Semi-implicit methods, nonlinear balance, and regularized equations*

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Abstract

A regularized time-staggered discretization of the shallow-water equations has recently been proposed. The form of the regularization operator is chosen such that linear equivalence with the semi-implicit method can be achieved. Here, we present a further generalization of the regularization that also takes into account nonlinear balance. The performance of the improved regularization procedure is demonstrated for a simple nonlinear test problem.

1 Introduction

A time-staggered discretization, combined with a regularization of the continuous governing equations, has recently been proposed as a solution method for the shallow-water equations (Frank et al., 2005). Unconditional linear stability of the method was shown for an appropriate regularization of the geopotential field. An improved regularization procedure, which additionally preserves any linear balance present in the shallow-water equations, is described in Wood et al. (2006). Numerical implementations for the shallow-water equations including the effect of advection and spatial discretization on an Arakawa C-grid have been proposed by Staniforth et al. (2006) and Reich (2006). In this note, we propose a further generalization of the regularization procedure that also takes into account nonlinear balance.

The concept of linear and nonlinear balance for shallow-water equations is summarized in §2 (for more details see, e.g., Haltiner (1971) section 3.12). Equivalence between the regularized time-staggered discretization of Wood et al. (2006) and the semi-implicit method (see, e.g.,(Durran, 1998)) for the linearized shallow-water equations is demonstrated in §3. This analysis is generalized to the nonlinear situation in §4 and motivates the new regularization procedure that takes into account nonlinear balance. Since equivalence between the semi-implicit method and the regularized time-staggered discretization is lost on the nonlinear equation level, a set of numerical experiments is conducted in §5 to demonstrate the performance of the newly proposed regularization procedure. Conclusions are drawn in §6.

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2 Shallow-water equations

For the purpose of this note it is convenient to write the shallow-water equations (SWEs) on an f-plane in Eulerian form

$$\frac{\partial u}{\partial t} + \mathcal{A}(u) - fv = -\frac{\partial \Phi}{\partial x}, \tag{1}$$

$$\frac{\partial v}{\partial t} + \mathcal{A}(v) + fu = -\frac{\partial \Phi}{\partial y}, \tag{2}$$

$$\frac{\partial \Phi}{\partial t} + \mathcal{A}(\Phi) = -\Phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \tag{3}$$

where Φ is the geopotential field and, for general X,

$$\mathcal{A}(X) \equiv DX/DT - \partial X/\partial t = u\,\frac{\partial X}{\partial x} + v\,\frac{\partial X}{\partial y} \tag{4}$$

is the advection operator. The effect of orographic forcing is ignored for simplicity.

2.1 Linearized equations and linear balance

The equations are linearized about a motionless reference state of constant geopotential Φ_0. The resulting linear SWEs are

$$\frac{\partial u}{\partial t} - fv = -\frac{\partial \Phi}{\partial x}, \tag{5}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{\partial \Phi}{\partial y}, \tag{6}$$

$$\frac{\partial \Phi}{\partial t} = -\Phi_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{7}$$

From these it follows that

$$\frac{\partial D}{\partial t} - f\zeta = -\nabla^2\Phi, \tag{8}$$

$$\frac{\partial \zeta}{\partial t} + fD = 0, \tag{9}$$

$$\frac{\partial \Phi}{\partial t} = -\Phi_0\,D, \tag{10}$$

where

$$D \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \qquad \zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{11}$$

The linear balance relation (Haltiner, 1971)

$$f\zeta - \nabla^2\Phi = 0 \tag{12}$$

is obtained by setting $\partial D/\partial t = 0$ in (8).

2.2 Nonlinear balance

We now generalize (12) to the nonlinear SWEs. The divergence equation (8) becomes

$$\frac{\partial D}{\partial t} + \left[\frac{\partial}{\partial x}\mathcal{A}(u) + \frac{\partial}{\partial y}\mathcal{A}(v)\right] - f\zeta = -\nabla^2\Phi. \tag{13}$$

Since

$$\frac{\partial}{\partial x}\mathcal{A}(u) + \frac{\partial}{\partial y}\mathcal{A}(v) = D^2 + \mathcal{A}(D) - 2J(u,v) \tag{14}$$

with Jacobian

$$J(u,v) \equiv \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}, \tag{15}$$

we can reformulate (13) as

$$\frac{\partial D}{\partial t} + D^2 + \mathcal{A}(D) = -\nabla^2\Phi + f\zeta + 2J(u,v). \tag{16}$$

A scale analysis for medium to large scale atmospheric motion (see, e.g., Haltiner (1971) chapter 3) reveals that the terms on the LHS of (16) are much smaller than the terms on the RHS and we obtain the nonlinear balance relation

$$\nabla^2\Phi - f\zeta - 2J(u,v) = 0, \tag{17}$$

where, formally, the velocity components in $J(u,v)$ are replaced by their rotational components.

