx 1$)

$$\mathscr{L}_\gamma h_\gamma(\sigma) \approx \Big(\kappa_\gamma * h_\gamma(\sigma) - h_\gamma(\sigma)\Big) - \frac{1}{3}\Big(\kappa_\gamma * h_\gamma(\sigma,\cdot)^3\Big) + \dots .$$

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Looks like an approximation to

$$h(t,x) \approx h(0,x) + \int_0^t \Big(\Delta h(s,x) - h(s,x)^3\Big)\, ds + \int_0^t \xi(s,x)\, ds.$$

In one space dimension proved by [Bertini et al. 93].

Some interesting models from Physics II

SOS Surface growth model

Bertini-Giacomin ’97 showed that a scaling limit of SOS-surface model is described by the KPZ -equation

$$\partial_t h = \partial_x^2 h + \lambda(\partial_x h)^2 + \xi.$$

- Introduced by Kardar-Parisi-Zhang (’86) to model fluctuation of a $1+1$ dimensional surface.
- Universal character: KPZ universality class.
- Recently, Spohn/Sasamoto and Amir/Corwin/Quastel ’11 gave an explicit formula for one-point distribution.

Regularity of the stochastic heat equation

Stochastic heat equation:

$$\partial_t X = \Delta X - X + \xi \qquad \text{(say on } [0,\infty] \times \mathbb{T}^n\text{).}$$

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$$\partial_t X_k(t) = -(|k|^2 + 1) X_k(t) + \dot{w}_k(t),$$

Ornstein-Uhlenbeck process. This implies

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$$\mathbb{E}\|X(t)\|_{H^s}^2 = \sum_{k\in\mathbb{Z}^n} |k|^{2s}\,\mathbb{E}|X_k(t)|^2 < \infty \Leftrightarrow s < \frac{2-n}{2}.$$

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In one dimension Brownian regularity. In $n \geq 2$ distribution valued.

Difficulty to define non-linearity I

KPZ:

$$\partial_t h = \partial_x^2 h + \lambda(\partial_x h)^2 + \xi.$$

- $h \in H^\alpha$ (actually $\mathcal{C}^\alpha$) for $\alpha < \frac{1}{2} \Rightarrow$ cannot define $(\partial_x h)^2$.
- Bertini,Giacomin introduce Hopf-Cole solution to KPZ.

  Write $h = \log(\lambda Z)$. Then formally

$$\partial_t Z = \partial_x^2 Z + Z\xi.$$

- Convergence result for SOS-model proved on level of Z.

Difficulty to define non-linearity II

ϕ^4_n:

$$\partial_t \phi = \Delta \phi - \phi^3 + \phi + \xi.$$

- One spatial dimension $\phi \in \mathcal{C}^\alpha$, $\alpha < \frac{1}{2} \Rightarrow$ No problem.
- Two and three spatial dimensions ϕ not a function $\Rightarrow$ classical methods do not work.
- Approximation for (Kac)-Ising only known in one spatial dimension.

Scope of theory

Metatheorem (Hairer 13)

Stable existence and uniqueness theory locally in time on compact domains for equations that are locally subcritical.

Locally subcritical: *On small scales the non-linear term is lower order*.

Example ϕ^4_n: scaling $x \mapsto \varepsilon x$, $t \mapsto \varepsilon^2 t$ and $\phi \mapsto \varepsilon^{\frac{n-2}{2}} \phi$, leaves stochastic heat equation invariant. Under this scaling ϕ^4_n equation becomes

$$\partial_t \hat{\phi} = \Delta \hat{\phi} - \varepsilon^{4-n} \hat{\phi}^3 + \hat{\xi}$$

locally subcritical in for $n \leq 3$.

The need for renormalisation I

Approximation by regularised noise

$$d\phi_\delta = \Big[\Delta\phi_\delta - (\phi_\delta^3 - \phi_\delta)\Big]\, dt + dW_\delta.$$

- From now on $n = 2$, $\Omega = \mathbb{T}^2$ torus.
- $W_\delta(t,x) := \sum_{|k|\leq\frac{1}{\delta}} e^{ik\cdot x} w^k(t)$. Noise white in time, spatial corellations $\sim \delta$.

