Paracontrolled distributions and the parabolic Anderson model

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The discrete parabolic Anderson model

- Particles in a random potential landscape:
  - independent continuous time random walks on $\mathbb{Z}^d$, in independent potential landscape $(\eta(x))_{x\in\mathbb{Z}^d}$;
  - in x, particle gets killed with rate $(\eta(x))^-$, particle branches with rate $(\eta(x))^+$; (two new particles that evolve independently).
- Model for many phenomena in physics, e.g. growth of magnetic fields in young stars.
- Expected number of particles at (t, x) solves stochastic heat equation:

$$\partial_t u(t,x) = \Delta_{\mathbb{Z}^d} u(t,x) + u(t,x)\eta(x).$$

Discrete PAM: long time behavior

$$\partial_t u(t,x) = \Delta_{\mathbb{Z}^d} u(t,x) + u(t,x)\eta(x).$$

- Mathematical interest in long time behavior of PAM: simple model which exhibits intermittency (largest part of the mass concentrated in few small "islands");
- countless results since early 90s, different universality classes depending on distribution of η.

Conjectured scaling limit

- To study long time behavior, and to obtain universality for different potentials η, would be interested in scaling limit:

$$\partial_t v^n(t,x) = \Delta_{\mathbb{Z}^d} v^n(t,x) + n^{-d/2} v^n(t,x)\eta(x);$$

$$u^n(t,x) = v^n(n^2 t, nx).$$

- diffusive scaling between t and x;
- scale down potential so noise is not overpowering (Gaussian scaling).
- Natural conjecture: limit solves

$$\partial_t u(t,x) = \Delta u(t,x) + u(t,x)\xi(x),$$

for spatial white noise ξ.

Continuous PAM

$$\partial_t u(t,x) = \Delta u(t,x) + u(t,x)\xi(x)$$

Problem: equation is ill posed, need some form of renormalization.
Existing solutions use Wick products (e.g. Hu 2002):

$$\partial_t u(t,x) = \Delta u(t,x) + u(t,x) \diamond \xi(x);$$

- apply formal chaos expansion to the solution;
- formally obtain solution as chaos series, $u = \sum_n I_n(f_n)$ for suitable deterministic f_n;
- see that for $d < 4$, the series indeed converges.

Interpretation? How does the Wick product transform the equation?
Scaling limit?

Continuous PAM continued

$$\partial_t u(t,x) = \Delta u(t,x) + u(t,x)\xi(x)$$

Suggest new notion of solution. Advantages:

- renormalization is very transparent;
- seems very suitable for proving that it is scaling limit;
- allows to handle nonlinear generalization:

$$\partial_t u(t,x) = \Delta u(t,x) + F(u(t,x))\xi(x)$$

  (this problem cannot be treated with usual Wick products);
- solution depends on potential in a pathwise continuous manner.

Disadvantages:

- can (so far) only treat $d = 2$;
- works on the torus, but not on $\mathbb{R}^2$;
- solutions are only local in time.

Generalized PAM and products of distributions

$$\partial_t u(t,x) = \Delta u(t,x) + F(u(t,x))\xi(x)$$

Problem: product $F(u)\xi$ not easy to define.

- In general, product fg of two distributions $f \in C^\alpha = B^\alpha_{\infty,\infty}$, $g \in C^\beta$ is only defined if $\alpha + \beta > 0$;
- here: spatial white noise in $C^{-d/2-}$; Laplacian gains two degrees of regularity; expect thus $u \in C^{2-d/2-}$;
- for $d > 1$: sum of regularities is negative.

Products of distributions and Littlewood-Paley theory

- Littlewood-Paley blocks give decomposition of distribution as series of smooth functions:

$$u = \sum_j \Delta_j u,$$

  where Δ_j is projection on Fourier modes of order 2^j.
- $\Delta_j u$ has Fourier transform of compact support and is thus smooth.
- Bony's paraproduct: $\alpha > 0$, $\beta < 0$, $u \in C^\alpha$, $v \in C^\beta$, then uv can be decomposed as $uv = u \prec v + u \circ v + u \succ v$, where

$$u \prec v = \sum_{i \ll j} \Delta_i u \Delta_j v, \quad u \circ v = \sum_{i \sim j} \Delta_i u \Delta_j v, \quad u \succ v = \sum_{i \gg j} \Delta_i u \Delta_j v.$$

- $\prec$ continuous from $C^\alpha \times C^\beta$ to C^β; $\succ$ continuous from $C^\alpha \times C^\beta$ to $C^{\alpha+\beta}$; $\circ$ continuous from $C^\alpha \times C^\beta$ to $C^{\alpha+\beta}$ – but only if $\alpha + \beta > 0$.

