Ensemble Kalman and H_∞ filters

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Abstract. The ensemble Kalman filter has become a popular method for nonlinear data assimilation. Standard ensemble Kalman filter implementations need to be modified to avoid filter divergence due to model and statistical errors. In this communication, we discuss ensemble inflation within the continuous ensemble Kalman filter approach and its link to H_∞ filtering.

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INTRODUCTION

We consider dynamical models given in form of ordinary differential equations (ODEs)

$$\dot{\mathbf{x}} = f(\mathbf{x}, t) \tag{1}$$

with state variable $\mathbf{x} \in \mathbb{R}^n$. Initial conditions at time t_0 are not precisely known and we assume instead that

$$\mathbf{x}(t_0) \sim \pi_0, \tag{2}$$

where $\pi_0(\mathbf{x})$ denotes a probability density function (PDF). In other words, our only information about the initial condition is that it is a realization of a random variable with PDF π_0. To compensate for the uncertainty in the initial conditions we assume that we obtain measurements $\mathbf{y}(t_q) \in \mathbb{R}^k$ at discrete times $t_q \geq t_0$, $q = 0, 1, \ldots, M$, subject to measurement errors. The measurement errors are assumed to be Gaussian distributed with zero mean and covariance matrix $\mathbf{R} \in \mathbb{R}^{k \times k}$, i.e.

$$\mathbf{y}(t_q) - \mathbf{H}\mathbf{x}(t_q) \sim \mathrm{N}(\mathbf{0}, \mathbf{R}). \tag{3}$$

Here $\mathbf{H} \in \mathbb{R}^{k \times n}$ is the (linear) measurement operator.

Data assimilation is the task to combine the model, our knowledge about its initial state and the incoming observations to an prediction about the current state of the model. A first step to perform data assimilation for nonlinear ODEs (1) is to approximate the time evolution of the initial PDF under the model dynamics without observations. The time evolution is provided by the classical Liouville's equation and particle methods are often used to approximate the exact solution by an evolving empirial measure [1].

More specifically, particle methods rely on the simultaneous propagation of m independent solutions $\mathbf{x}_i(t)$, $i = 1, \ldots, m$, of (1). We associate the empirical measure

$$\pi_{\mathrm{em}}(\mathbf{x}, t) = \frac{1}{m} \sum_{i=1}^{m} \delta(\mathbf{x} - \mathbf{x}_i(t)) \tag{4}$$

Many methods have been proposed to combine particle methods for Liouville's equation with the assimilation of observations. A common class is provided by, so called, particle filters or sequential Monte Carlo methods [2, 3]. More recently an interesting class of particle-based methods has been proposed which combines (4) with a standard Kalman filter step. These, so called, ensemble Kalman filters (EnKF) [1] are now widely being used, for example, in meteorology.

In this communication, we describe a continuous formulation of EnKF [4, 5] and explore the ensemble inflation technique [6] from the perspective of H_∞ filtering [7].

CONTINUOUS ENSEMBLE KALMAN FILTER FORMULATION

In this paper we focus on ensemble square root filter [8, 1] implementations of an EnKF. For notational convenience, the ensemble members $\{\mathbf{x}_i\}_{i=1}^m$ are collected in a matrix $\mathbf{X}(t) \in \mathbb{R}^{n\times m}$. In terms of $\mathbf{X}$, the ensemble mean is given by

$$\overline{\mathbf{x}}(t) = \frac{1}{m}\mathbf{X}(t)\mathbf{e} \in \mathbb{R}^n \tag{5}$$

and we introduce the ensemble deviation matrix

$$\mathbf{Y}(t) = \mathbf{X}(t) - \overline{\mathbf{x}}(t)\mathbf{e}^T \in \mathbb{R}^{n\times m}, \tag{6}$$

where $\mathbf{e} = (1,\dots,1)^T \in \mathbb{R}^m$. The implied covariance matrix is

$$\mathbf{P} = \frac{1}{m-1}\sum_{i=1}^{m}(\mathbf{x}_i - \overline{\mathbf{x}})(\mathbf{x}_i - \overline{\mathbf{x}})^T = \frac{1}{m-1}\mathbf{Y}\mathbf{Y}^T. \tag{7}$$

As proposed in [4, 5], the ensemble Kalman analysis step can be formulated as a continuous process characterized by the differential equations

$$\frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}s} = -\mathbf{P}\nabla_{\mathbf{x}_i}\mathscr{V}_q(\mathbf{X}) = -\frac{1}{2}\mathbf{P}\mathbf{H}^T\mathbf{R}^{-1}\left(\mathbf{H}\mathbf{x}_i + \mathbf{H}\overline{\mathbf{x}} - 2\mathbf{y}(t_q)\right), \tag{8}$$