For $\delta > 0$ equation is well-posed. What happens if $\delta \to 0$?

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Related results: Small noise, pass to this limit on the level of large deviations (Kohn & Otto & Westdickenberg (Reznikoff) & Vanden-Eijnden – Cerrai & Freidlin – Barret & Bovier & Meleard).

Renormalised powers I

Constructive field theory ('70s)

$X_\delta =$ stationary solution of (approximated) stochastic heat equation

$$dX_\delta = \Delta X_\delta - X_\delta + dW_\delta.$$

$$\mathbb{E}[X_\delta(x)X_\delta(y)] \lesssim |\log|x-y|| \wedge \log(\delta).$$

Question:

- Does X_δ^3 converge to a random distribution?
- Does $\langle X_\delta^3, \varphi\rangle$ converge to a random variable for smooth φ?
- Does $\mathbb{E}[\langle X_\delta^3, \varphi\rangle^2]$ remain bounded as $\delta \to 0$?

Renormalised powers II

$$\mathbb{E}[\langle X_\delta^3, \varphi\rangle^2] = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \varphi(x)\,\varphi(y)\,\mathbb{E}\Big[X_\delta^3(x) X_\delta^3(y)\Big]\,dx\,dy$$

Gaussian Moments

$$\begin{aligned}\mathbb{E}\Big[X_\delta^3(x)\,X_\delta^3(y)\Big] \\ &= 6\,\mathbb{E}[X_\delta(x)X_\delta(y)]^3 + 9\,\mathbb{E}[X_\delta(x)X_\delta(y)]\,\mathbb{E}[X_\delta(x)X_\delta(x)]^2 \\ &\lesssim |\log(x-y)|^3 + |\log(\delta)|^2 |\log(x-y)|\end{aligned}$$

- $|\log(x-y)|$ term is integrable. $|\log(\delta)|$ term diverges.
- $\mathbb{E}[\langle X_\delta^3, \varphi\rangle^2]$ diverges as $\delta \to 0$.

Renormalised powers III

$$: X_\delta^3(x) := X_\delta^3(x) - 3C_\delta X_\delta(x) \quad \textit{where } C_\delta = \mathbb{E}[X_\delta(x)^2] \sim |\log(\delta)|.$$

$$\Rightarrow \mathbb{E}\Big[: X_\delta^3(x) :: X_\delta^3(y) : \Big] = 6\, \mathbb{E}[X_\delta(x) X_\delta(y)]^3.$$

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Theorem (Glimm, Jaffe, Nelson, Gross,... 70s)

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Theorem (Glimm, Jaffe, Nelson, Gross,... 70s)

$:X_\delta^3:$ converges to a random distribution $:X^3:$ in every negative Sobolev space.

- $:X^3:$ called third Wick power.

Trivial limit

$$d\phi_\delta \;=\; \Big[\Delta\phi_\delta - (\phi_\delta^3 - \phi_\delta)\Big]\, dt + dW_\delta$$

Theorem (Hairer, Ryser, W' 12)

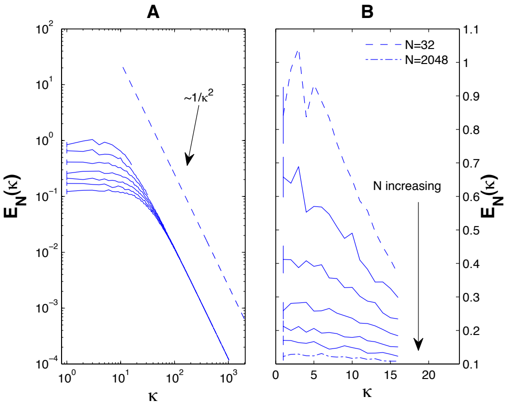
ϕ_δ^0 *bounded in* $C(\mathbb{T}^2)$. *Then for* $t_* > 0$, $s > 0$ *almost surely*

$$\big\|\phi_\delta\big\|_{C\left([t_*,T],H^{-s}\right)} \to 0 \qquad \textit{for } \delta \to 0!$$

- Convergence is slow (logarithmic).

Numerics

Ryser & Nigam & Tupper *preprint '11*



N number of grid points $\sim \frac{1}{\delta^2}$.

$E_N(\kappa) \sim$ strength of κ Fourier mode.

