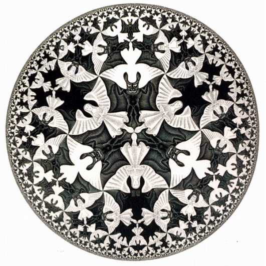
Analytic discs in Complex Analysis

Florian Bertrand

1866 AUB
American University of Beirut
الجامعة الأميركية في بيروت

Plan of the talk

- Set up
- Discs and the Poincaré metric
- Discs and an invariant Finsler metric in higher dimension.
- From metric properties to complex geometric properties



Angels and Devils, M.C.Escher

Set up

Let $\Omega \subset \mathbb{C}$ be a domain.

Definition

A function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is holomorphic if it is complex differentiable at each point of Ω, i.e.

$$\lim_{h \to 0, h \neq 0} \frac{f(\zeta + h) - f(\zeta)}{h}$$

exists at each point $\zeta \in \Omega$.

Examples:

- $f(\zeta) = e^{i\theta}\zeta$ where $\theta \in \mathbb{R}$.
- $f(\zeta) = \overline{\zeta}$

Set up

Let $\Omega \subset \mathbb{C}$ be a domain.

Definition

A function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is holomorphic if it is complex differentiable at each point of Ω, i.e.

$$\lim_{h \to 0, h \neq 0} \frac{f(\zeta + h) - f(\zeta)}{h}$$

exists at each point $\zeta \in \Omega$.

Examples:

- $f(\zeta) = e^{i\theta}\zeta$ where $\theta \in \mathbb{R}$.
- $f(\zeta) = \overline{\zeta}$

Definition

A map $f = (f_1, f_2, \ldots, f_n) : \Omega \subset \mathbb{C} \to \mathbb{C}^n$ is holomorphic if f_j, $j = 1, \ldots, n$, is a holomorphic function.

Set up

Denote by $\Delta = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ the unit disc in $\mathbb{C}$.

We are interested in holomorphic maps $f : \Delta \to \mathbb{C}^n$; such a map is called a *holomorphic disc*.

Set up

Denote by $\Delta = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ the unit disc in $\mathbb{C}$.

We are interested in holomorphic maps $f : \Delta \to \mathbb{C}^n$; such a map is called a *holomorphic disc*.

Let $M \subset \mathbb{C}^n$ be a real hypersurface (e.g. boundary of a domain).

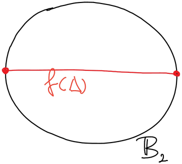
Definition

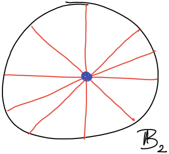
An analytic disc f attached to M is a continuous map $f : \overline{\Delta} \to \mathbb{C}^n$, holomorphic on Δ and such that $f(\partial\Delta) \subset M$.

Question: Understand the family, or subfamilies, of analytic discs attached to M; and accordingly deduce analytic or geometric properties of M

Two examples

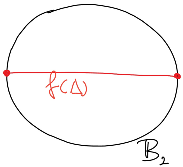
- $\mathbb{B}_2 = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}$, $f_v(\zeta) = \zeta v$ where $v \in \mathbb{C}^n$ is a unit vector.

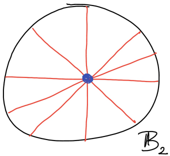




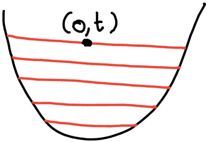
Two examples

- $\mathbb{B}_2 = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}$, $f_v(\zeta) = \zeta v$ where $v \in \mathbb{C}^n$ is a unit vector.





- $\Omega = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid \Re e z_2 - |z_1|^2 > 0\}$, $f_t(\zeta) = (\sqrt{t}\zeta, t)$ where $t \geq 0$.



Set up

Let $M \subset \mathbb{C}^n$ be a real hypersurface (e.g. boundary of a domain).

Nonlinear boundary Riemann-Hilbert problem

A continuous map $f : \overline{\Delta} \to \mathbb{C}^n$ is an analytic disc attached to M iff

$$\begin{cases} f \text{ is holomorphic on } \Delta \\ f(\partial\Delta) \subset M \end{cases}$$

History: Riemann 1851, Plemelj 1908, Hilbert 1912, Bishop 1965, Lempert 1981, Forstnerič 1987, Globevnik 1993...

