An ensemble Kalman-Bucy filter for continuous data assimilation

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Abstract

The ensemble Kalman filter has emerged as a promising filter algorithm for nonlinear differential equations subject to intermittent observations. In this paper, we extend the well-known Kalman-Bucy filter for linear differential equations subject to continous observations to the ensemble setting and nonlinear differential equations. The proposed filter is called the ensemble Kalman-Bucy filter and its performance is demonstrated for a simple mechanical model (Langevin dynamics) subject to incremental observations of its position.

1 Introduction

We consider stochastic differential equations of type

$$d\mathbf{x} = f(\mathbf{x}, t)dt + \mathbf{\Omega}^{1/2} d\mathbf{w}(t) \tag{1.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^N$, $\mathbf{\Omega} \in \mathbb{R}^{N\times N}$ is a symmetric positive semi-definite matrix, and $\mathbf{w}(t)$ denotes N-dimensional standard Brownian motion (GARDINER, 2004).

The standard intermittent data assimilation problem for (1.1) can be stated as follows. Given initial conditions $\mathbf{x}(t_0)$ at time t_0 and measurements

$$\mathbf{y}(t_q) = \mathbf{H}\mathbf{x}(t_q) + \eta_q \tag{1.2}$$

at discrete times t_q, $q = 1, \ldots, J$, find the optimal estimate for solutions at times $t > t_0$ provided measurements at all instances $t_q < t$ are available (filtering/prediction problem). Here $\mathbf{H} \in \mathbb{R}^{K\times N}$ is the forward operator, $\mathbf{y}(t_q) \in \mathbb{R}^K$ is the measurement at time t_q and $\eta_q \in \mathbb{R}^K$ are realizations of a Gaussian random variable with mean zero and covariance matrix $\mathbf{R} \in \mathbb{R}^{K\times K}$.

In recent years, the ensemble Kalman filter (EVENSEN, 2006) has emerged as a powerful tool to approximately solve the filtering/prediction problem; see also related work on the unscented Kalman filter (JULIER and UHLMANN, 1997). Implementations of the ensemble Kalman filter can be grouped into ensemble transform/square root filters (TIPPETT et al., 2003) and ensemble filters based on perturbed observations (EVENSEN, 2006). BERGEMANN et al. (2009), BERGEMANN and REICH (2010a), BERGEMANN and REICH (2010b), and REICH (2011a) have developed associated continuous ensemble Kalman filter implementations which are closely related to the Kalman-Bucy filter for linear differential equations (JAZWINSKI, 1970). Implementation issues and a comparison to the local ensemble transform Kalman filter (HUNT et al., 2007) have been addressed in AMEZCUA et al. (2011).

In this paper, we demonstrate how to use the continuous formulations of BERGEMANN and REICH (2010a,b); REICH (2011a) to tackle assimilation problems for which observations of incrementally changing variables are to be assimilated. A typical example is provided by Lagrangian data, i.e. ocean drifter positions which are used to correct Eulerian velocity fields forecasted by models (KUZNETSOV et al., 2003; MOLCARD et al., 2003). To be more specific, let $\mathbf{X}(t) \in \mathbb{R}^2$ denote the horizontal position of a drifter at time t and $\mathbf{X}(t+\tau)$ its position at $t+\tau$, where τ is the (assumed to be small) observation interval. Then also assuming that the drifter's velocity is subject to random perturbations to its surrounding horizontal fluid velocity field $\mathbf{v}(\mathbf{X}, t) \in \mathbb{R}^2$, we obtain the approximation

$$\mathbf{X}(t+\tau) - \mathbf{X}(t) = \mathbf{v}(\mathbf{X}, t)\tau + \sqrt{\tau}\sigma\mathbf{r}_t$$

where $\sigma > 0$ is a constant and $\mathbf{r}_t \in \mathbb{R}^2$ is a random vector with mean zero and covariance equal to the identity matrix $\mathbf{I}$. In the limit $\tau \to 0$, one obtains the stochastic differential equation

$$d\mathbf{X} = \mathbf{v}(\mathbf{X}, t)dt + \sigma d\mathbf{s}(t),$$

where $\mathbf{s}(t) \in \mathbb{R}^2$ denotes two dimensional standard Brownian motion.