Small noise

Small noise $\sigma(\delta) \leq 1$

$$d\bar{u}_\delta = \Big[\Delta \bar{u}_\delta - (\bar{u}_\delta^3 - \bar{u}_\delta)\Big]\, dt + \sigma(\delta) dW_\delta$$

Theorem

$\phi_\delta^0 \to u^0$ in $C(\mathbb{T}^2)$. Then for $t_* > 0$, $s > 0$ almost surely in H^{-s}

$$\|\bar{u}_\delta\|_{C\left([t_*,T],H^{-s}\right)} \to \begin{cases} 0 & \text{if } |\log(\delta)|^{-\frac{1}{2}} \ll \sigma(\delta) \ll 1 \\ \bar{u}^* & \text{if } |\log(\delta)|^{-\frac{1}{2}} = \sigma(\delta) \\ \bar{u} & \text{if } 0 \ll \sigma(\delta) \ll |\log(\delta)|^{-\frac{1}{2}} \end{cases} .$$

- $\bar{u}$ solution to deterministic Allen-Cahn equation.
- $\bar{u}^*$ solution to $\dot{u} = \Delta \bar{u}_\delta - (\bar{u}_\delta^3 - \bar{u}_\delta) - C_* u$.

The scheme needs to be modified

Approximation by regularised noise

$$d\phi_\delta = \Big[\Delta\phi_\delta - (\phi_\delta^3 - C_\delta\phi_\delta - \phi_\delta)\Big]\,dt + dW_\delta.$$

$C_\delta \sim |\log\delta|$.

Theorem (da Prato/Debussche '03)

ϕ_δ *converge to a limit. This is limit is called solution to*

$$du = \Big[\Delta u - (\,:u^3:\, - u)\Big]\,dt + dW,$$

or dynamic ϕ_2^4 *model.*

Strategy I:

1.) Lift Gaussian process

Lemma

For every $t \in [0, T]$, $p \geq 1$, $s > 0$

- $X_\delta \to X$,
- $X_\delta^2 - C_\delta \to :X^2:$,
- $X_\delta^3 - C_\delta X_\delta \to :X^3:$,

in $L^p(\mathcal{B}^{-s}_{\infty,\infty})$.

- Gaussian moment calculation.
- equivalence of moments in fixed Wiener chaos - Nelson estimate.
- Regularity measured in Besov spaces.

Strategy II

2.) Non-linear evolution as continuous function of lifted Gaussian process

Standard regularisation trick: $v_\delta = \phi_\delta - X_\delta$.

$$\begin{aligned}\frac{dv_\delta}{dt} &= \Delta v_\delta - \left((X_\delta + v_\delta)^3 - 3\,C_\delta(X_\delta + v_\delta)\right) \\ &= \Delta v_\delta - \left(:X_\delta^3: + 3 :X_\delta^2:\, v_\delta + 3 X_\delta\, v_\delta^2 + v_\delta^3\right).\end{aligned}$$

Multiplicative inequality: If $s < 0 < \alpha$ and $s + \alpha > 0$. Then

$$\|u\, v\|_{\mathcal{B}^s_{\infty,\infty}} \lesssim \|u\|_{\mathcal{B}^s_{\infty,\infty}} \|v\|_{\mathcal{B}^\alpha_{\infty,\infty}}.$$

Used to deal with nonlinearity.

Comments:

- Only works on bounded domains.
- Extra argument needed for global in time solutions. Extra argument required (invariant measures à la Bourgain).
- In 3-d the normalisation is more tricky. Careful with word "Wick".
- More term in expansion necessary, one extra term diverges. Approximations of type

$$d\phi_\delta = \Big[\Delta\phi_\delta - \big(\phi_\delta^3 - C_\delta\phi_\delta - \phi_\delta\big)\Big]\,dt + dW_\delta,$$

for $C_\delta = \frac{C_1}{\delta} + C_2|\log(\delta)|$.

Summary/Outlook:

- Non-linear white noise driven SPDEs arise as scaling limits for particle systems in interesting regimes.
- Solutions to these equations have very poor regularity properties and it is not always clear how to treat non-linear terms.
- Infinite constants have to be dealt with.

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From friday: How to implement renormalisation for more complicated equations.