$i = 1,\dots,m$, where $\mathscr{V}_q$ is the potential

$$\mathscr{V}_q(\mathbf{X}) = \frac{m}{2}\left\{S(\overline{\mathbf{x}}|\mathbf{y}(t_q)) + \frac{1}{m}\sum_{i=1}^{m}S(\mathbf{x}_i|\mathbf{y}(t_q))\right\} \tag{9}$$

with observational cost function $S(\mathbf{x}|\mathbf{y}(t_q))$ given by

$$S(\mathbf{x}|\mathbf{y}(t_q)) = \frac{1}{2}\left(\mathbf{H}\mathbf{x} - \mathbf{y}(t_q)\right)^T\mathbf{R}^{-1}\left(\mathbf{H}\mathbf{x} - \mathbf{y}(t_q)\right) \tag{10}$$

for a set of observations at t_q. The solutions of (8) over a unit time interval provide a particular analysis step for an ensemble square root filter.

The continuous formulation (8) allows for a concise formulation for the analysis of a sequences of observations at time instances t_q, $q = 1,\dots,M$, and intermediate propagation of the ensemble under the dynamics (1). Specifically, we obtain the differential equation

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i, t) - \sum_{q=1}^{M}\delta(t - t_q)\,\mathbf{P}\nabla_{\mathbf{x}_i}\mathscr{V}_q(\mathbf{X}) \tag{11}$$

in each ensemble member, where $\delta(\cdot)$ denotes the standard Dirac delta function.

While the Kalman filter is optimal for linear systems, the ensemble Kalman filter provides only an approximative solution to the filtering problem which furthermore is subject to errors induced by the finite ensemble size m. The later leads to the underestimation of short range correlations and spurious long range correlations, which can destabilize the filter. Techniques such as ensemble inflation [6] and localization [9, 10] have been proposed to circumvent filter divergence and have been successfully applied to meteorological applications. It has been shown in [4] that localization can easily be combined with the continuous EnKF formulation (8). Mollification as a further filter stabilization techniques has been considered in [5]. In the following section, we demonstrate how ensemble inflation can be put into the context of the continuous EnKF formulation (8) and how it is related to H_∞ filtering [7].

Ensemble inflation and H_∞ filtering

Ensemble inflation [6] is another popular technique to correct for pour statistics from small ensemble sizes, i.e $m \ll n$. Ensemble inflation is performed either before or after a data analysis step and consists in replacing $\mathbf{P}$ by $\delta\mathbf{P}$ with factor $\delta > 1$.

We now point to an interesting link between ensemble inflation and H_∞ filtering [7]. We first introduce a (negative) cost function

$$D(\mathbf{x}) = -\frac{\theta}{2}(\mathbf{L}\mathbf{x})^T\mathbf{S}^{-1}\mathbf{L}\mathbf{x} \tag{12}$$

for given matrices $\mathbf{L}$ and $\mathbf{S}$ and parameter $\theta > 0$. The matrix $\mathbf{S}$ is assumed to be symmetric positive definite. We also introduce the (negative) potential

$$\mathscr{W}(\mathbf{X}) = \frac{m}{2}\left\{\frac{1}{m}\sum_{i=1}^{m} D(\mathbf{x}_i) - D(\overline{\mathbf{x}})\right\} \tag{13}$$

and the modified data assimilation equations

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i,t) - \sum_{q=1}^{M} \delta_\varepsilon(t-t_q)\mathbf{P}\left[\nabla_{\mathbf{x}_i}\mathscr{V}_q(\mathbf{X}) + \nabla_{\mathbf{x}_i}\mathscr{W}(\mathbf{X})\right] \tag{14}$$

$$= f(\mathbf{x}_i,t) - \sum_{q=1}^{M} \delta_\varepsilon(t-t_q)\mathbf{P}\left[\nabla_{\mathbf{x}_i}\mathscr{V}_q(\mathbf{X}) - \frac{\theta}{2}\mathbf{L}^T\mathbf{S}^{-1}\mathbf{L}\left(\mathbf{x}_i - \overline{\mathbf{x}}\right)\right]. \tag{15}$$

The additional term does not directly affect the evolution of the mean. Instead an additional contribution to the ensemble deviations is introduced that exactly mirrors the term found in continuous H_∞ filter formulations.

If one formally sets $\mathbf{L}^T\mathbf{S}^{-1}\mathbf{L} = \mathbf{P}^{-1}$, then the additional term can be interpreted as a continuous formulation of the ensemble inflation technique, i.e.