The paper is organized as follows. The continuous ensemble Kalman filter formulation of BERGEMANN and REICH (2010a,b) is reviewed in Section 2 in the context of intermittent data assimilation. The ensemble Kalman-Bucy filter for continuous data assimilation is derived in Section 3 and implemented in Section 4 for a simple mechanical system to assess the performance of the proposed ensemble Kalman-Bucy filter relative to that of the extended Kalman-Bucy filter (JAZWINSKI, 1970).

2 Background

Ensemble Kalman filters provide a generalization of the classic linear Kalman filter to nonlinear models such as (1.1). It is based on propagating an ensemble of independent solutions $\mathbf{x}_i, \quad i = 1, \ldots, m$, of our model (1.1) and using it to extract an empirical mean

$$\overline{\mathbf{x}} = \frac{1}{m}\sum_{i=1}^{m} \mathbf{x}_i \tag{2.1}$$

and an empirical error covariance matrix

$$\mathbf{P} = \frac{1}{m-1}\sum_{i=1}^{m}(\mathbf{x}_i - \overline{\mathbf{x}})(\mathbf{x}_i - \overline{\mathbf{x}})^T. \tag{2.2}$$

Several different variants of the ensemble Kalman filter have been proposed in recent years, like the perturbed observations EnKF (BURGERS et al., 1998), the EnSRF/ETKF (TIPPETT et al., 2003) or the LETKF (HUNT et al., 2007), which all use different methods to modify the ensemble members $\mathbf{x}_i$ every time new observations $\mathbf{y}(t_q)$ become available such that the resulting ensemble mean (2.1) and covariance matrix (2.2) satisfy the Kalman analysis equations (JAZWINSKI, 1970).

BERGEMANN and REICH (2010a,b) proposed a continuous ensemble Kalman filter where the ensemble members $\mathbf{x}_i$ are modified at every observation time t_q using the ordinary differential equation

$$\frac{d}{ds}\mathbf{x}_i = -\frac{1}{2}\mathbf{P}\mathbf{H}^T\mathbf{R}^{-1}(\mathbf{H}\mathbf{x}_i + \mathbf{H}\overline{\mathbf{x}} - 2\mathbf{y}(t_q)), \tag{2.3}$$

which is solved in a ficticious time s over the interval $[0, 1]$, i.e. we use the forecast ensemble members as initial conditions $\mathbf{x}_i(s = 0) = \mathbf{x}_{i,f}$ for (2.3) and obtain analysed ensemble members $\mathbf{x}_{i,a} = \mathbf{x}_i(s = 1)$. This reformulation of the Kalman analysis step is attractive since it allows for localization (BERGEMANN and REICH, 2010a) and mollification (BERGEMANN and REICH, 2010b) in a straightforward manner as well as providing the starting point for non-Gaussian extensions (REICH, 2011b). Time-stepping methods for (2.3) are discussed in AMEZCUA et al. (2011).

Furthermore, BERGEMANN and REICH (2010b) demonstrated that (1.1) with $\mathbf{\Omega} = \mathbf{0}$ and (2.3) can in fact be combined into a single differential equation via

$$\frac{d}{dt}\mathbf{x}_i = f(\mathbf{x}_i, t) - \frac{1}{2}\sum_{q=1}^{J}\delta(t - t_q)\mathbf{P}\mathbf{H}^T\mathbf{R}^{-1}(\mathbf{H}\mathbf{x}_i + \mathbf{H}\overline{\mathbf{x}} - 2\mathbf{y}(t_q)), \tag{2.4}$$

where $\delta(t)$ denotes the Dirac delta function.

Although ensemble Kalman filters use the full nonlinear model to propagate the ensemble, the analysis is still based on the standard Kalman equations which require the background, measurement and possible model errors to be Gaussian distributed. This cannot be guaranteed in a nonlinear setting because even if the initial background errors are Gaussian the nonlinear dynamics of the model will destroy Gaussianity. Therefore ensemble Kalman filters are not nonlinear filters in a strict sense.