The Schwarz Lemma

Theorem (Schwarz Lemma)

Let $f : \Delta \to \Delta$ be a holomorphic function s.t. $f(0) = 0$. Then

$$|f'(0)| \leq 1,$$

with equality iff f is a rotation.

The Schwarz Lemma

Theorem (Schwarz Lemma)

Let $f : \Delta \to \Delta$ be a holomorphic function s.t. $f(0) = 0$. Then

$$|f'(0)| \leq 1,$$

with equality iff f is a rotation.

Application: $Aut(\Delta) = \{R_\theta \circ B_a \mid \theta \in [0, 2\pi), a \in \Delta\}$, where

$$R_\theta(\zeta) = e^{i\theta}\zeta \quad \text{and} \quad B_a(\zeta) = \frac{\zeta - a}{1 - \bar{a}\zeta}.$$

The Schwarz-Pick Lemma and the Poincaré metric on Δ

Theorem (Schwarz-Pick Lemma)

Let $f : \Delta \to \Delta$ be a holomorphic function. Then

$$\frac{|f'(\zeta)|}{1-|f(\zeta)|^2} \leq \frac{1}{1-|\zeta|^2}$$

with equality iff $f \in Aut(\Delta)$.

The Schwarz-Pick Lemma and the Poincaré metric on Δ

Theorem (Schwarz-Pick Lemma)

Let $f : \Delta \to \Delta$ be a holomorphic function. Then

$$\frac{|f'(\zeta)|}{1 - |f(\zeta)|^2} \leq \frac{1}{1 - |\zeta|^2}$$

with equality iff $f \in Aut(\Delta)$.

Definition (Poincaré metric)

For $\zeta \in \Delta$ and $v \in \mathbb{C}$

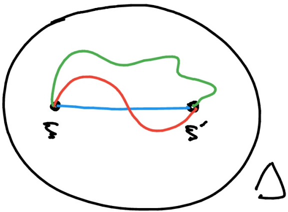
$$K_\Delta(\zeta, v) = \frac{|v|}{1 - |\zeta|^2} = \frac{|v|}{d(\zeta, \partial\Delta)}$$

The Poincaré distance on Δ

Define the *Poincaré distance* $d_\Delta(\zeta, \zeta')$:

$$d_\Delta(\zeta, \zeta') = \inf \int_0^1 K_\Delta(\gamma(t), \gamma'(t))dt,$$

where $\gamma : [0,1] \to \Delta$ are such that $\gamma(0) = \zeta$ and $\gamma(1) = \zeta'$.



The Poincaré distance on Δ

Interpretation of Schwarz-Pick Lemma:

- Holomorphic functions $f : \Delta \to \Delta$ are decreasing the distance:

$$d_\Delta(f(\zeta), f(\zeta')) \leq d_\Delta(\zeta, \zeta').$$

- Automorphisms $f \in Aut(\Delta)$ are isometries.

The Poincaré distance on Δ

Interpretation of Schwarz-Pick Lemma:

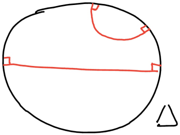
- Holomorphic functions $f : \Delta \to \Delta$ are decreasing the distance:

$$d_\Delta(f(\zeta), f(\zeta')) \leq d_\Delta(\zeta, \zeta').$$

- Automorphisms $f \in Aut(\Delta)$ are isometries.

Some facts about the Poincaré disc:

- (Δ, d_Δ) is a complete metric space.
- Geodesic paths between two points are intersecting $\partial\Delta$ orthogonally.



- Gauss curvature of the Poincaré disc is constant and negative.
- Isometries of (Δ, d_Δ): $Aut(\Delta)$ or $\overline{Aut(\Delta)}$.

Why is the unit disc special ?

Theorem (Riemann mapping Theorem)

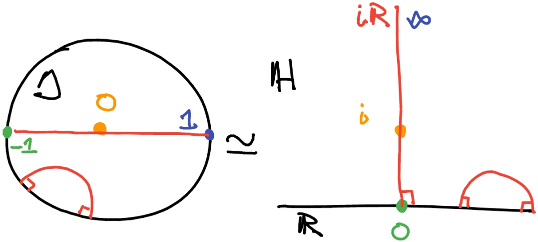
1. *Let* $\Omega \subsetneq \mathbb{C}$ *be a simply connected domain. Then* Ω *is biholomorphic to* Δ.
2. *Compact Riemann surfaces with genus* ≥ 2 *admit* Δ *as universal cover.*

Why is the unit disc special ?