Paracontrolled distributions

- $u = u \prec v + u \circ v + u \succ v$, where $u \prec v$ and $u \succ v$ are always defined, but $u \circ v$ is only defined if sum of regularities is positive.
- Example: $u, v \in C^\alpha(\mathbb{R})$ with $\alpha > 1/2$, then

$$u\partial_t v = \underbrace{u \prec \partial_t v}_{C^{\alpha-1}} + \underbrace{u \circ \partial_t v}_{C^{2\alpha-1}} + \underbrace{u \succ \partial_t v}_{C^{2\alpha-1}},$$

  so $u\partial_t v - (u \prec \partial_t v) \in C^{2\alpha-1}$.
- Example: $u \in C^\alpha$, $\alpha > 0$, F a smooth function. Paralinearization (Bony 1981): $F(u) - (F'(u) \prec u) \in C^{2\alpha}$.
- Thus, we call u paracontrolled by f if there exists "derivative" u^f s.t. $u^\sharp = u - (u^f \prec f)$ is "smoother".

Product of paracontrolled distributions

Let $\alpha > 1/3$, $f, g \in C^\alpha$, and u be controlled by f.
Aim: Define $u\partial_t g$. Only problem is $u \circ \partial_t g$. But

$$u \circ \partial_t g = u^\sharp \circ \partial_t g + (u^f \prec f) \circ \partial_t g.$$

$u^\sharp \in C^{2\alpha}$ and $\partial_t g \in C^{\alpha-1}$.
$2\alpha + \alpha - 1 > 0$, i.e. $u^\sharp \circ \partial_t g$ well defined.
Only remaining problem: $(u^f \prec f) \circ \partial_t g$.

Besov area and commutator estimate

Definition of $u\partial_t g$: only difficulty is $(u^f \prec f) \circ \partial_t g$.
Assumption: $f \circ \partial_t g \in C^{2\alpha-1}$ is given exogenously - here probabilistic estimates will be needed!

Lemma (Gubinelli, Imkeller, P. (2012))

Let $\alpha > 1/3$ and $u^f, f, g \in C^\alpha$. Then the "commutator"

$$C(u^f, f, \partial_t g) = (u^f \prec f) \circ \partial_t g - u^f (f \circ \partial_t g)$$

is well-defined and in $C^{3\alpha-1}$

Set $(u^f \prec f) \circ \partial_t g := C(u^f, f, \partial_t g) + u^f (f \circ \partial_t g)$.
Note: $u^f (f \circ \partial_t g)$ well-defined, because $u^f \in C^\alpha$, $(f \circ \partial_t g) \in C^{2\alpha-1}$, $3\alpha - 1 > 0$.
Remark: Not important that index set is one-dimensional or that $\partial_t g$ is a derivative. Can define ug for $u, g \in \mathcal{S}'(\mathbb{R}^d)$, as long as $f \circ g$ is given.

Generalized PAM continued

$Lu(t,x) = F(u(t,x))\xi(x)$, where $L = \partial_t - \Delta$.

- Assume smooth ξ for now.
- Paracontrolled ansatz:

$$u = F(u) \prec \theta + u^\sharp,$$

where θ solves $L\theta = \xi$ and where $u^\sharp \in C^{2\alpha}$, $u, \theta \in C^\alpha$, $\alpha \in (2/3, 1)$.

- Use paralinearization (i.e. $F(u) - (F'(u) \prec u) \in C^{2\alpha}$) and our commutator lemma to derive equation for $u^\sharp$:

$$Lu^\sharp = G(u,\xi) + F'(u)F(u)(\theta \circ \xi) + F'(u)(u^\sharp \circ \xi)$$

for suitable G.

- u can be bounded in terms of $u^\sharp$, so equation for $u^\sharp$ leads to a priori bound on $\|u^\sharp\|_{2\alpha}$ in terms of $\|\xi\|_{\alpha-2}$ and $\|\theta \circ \xi\|_{2\alpha-2}$.

Generalized PAM continued

- For smooth ξ: have a priori bound on solution u of $Lu(t,x) = F(u(t,x))\xi(x)$ in terms of $\|\xi\|_{\alpha-2}$, $\|\theta \circ \xi\|_{2\alpha-2}$.
- Irregular ξ: approximate it so that $(\xi^\varepsilon, \theta^\varepsilon \circ \xi^\varepsilon) \to (\xi, \theta \circ \xi)$, obtain existence of solutions.
- Similar arguments show that u depends continuously on $(\xi, \theta \circ \xi)$, thus uniqueness of solutions.