3 Ensemble Kalman-Bucy filter for continuous data assimilation

The Kalman-Bucy filter is concerned with continuously assimilating observations of trajectory increments rather than trajectory values at discrete points in time. In other words, measurements in a Kalman-Bucy framework are formulated mathematically as a stochastic differential equation

$$d\mathbf{z} = \mathbf{G}\mathbf{x}dt + \mathbf{C}^{1/2}d\mathbf{r}(t), \tag{3.1}$$

in terms of the state vector $\mathbf{x}(t)$ where $\mathbf{G} \in \mathbb{R}^{K\times N}$ is the linear forward operator, $\mathbf{r}(t)$ denotes standard K-dimensional Brownian motion, and $\mathbf{C} \in \mathbb{R}^{K\times K}$ is a positive definite matrix. See JAZWINSKI (1970) for more details.

As already discussed in Section 1, $d\mathbf{z}(t)$ is not observable in practice and instead increments $\Delta\mathbf{z}(t) = \mathbf{z}(t+\tau) - \mathbf{z}(t)$ of $\mathbf{z}(t)$ with $\tau > 0$ being small relative to the model dynamics are being measured. However, (3.1) is nevertheless useful as a mathematical abstraction since it indicates how an appropriate likelihood function for observing $\Delta\mathbf{z}$ given $\mathbf{x}$ should scale with regard to the measurement interval τ. To gain a better intuitive understanding of the problem, we replace the measurement equation by its forward Euler discretization

$$\Delta\mathbf{z}^n = \mathbf{G}\mathbf{x}^n\tau + \tau^{1/2}\xi^n \tag{3.2}$$

with ξ^n independent and identically distributed Gaussian random variables with mean zero and covariance matrix $\mathbf{C}$ and observed $\Delta\mathbf{z}^n = \mathbf{z}^{n+1} - \mathbf{z}^n$, where the superscript n denotes the nth time-step.

It should be noted that the error model (3.2) entails that measurement errors are now assumed to be Gaussian in the increments between two subsequent measurements of $\mathbf{z}(t)$ rather than in the values $\mathbf{z}(t)$ itself. This error model might also provide an alternative for the assimilation of measurements arriving at a high sampling rate since the point-wise measurement errors in (1.2) should be correlated in time, in general. In line with standard filter theory, we assume that the model and measurement errors are uncorrelated.

Upon accepting the measurement error model (3.2) we obtain the likelihood function

$$\pi(\Delta\mathbf{z}^n|\mathbf{x}^n) \propto \exp\left(-\frac{1}{2\tau}(\Delta\mathbf{z}^n - \mathbf{G}\mathbf{x}^n\tau)^T\mathbf{C}^{-1}(\Delta\mathbf{z}^n - \mathbf{G}\mathbf{x}^n\tau)\right),$$

which replaces the likelihood function of a standard ensemble Kalman filter. As for the standard ensemble Kalman filter, the ensemble update is obtained by combining the likelihood with the Gaussian prior

$$\pi_{\text{prior}}(\mathbf{x}^n) \propto \exp\left(-\frac{1}{2}(\mathbf{x}^n - \bar{\mathbf{x}}^n)^T(\mathbf{P}^n)^{-1}(\mathbf{x}^n - \bar{\mathbf{x}}^n)\right)$$

using Bayes' theorem. The ensemble covariance matrix $\mathbf{P}^n$ and the ensemble mean $\bar{\mathbf{x}}^n$ are defined as for the standard ensemble Kalman filter (EVENSEN, 2006). The continuous ensemble Kalman filter formulation (2.3) of BERGEMANN and REICH (2010a,b) for the ith ensemble member becomes

$$\frac{d\mathbf{x}_i^n}{ds} = -\frac{1}{2}\mathbf{P}^n\mathbf{G}^T\mathbf{C}^{-1}(\mathbf{G}\mathbf{x}_i^n\tau + \mathbf{G}\bar{\mathbf{x}}^n\tau - 2\Delta\mathbf{z}^n).$$

Note that $\mathbf{R} = \mathbf{C}/\tau$, $\mathbf{H} = \mathbf{G}$, and $\mathbf{y} = \Delta\mathbf{z}/\tau$ formally lead back to the continuous ensemble Kalman filter for intermittent data assimilation (BERGEMANN and REICH, 2010a,b). With this choice for $\mathbf{R}$ and $\mathbf{H}$ one could also implement a standard ensemble Kalman filter (EVENSEN, 2006). However, necessary matrix inversions and factorizations would introduce an unnecessary computational overhead.