Theorem (Riemann mapping Theorem)

1. *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain. Then Ω is biholomorphic to Δ.*
2. *Compact Riemann surfaces with genus ≥ 2 admit Δ as universal cover.*

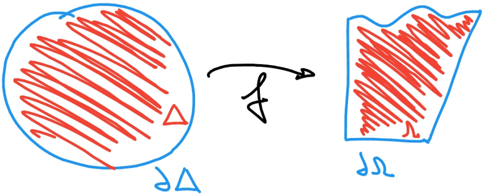
Example: The upper half plane $\mathbb{H} = \{\zeta \in \mathbb{C} \mid \Im m\zeta > 0\}$ is biholomorphic to Δ. Isometry from Δ to $\mathbb{H}$ is $\zeta \mapsto i\dfrac{1+\zeta}{1-\zeta}$.



Analytic discs and the Riemann mapping Theorem

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain.

Assume the boundary $\partial\Omega$ of Ω is continuous. "The" Riemann mapping between Δ and Ω extends continuously up to $\partial\Delta$; it is an analytic disc attached to $\partial\Omega$.



An observation due to Poincaré

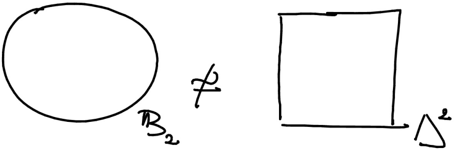
Theorem (Poincaré 1907)

For $n \geq 2$, the unit ball

$$\mathbb{B}_n = \{z \in \mathbb{C}^n \mid |z|^2 = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 < 1\}$$

is not biholomorphic to the unit polydisc

$$\Delta^n = \{z \in \mathbb{C}^n \mid |z_j| < 1 \textit{ for } j = 1, \cdots, n\}.$$



Obstruction: Geometry of the boundaries (presence of complex objects).

Poincaré equivalence problem 1907

Questions:

- Determine when and how domains of $\mathbb{C}^n$ can be mapped into one another by means of a holomorphic mapping.

  Carathéodory 1926, Bergman 1950, Kobayashi 1967: theory of invariant metrics.

Poincaré equivalence problem 1907

Questions:

- Determine when and how domains of $\mathbb{C}^n$ can be mapped into one another by means of a holomorphic mapping.

  Carathéodory 1926, Bergman 1950, Kobayashi 1967: theory of invariant metrics.

- Determine when and how real submanifolds of $\mathbb{C}^n$ can be mapped into one another by means of a holomorphic mapping.

  Poincaré 1907, Segre 1931, E. Cartan 1932, Chern-Moser 1975: CR geometry. Invariants by means of Taylor series coefficients.

Kobayashi pseudometric

Let $\Omega \subset \mathbb{C}^n$ be a domain.

Definition (Kobayashi pseudometric)

Let $z \in \Omega$ and $v \in \mathbb{C}^n$:

$$K_\Omega(z,v) = \inf\left\{\frac{1}{r} > 0 \mid \ f : \Delta \to \Omega \textit{ holomorphic}, \ f(0) = z, \ f'(0) = rv\right\}.$$

Kobayashi pseudometric

Let $\Omega \subset \mathbb{C}^n$ be a domain.

Definition (Kobayashi pseudometric)

Let $z \in \Omega$ and $v \in \mathbb{C}^n$:

$$K_\Omega(z,v) = \inf\left\{\frac{1}{r} > 0 \mid \; f : \Delta \to \Omega \textit{ holomorphic}, \; f(0) = z, \; f'(0) = rv\right\}.$$

Remarks:

- Biholomorphic invariant.
- Natural extension of the Poincaré metric in higher dimension.
- Measures the size of holomorphic discs contained in Ω.
- Can be degenerate.

Kobayashi hyperbolicity

Define the *Kobayashi pseudodistance* $d_\Omega(z, z')$ by considering lengths of smooth paths joining z and z'.

Definition

- Ω *is hyperbolic if the Kobayashi pseudodistance* d_Ω *is a distance.*
- Ω *is complete hyperbolic if* (Ω, d_Ω) *is a complete metric space.*

Kobayashi hyperbolicity

Define the *Kobayashi pseudodistance* $d_\Omega(z, z')$ by considering lengths of smooth paths joining z and z'.