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i,t) - \sum_{q=1}^{M} \delta_\varepsilon(t-t_q)\left[\mathbf{P}\nabla_{\mathbf{x}_i}\mathscr{V}_q(\mathbf{X}) + \frac{\theta}{2}(\mathbf{x}_i - \overline{\mathbf{x}})\right]. \tag{16}$$

Alternatively, one can chose $\mathbf{L} = \mathbf{H}$ and $\mathbf{S} = \mathbf{R}$ in which case the combined potential $\mathscr{V}_q + \mathscr{W}$ becomes

$$\mathscr{V}_q(\mathbf{X}) + \mathscr{W}(\mathbf{X}) = \frac{m}{2}\left\{(1+\theta)S(\overline{\mathbf{x}}|\mathbf{y}(t_q)) + \frac{1-\theta}{m}\sum_{i=1}^{m} S(\mathbf{x}_i|\mathbf{y}(t_q))\right\}. \tag{17}$$

This modification implies that the "observed" ensemble deviations $\mathbf{H}\mathbf{x}_i' = \mathbf{H}(\mathbf{x}_i - \overline{\mathbf{x}})$ are pulled less strongly towards zero during an assimilation step while the dynamics in the ensemble mean remains unaltered.

NUMERICAL EXPERIMENTS

We now report numerical results for the standard implementation of the Lorenz-96 model [11, 12] with state vector $\mathbf{x} = (x_1,\ldots,x_n)^T \in \mathbb{R}^n$, $n = 40$, and time evolution given by the differential equations

$$\dot{x}_j = (x_{j+1} - x_{j-2})x_{j-1} - x_j + 8 \tag{18}$$

for $j = 1,\ldots,n$. The equations are closed through the boundary conditions $x_{-1} = x_{39}$, $x_0 = x_{40}$, and $x_{41} = x_1$.

The numerical experiments are conducted precisely as described in [4]. In particular, all methods use $m = 10$ ensemble members and localization to reduce long range correlations [9, 10, 4].

Experiments are conducted using the continuous inflation techniques (16) and (17), respectively. The results can be found in Figure 1, where they are compared to a standard inflation approach and an EnKF implementation with perturbed observations and multiple ensembles [9, 13]. It is found that (17) leads to a less robust performance of the filter while (16) slightly improves the standard inflation technique for small localization radii. Hence it is preferable to inflate all directions instead of only inflation in the direction of the observations. This conclusion might no longer hold for more complex models with a multi-scale behavior.

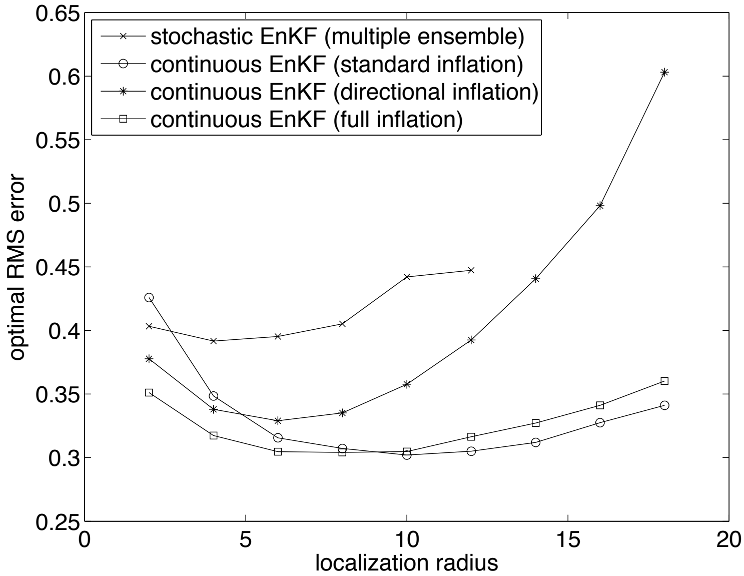

FIGURE 1. Numerical results for the Lorenz-96 model with $n = 40$ state variables and $m = 10$ ensemble members. Different inflation methods are compared for varying values of the localization radius. Results are displayed for an optimal choice of the inflation factors for the following three approaches: (i) standard inflation [6], (ii) full inflation (16) and (iii) directional inflation (17). Results from a EnKF with perturbed observations are inlcuded as a further reference.

SUMMARY

We have summarized the continuous EnKF formulation of [4, 5]. It has been demonstrated that the popular ensemble inflation technique can be put into the context of the continuous EnKF formulation and leads to additional terms closely related to H_∞ filtering for linear systems. Implications of this link need to be explored further for more complex model problems. The findings should provide further insight into how to make EnKF more robust against statistical and model errors.

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