To formally derive a continuous filter equation, we let $\tau \to 0$ and simultaneously apply forward Euler to the continuous formulation with $\Delta s = 1$. Then, first (forward Euler with $\Delta s = 1$)

$$\Delta \mathbf{x}_i^n = -\frac{1}{2}\mathbf{P}^n \mathbf{G}^T \mathbf{C}^{-1}(\mathbf{G}\mathbf{x}_i^n \tau + \mathbf{G}\bar{\mathbf{x}}^n \tau - 2\Delta \mathbf{z}^n)$$

and, second ($\tau \to 0$)

$$d\mathbf{x}_i = -\frac{1}{2}\mathbf{P}\mathbf{G}^T \mathbf{C}^{-1}(\mathbf{G}\mathbf{x}_i dt + \mathbf{G}\bar{\mathbf{x}} dt - 2d\mathbf{z}(t)).$$

We emphasize that these limits have to be taken with great care in general since we are dealing with stochastic differential equations (JAZWINSKI, 1970). In our context, however, our simplified derivation can be justified.

Since data is now (formally) collected continuously in time, we can — similarly to (2.4) — combine the assimilation step with the model dynamics to yield the final result

$$d\mathbf{x}_i = f(\mathbf{x}_i, t)dt + \mathbf{\Omega}^{1/2} d\mathbf{w}_i - \frac{1}{2}\mathbf{P}\mathbf{G}^T \mathbf{C}^{-1}(\mathbf{G}\mathbf{x}_i dt + \mathbf{G}\bar{\mathbf{x}} dt - 2d\mathbf{z}(t)), \tag{3.3}$$

which constitutes an ensemble Kalman-Bucy filter formulation. The corresponding continuous formulation of the EnKF with perturbed observations instead (see REICH (2011a) for details) is given by

$$d\mathbf{x}_i = f(\mathbf{x}_i, t)dt + \mathbf{\Omega}^{1/2} d\mathbf{w}_i - \mathbf{P}\mathbf{G}^T \mathbf{C}^{-1}(\mathbf{G}\mathbf{x}_i dt - d\mathbf{z}(t) + \mathbf{C}^{1/2} d\mathbf{u}_i(t)).$$

Here $\mathbf{u}_i(t)$ denotes K-dimensional Brownian motion and a different realization is chosen for each ensemble member. Nonlinear forward operators $g(\mathbf{x})$ can be treated by a modification of the covariance matrix $\mathbf{P}$ in the same manner as for ensemble Kalman filters (EVENSEN, 2006), i.e. $\mathbf{P}\mathbf{G}^T$ is replaced by

$$\mathbf{P}\mathbf{G}^T \leftarrow \frac{1}{m-1}\sum_{i=1}^{m}(\mathbf{x}_i - \overline{\mathbf{x}})(g(\mathbf{x}_i) - \overline{\mathbf{g}})^T.$$

with

$$\overline{\mathbf{g}} = \frac{1}{m}\sum_{i=1}^{m} g(\mathbf{x}_i).$$

The same applies to localization techniques (EVENSEN, 2006; BERGEMANN and REICH, 2010a).

4 Numerical demonstration

We now discuss an example from classical mechanics. The evolution of the state $\mathbf{x} = (q, v) \in \mathbb{R}^2$ is described by Langevin dynamics (GARDINER, 2004) with equations of motion

$$\begin{aligned} dq &= v\,dt, \\ dv &= -V'(q)\,dt - \gamma v\,dt + \sigma dw(t), \end{aligned}$$

where the potential $V(q)$ is given by

$$V(q) = \cos(q) + \frac{3}{4}(q/6)^4 + \frac{q}{10}$$

(see Figure 1), the friction coefficient is $\gamma = 0.25$, $w(t)$ denotes standard Brownian motion, and $\sigma^2 = 0.35$. A reference solution, denoted by $(q_r(t), v_r(t))$, is obtained for initial condition $(q_0, v_0) = (1, 1)$ and a particular realization of $w(t)$.

Let us address the situation that the reference solution is not directly accessible to us and that instead we are only able to observe $Q(t)$ subject to

$$dQ(t) = v_r(t)\,dt + c^{1/2} d\xi(t), \tag{4.1}$$

where $\xi(t)$ denotes again standard Brownian motion and c the measurement error variance.