Definition

- Ω *is hyperbolic if the Kobayashi pseudodistance* d_Ω *is a distance.*
- Ω *is complete hyperbolic if* (Ω, d_Ω) *is a complete metric space.*

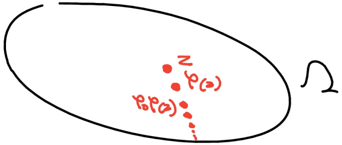
Examples:

- Δ, $\mathbb{H}$, $\Delta \setminus \{0\}$, $\mathbb{B}_n$ and Δ^n are complete hyperbolic.
- Any bounded domain in $\mathbb{C}^n$ is hyperbolic.
- $\{z \in \mathbb{C}^2 \mid 1 < |z|^2 < 4\}$ is hyperbolic but not complete.
- $\{z \in \mathbb{C}^2 \mid \Re e z_2 + |z_1|^2 < 0\}$ is unbounded complete hyperbolic.
- $\mathbb{C}^n$ is not hyperbolic. Any domain containing a complex line is not hyperbolic.

An important rigidity result

Theorem (Wong 1977, Rosay 1979)

Let Ω be a smoothly bounded strictly pseudoconvex domain of $\mathbb{C}^n$. Assume that $Aut(\Omega)$ acts transitively on Ω (resp. is noncompact). Then Ω is biholomorphic to the unit ball $\mathbb{B}_n$.



Remark: Estimates of the Kobayashi metric near the boundary (Graham 1975)

Extremal discs

Recall that for $z \in \Omega$ and $v \in \mathbb{C}^n$:

$$K_\Omega(z,v) = \inf\left\{\frac{1}{r} > 0 \mid f : \Delta \to \Omega \text{ holomorphic}, f(0) = z,\ f'(0) = rv\right\}.$$

Remark: When Ω is bounded, there is a disc $f : \Delta \to \Omega$ with $f(0) = z$ and $f'(0) = \frac{1}{K_\Omega(z,v)} v$. Such a disc is called an *extremal disc of Ω for (z,v)*.

Extremal discs

Recall that for $z \in \Omega$ and $v \in \mathbb{C}^n$:

$$K_\Omega(z,v) = \inf\left\{\frac{1}{r} > 0 \mid f:\Delta \to \Omega \text{ holomorphic}, f(0)=z,\ f'(0)=rv\right\}.$$

Remark: When Ω is bounded, there is a disc $f:\Delta \to \Omega$ with $f(0)=z$ and $f'(0)=\frac{1}{K_\Omega(z,v)}v$. Such a disc is called an *extremal disc of Ω for (z,v)*.

Examples:

- The set of extremal discs of Δ is $Aut(\Delta)$.
- Extremal discs of $\mathbb{B}_n$ centered at the origin are linear discs $f(\zeta)=\zeta v$ (here $\|v\|=1$). *Extremal discs of $\mathbb{B}_n$ are the holomorphic isometries $f:(\Delta,d_\Delta)\to(\mathbb{B}_n,d_{\mathbb{B}_n})$.*
- $f_0(\zeta)=(\zeta,0)$ and $f_1(\zeta)=(\zeta,\zeta^2)$ are extremal discs of Δ^2 for $((0,0),(1,0))$.

Lempert theory of extremal discs

Let Ω be a bounded smooth strongly convex domains in $\mathbb{C}^n$.

Theorem (Lempert 1981)

Ω admits a singular foliation through any point by images of extremal discs.

Lempert theory of extremal discs

Let Ω be a bounded smooth strongly convex domains in $\mathbb{C}^n$.

Theorem (Lempert 1981)

Ω admits a singular foliation through any point by images of extremal discs.

Consequences and remarks:

- Extremal discs are holomorphic isometries, are smooth up to the boundary, and are isolated.
- Allows to construct a circular representation of Ω: $\Phi_z : \Omega \to \mathbb{B}^n$.
- Extremal discs are stationary (Poletskii 1983: stationarity = Euler-Lagrange)

Birth of Stationary discs.

a few words on stationary discs

- They are analytic discs
- Biholomorphically invariant
- Their existence is well understood for "nondegenerate" hypersurfaces and relies on nonlinear Riemann-Hilbert problems (Forstnerič 1987, Globevnik 1993-1994)
- They usually form a submanifold of finite dimension (of the infinitely dimensional Banach submanifold of analytic discs).
- Well adapted to study mapping problems (Lempert 1981, Huang 1994, Tumanov 2001), and to study the question "how to distinguish maps from one another?" (B-Blanc-Centi 2014, also with Della Sala, Lamel, Meylan).