The need for renormalization

Have obtained unique pathwise solution to $Lu(t,x) = F(u(t,x))\xi(x)$ if $(\xi, \theta \circ \xi)$ is given as limit of $(\xi^\varepsilon, \theta^\varepsilon \circ \xi^\varepsilon)$.
Problem: If ξ is spatial white noise on $\mathbb{T}^2$, ψ mollifier, $\xi^\varepsilon = \varepsilon^{-2}\psi(\varepsilon^{-1}\cdot) * \xi$, $\theta^\varepsilon = \varepsilon^{-2}\psi(\varepsilon^{-1}\cdot) * \theta$, then

$$\theta^\varepsilon \circ \xi^\varepsilon$$

diverges as $\varepsilon \to 0$! Easy to see:

$$\theta^\varepsilon \circ \xi^\varepsilon = \theta^\varepsilon \circ (L\theta^\varepsilon) = \frac{1}{2} L(\theta^\varepsilon \circ \theta^\varepsilon) - (\mathrm{D}_x\theta^\varepsilon \circ \mathrm{D}_x\theta^\varepsilon).$$

- $L(\theta^\varepsilon \circ \theta^\varepsilon)$ converges nicely as $\varepsilon \to 0$;
- convergence of $(\mathrm{D}_x\theta^\varepsilon \circ \mathrm{D}_x\theta^\varepsilon)$ equivalent to convergence of positive term $|\mathrm{D}_x\theta^\varepsilon|^2$; thus no stochastic cancellations!

Renormalized product

- $(\theta^\varepsilon \circ \xi^\varepsilon)$ diverges as $\varepsilon \to 0$.
- But: for suitable diverging constants $C_\varepsilon = C_\varepsilon(\psi) = O(\log(\varepsilon))$

$$(\theta^\varepsilon \circ \xi^\varepsilon - C_\varepsilon)$$

  converges (roughly: $C_\varepsilon = \mathbb{E}[\theta^\varepsilon \circ \xi^\varepsilon]$).
- How to make this renormalization appear in the equation? Recall paracontrolled ansatz $u = F(u) \prec \theta + u^\sharp$ and equation for $u^\sharp$:

$$Lu^\sharp = G(u,\xi) + F'(u)F(u)(\theta \circ \xi) + F'(u)(u^\sharp \circ \xi).$$

- Analogous: if u solves

$$Lu = F(u)\xi - F'(u)F(u)C, \qquad \text{then}$$

$$Lu^\sharp = G(u,\xi) + F'(u)F(u)(\theta \circ \xi - C) + F'(u)(u^\sharp \circ \xi).$$

Renormalization and generalized PAM

Obtain unique solution to

$$Lu = F(u) \diamond \xi.$$

Pathwise limit of solutions u^ε to

$$Lu^\varepsilon = F(u^\varepsilon) \diamond \xi^\varepsilon = F(u^\varepsilon)\xi^\varepsilon - F'(u^\varepsilon)F(u\varepsilon)C_\varepsilon,$$

as long as $(\xi^\varepsilon, \theta^\varepsilon \circ \xi^\varepsilon - C_\varepsilon)$ converges to $(\xi, \theta \diamond \xi)$.

Interpreting the renormalization

- Renormalization was required for mathematical reasons, obtained by arguing in the continuous setting.
- Is the renormalized equation the correct candidate for the scaling limit? Is there a physical reasoning for introducing the renormalization, obtained by arguing in the discrete setting?
- In the linear setting:

$$Lu = u \diamond \xi = u\xi - u\infty.$$

Want scaling limit of $u^n(t,x) = v^n(n^2 t, nx)$, where

$$\partial_t v^n(t,x) = \Delta_{\mathbb{Z}^2} v^n(t,x) + \frac{1}{n} v^n(t,x)\eta(x).$$

- Expect that total mass $U^n(t) = n^{-2} \sum_x u^n(t,x)$ explodes for $n \to \infty$.

Interpreting the renormalization II

Trying to show that total mass $U^n(t) = n^{-2} \sum_x u^n(t, x)$ explodes:

- To simplify our live: assume $\mathbb{P}(\eta(x) = 1) = \mathbb{P}(\eta(x) = -1) = 1/2$.
- Feynman-Kac representation:

$$u^n(t, x) = \mathbb{E}^{nx} \left[u_0(X_{n^2 t}) \exp \left(\frac{1}{n} \int_0^{n^2 t} \eta(X_s) \mathrm{d}s \right) \right].$$

  Should allow us to show by simple calculation that U^n explodes.

- So far no success. But: we have seen that $\theta\xi$ plays important role. Can show blowup for $\theta^n \xi^n$, where $\xi^n(x) = n^{-1} \eta(nx)$.

Scaling limits and paracontrolled distributions:

- Proving that object is scaling limit is usually difficult.
- Two cases with well developed techniques: Markov processes (e.g. martingale problem), and Gaussian processes (e.g. direct calculation, chaos series, characteristic functions, ...).
- Linear PAM not Gaussian, but might still be accessible by "Gaussian methods". Generalized PAM certainly not. Also not Markovian.

Our notion of solution seems suitable for proving that it is scaling limit. We decompose the problem into two parts:

- an analytic part, which studies the nonlinear PDE in a pathwise manner and shows that it suffices to control $(\xi^n, \theta^n \circ \xi^n)$;
- a probabilistic part, which derives "Gaussian" scaling limits for $(\xi^n, \theta^n \circ \xi^n)$.
- We can combine both steps using Skorokhod representation.

Thank You