We got two independent sources of information and each individually allows us to guess the $q(t)$ component of the reference solution. Firstly, we may use the observations $dQ(t)$ to obtain an estimate $Q(t)$ via

$$Q(t) = q_0 + \int_0^t dQ(t).$$

So by “summing up” the infinitesimal increments $dQ(t)$ and adding them to the known initial position q_0 we obtain an approximation for $q_r(t)$. Secondly, we may solve the Langevin equations with initial condition $(q_0, p_0) = (1, 1)$ and some realization $w(t)$ of Brownian motion and thus estimate $q_r(t)$ via a free model run. However both approaches diverge from the reference solution $q_r(t)$ fairly quickly.

We now combine the model equations and the observations within the ensemble Kalman-Bucy framework. The ensemble filter relies on the simultaneous propagation of an ensemble of solutions $x_i(t) = (q_i(t), v_i(t))$, $i = 1, \ldots, m$. The filter equation (3.3) becomes

$$\begin{aligned} dq_i &= v_i\,dt - \frac{P_{qv}}{2c}(v_i\,dt + \bar{v}\,dt - 2dQ(t)), \\ dv_i &= -V'(q_i)\,dt - \gamma v_i\,dt + \sigma dw_i(t) - \frac{P_{vv}}{2c}(v_i\,dt + \bar{v}\,dt - 2dQ(t)) \end{aligned}$$

with mean

$$\bar{v} = \frac{1}{m}\sum_i v_i, \qquad \bar{q} = \frac{1}{m}\sum_i q_i$$

and variance/covariance

$$P_{vv} = \frac{1}{m-1}\sum_i (v_i - \bar{v})^2, \qquad P_{qv} = \frac{1}{m-1}\sum_i (q_i - \bar{q})(v_i - \bar{v}).$$

The equations are solved for each ensemble member with different realizations $w_i(t)$ of standard Brownian motion and step-size $\Delta t = 0.01$. The observation interval in (3.2) is $\tau = \Delta t$.

The extended Kalman-Bucy filter (JAZWINSKI, 1970) relies on the equations

$$\begin{aligned} d\bar{q} &= \bar{v}\,dt - \frac{P_{qv}}{c}(\bar{v}\,dt - dQ(t)), \\ d\bar{v} &= -V'(\bar{q})\,dt - \gamma\bar{v}\,dt - \frac{P_{vv}}{c}(\bar{v}\,dt - dQ(t)) \end{aligned}$$

for the mean and

$$\frac{d\mathbf{P}}{dt} = \mathbf{A}(t)\mathbf{P} + \mathbf{P}\mathbf{A}(t)^T + \mathbf{\Omega} - \frac{1}{c}\mathbf{P}\mathbf{e}_2\mathbf{e}_2^T\mathbf{P}$$

with $\mathbf{e}_2 = (0,1)^T$,

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ -V''(\bar{q}(t)) & -\gamma \end{bmatrix}$$

and

$$\mathbf{\Omega} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma^2 \end{bmatrix}$$

for the covariance.

Figure 2 shows the reference solution $q_r(t)$ that was used for all experiments. First we performed experiments using a small ensemble size $m = 3$ and different measurement error variances varying from $c = 0.02$ to $c = 0.5$. For $c = 0.02$ the ensemble Kalman-Bucy filter is able to track the reference solution, missing transitions from one local potential minimum to the other only in some instances and only very shortly (Figure 3, left panel). The extended Kalman-Bucy filter however is unable to track the reference solution and is trapped in one of the local potential minima (Figure 3, right panel). With increasing measurement error variance the performance of the ensemble Kalman-Bucy deteriorates. For $c = 0.1$ the ensemble Kalman-Bucy filter still tracks the reference most of the time but exhibits short episodes where the filter solution sojourns into the wrong potential minimum. The extended Kalman-Bucy filter is again trapped in one minimum and therefore unable to track the reference solution at all (Figure 4). Finally, for $c = 0.5$, the ensemble Kalman-Bucy filter cannot track the reference reliably either (Figure 5).

In case of a larger ensemble size $m = 10$ the results are similar to the small ensemble scenario. For small measurement error variance $c = 0.02$ the reference solution is tracked very well and, as was to be expected, the variance of the estimation error $q_r(t) - \bar{q}(t)$ is smaller than the estimation error in the previous scenario (Figure 6). For $c = 0.1$ the ensemble Kalman-Bucy filter still loses track of the reference solution in some cases but the duration of these episodes is greatly reduced (Figure 7). Again for $c = 0.5$ although the performance has improved it is not reliable yet (Figure 8). Please note that, since the results of the extended Kalman-Bucy filter do not depend on the ensemble size, we do not include them in the Figures 6–8. Instead we show the results for the velocity component v. The fact that the assimilation of data does not improve the tracking of the reference solution in the velocity component is due to the error model (4.1) which implies that the measurement error in the velocity component is unbounded. This follows from the fact that formally $dQ/dt = v_r(t) + c^{1/2} d\xi/dt$ and $d\xi/dt$ is the derivative of Brownian motion.

The filter behavior of the proposed ensemble Kalman-Bucy filter is clearly superior to the standard extended Kalman-Bucy filter for this highly nonlinear model problem. The performance of the ensemble Kalman-Bucy filter is limited by two factors. Firstly, limited ensemble sizes lead to estimation errors in the empirical ensemble mean and covariance matrix. Secondly, the Gaussian approximation in the data assimilation step is not appropriate in case of poorly observed systems. In our model system, this has been demonstrated for increasing measurement noise levels. The Gaussian approximation could be overcome by combining the proposed ensemble Kalman-Bucy filter with the Gaussian mixture approach of REICH (2011b).

5 Conclusions

We have extended the popular ensemble Kalman filter methodology to assimilation problems with continuously arriving measurements. The resulting ensemble filter is called ensemble Kalman-Bucy filter.

It has been demonstrated by means of a simple Langevin dynamics model that an ensemble Kalman-Bucy filter can track the solutions of a mechanical system using noisy observations of incremental changes in the positions.

A possible application is provided by Lagrangian data assimilation in ocean models (KUZNETSOV et al., 2003; MOLCARD et al., 2003). In this context $\Delta \mathbf{z}$ could, for example, correspond to the incremental change in the position of a Lagrangian drifter and $\mathbf{Gx}$ would equal the velocity of the surrounding fluid. We would assume that the drifter transmits its position in time intervals comparable to the model's time-step. If that is not the case, some form of interpolation would be required. An alternative implementation of the ensemble Kalman filter for Lagrangian data assimilation has been proposed by SALMAN et al. (2005), where a state augmentation technique combined with intermittent data assimilation is used.

The proposed ensemble Kalman-Bucy filter should more genererally be useful for data arriving at a high sampling rate. For such data types the error model (3.2) might be more appropriate than the error model (1.2), which is normally used for intermittent data assimilation. An example is provided by the reentry tracking problem, where a radar is used for tracking a space vehicle reentering the atmosphere (see, for example, JULIER and UHLMANN (1997)).

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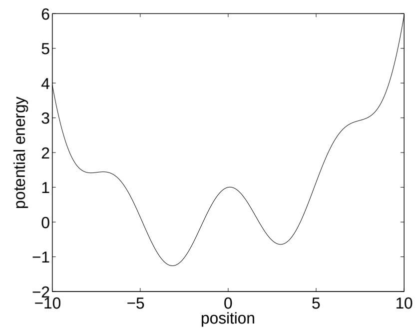

Figure 1: Potential energy function of the Langevin dynamics model.

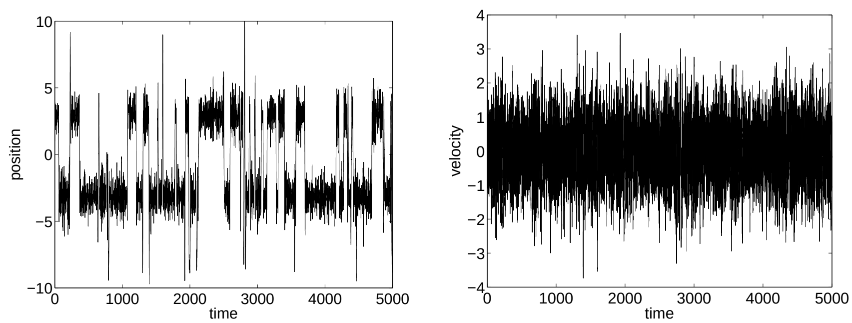

Figure 2: Reference solution of the Langevin dynamics model. Shown here is the position $q_r(t)$ (left panel) and the velocity $v_r(t)$ (right panel).

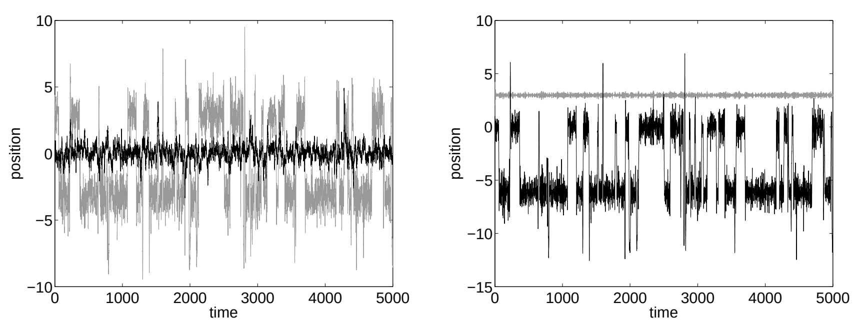

Figure 3: Left panel: The estimated (ensemble mean) solution $\bar{q}(t)$ (gray) and the difference $q_r(t) - \bar{q}(t)$ (black) for an ensemble size $m = 3$ and a measurement error variance of $c = 0.02$. Right panel: The solution from the extended Kalman-Bucy filter (gray) and the difference between reference and extended Kalman-Bucy solution (black) for a measurement error variance of $c = 0.02$.

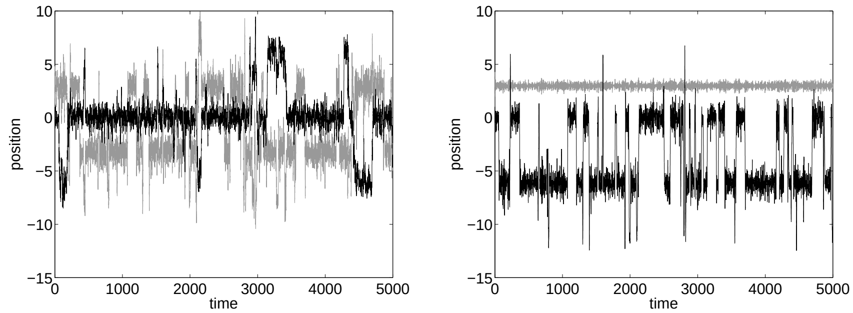

Figure 4: Left panel: The estimated (ensemble mean) solution $\bar{q}(t)$ (gray) and the difference $q_r(t) - \bar{q}(t)$ (black) for an ensemble size $m = 3$ and a measurement error variance of $c = 0.1$. Right panel: The solution from the extended Kalman-Bucy filter (gray) and the difference between reference and extended Kalman-Bucy solution (black) for a measurement error variance of $c = 0.1$.

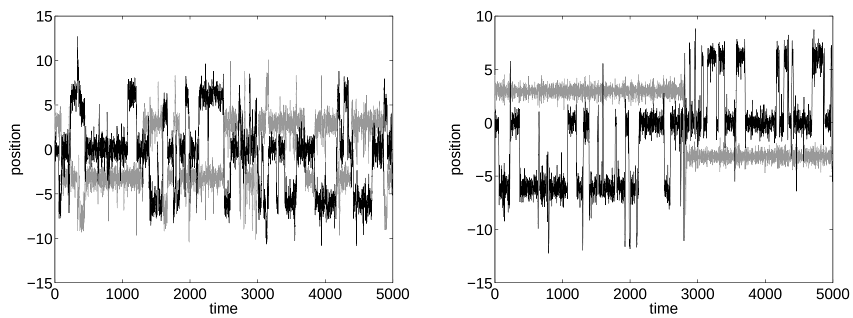

Figure 5: Left panel: The estimated (ensemble mean) solution $\bar{q}(t)$ (gray) and the difference $q_r(t) - \bar{q}(t)$ (black) for an ensemble size $m = 3$ and a measurement error variance of $c = 0.5$. Right panel: The solution from the extended Kalman-Bucy filter (gray) and the difference between reference and extended Kalman-Bucy solution (black) for a measurement error variance of $c = 0.5$.

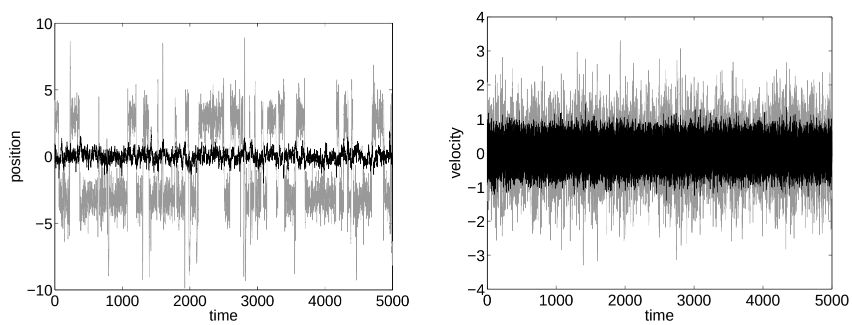

Figure 6: Left panel: The estimated (ensemble mean) solution $\bar{q}(t)$ (gray) and the difference $q_r(t) - \bar{q}(t)$ (black) for an ensemble size $m = 10$ and a measurement error $c = 0.02$. Right panel: The estimated (ensemble mean) solution $\bar{v}(t)$ (gray) and the difference $v_r(t) - \bar{q}(t)$ (black) for an ensemble size $m = 10$ and a measurement error $c = 0.02$.

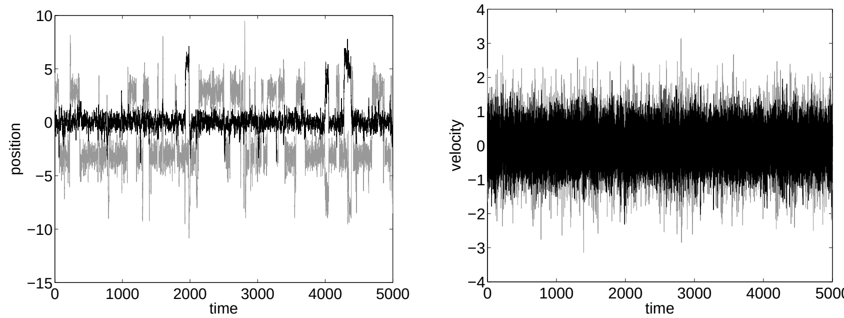

Figure 7: Left panel: The estimated (ensemble mean) solution $\bar{q}(t)$ (gray) and the difference $q_r(t) - \bar{q}(t)$ (black) for an ensemble size $m = 10$ and a measurement error $c = 0.1$. Right panel: The estimated (ensemble mean) solution $\bar{v}(t)$ (gray) and the difference $v_r(t) - \bar{q}(t)$ (black) for an ensemble size $m = 10$ and a measurement error $c = 0.1$.

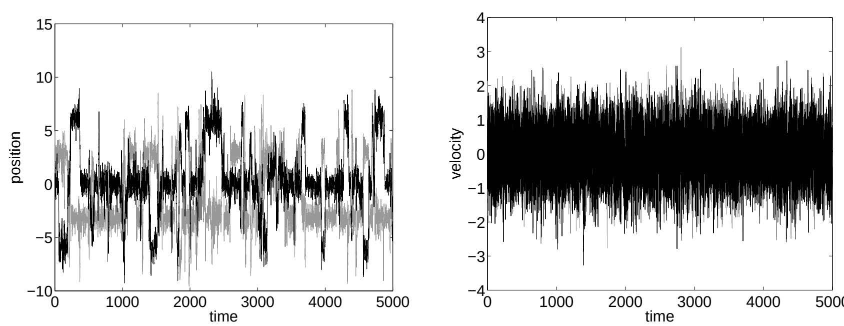

Figure 8: Left panel: The estimated (ensemble mean) solution $\bar{q}(t)$ (gray) and the difference $q_r(t) - \bar{q}(t)$ (black) for an ensemble size $m = 10$ and a measurement error $c = 0.5$. Right panel: The estimated (ensemble mean) solution $\bar{v}(t)$ (gray) and the difference $v_r(t) - \bar{q}(t)$ (black) for an ensemble size $m = 10$ and a measurement error $c = 0.5$.