3 Regularized shallow-water equations

The SWEs are unchanged except that the geopotential, Φ, is replaced by a regularized geopotential, $\widetilde{\Phi}$, in the momentum equations (Frank et al., 2005):

$$\frac{\partial u}{\partial t} + \mathcal{A}(u) - fv = -\frac{\partial \widetilde{\Phi}}{\partial x}, \tag{18}$$

$$\frac{\partial v}{\partial t} + \mathcal{A}(v) + fu = -\frac{\partial \widetilde{\Phi}}{\partial y}, \tag{19}$$

$$\frac{\partial \Phi}{\partial t} + \mathcal{A}(\Phi) = -\Phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{20}$$

As proposed in Wood et al. (2006), the regularized geopotential is defined by

$$\left[1 - \alpha^2\nabla^2\right]\left(\widetilde{\Phi} - \Phi\right) = \alpha^2\left(\nabla^2\Phi - f\zeta\right), \tag{21}$$

and $\widetilde{\Phi} = \Phi$ under linear balance (12). The choice of the parameter $\alpha > 0$ will be discussed in §3.2 (see also Wood et al. (2006)).

3.1 Staggered time-stepping for linearized equations

The linearized regularized equations

$$\frac{\partial u}{\partial t} - fv = -\frac{\partial \widetilde{\Phi}}{\partial x}, \tag{22}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{\partial \widetilde{\Phi}}{\partial y}, \tag{23}$$

$$\frac{\partial \Phi}{\partial t} = -\Phi_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \tag{24}$$

with regularized geopotential, $\widetilde{\Phi}$, given by (21) are discretized by the staggered time-stepping method

$$\begin{aligned}
\frac{u^{n+1}-u^n}{\Delta t} - \frac{f}{2}\left(v^{n+1}+v^n\right) &= -\frac{\partial \widetilde{\Phi}^{n+1/2}}{\partial x}, && (25)\\
\frac{v^{n+1}-v^n}{\Delta t} + \frac{f}{2}\left(u^{n+1}+u^n\right) &= -\frac{\partial \widetilde{\Phi}^{n+1/2}}{\partial y}, && (26)\\
\frac{\Phi^{n+1/2}-\Phi^{n-1/2}}{\Delta t} &= -\Phi_0\left(\frac{\partial u^n}{\partial x}+\frac{\partial v^n}{\partial y}\right), && (27)
\end{aligned}$$

and

$$\left[1-\alpha^2\nabla^2\right]\left(\widetilde{\Phi}^{n+1/2}-\Phi^{n+1/2}\right) = \alpha^2\left(\nabla^2\Phi^{n+1/2} - f\widehat{\zeta}^{n+1/2}\right), \tag{28}$$

where

$$\widehat{\zeta}^{n+1/2} \equiv \zeta^n - \frac{\Delta t f}{2}D^n \tag{29}$$

is an explicit midpoint approximation (predictor) to the vorticity, ζ, obtained from a discrete form of (9).

3.2 Linear equivalence to semi-implicit method

Equivalence between the semi-implicit method applied to the linear equations (5)-(7) and the staggered time discretization (25)-(28) has been shown by Wood et al. (2006) for the particular choice

$$\alpha^2 = \left(1+\frac{\Delta t^2 f^2}{4}\right)^{-1}\frac{\Delta t^2 \Phi_0}{4}. \tag{30}$$

We now provide an alternative derivation of this result. The semi-implicit method applied to (5)-(7) yields

$$\begin{aligned}
\frac{u^{n+1}-u^n}{\Delta t} - f v^{n+1/2} &= -\frac{\partial \Phi^{n+1/2}}{\partial x}, && (31)\\
\frac{v^{n+1}-v^n}{\Delta t} + f u^{n+1/2} &= -\frac{\partial \Phi^{n+1/2}}{\partial y}, && (32)\\
\frac{\Phi^{n+1}-\Phi^{n}}{\Delta t} &= -\Phi_0\left(\frac{\partial u^{n+1/2}}{\partial x}+\frac{\partial v^{n+1/2}}{\partial y}\right), && (33)
\end{aligned}$$

where midpoint values are defined, for general X, by

$$X^{n+1/2} \equiv \frac{1}{2}\left(X^{n+1}+X^n\right). \tag{34}$$

From these equations it follows that

$$\begin{aligned}
D^{n+1/2} &= D^n + \frac{\Delta t f}{2}\zeta^{n+1/2} - \frac{\Delta t}{2}\nabla^2\Phi^{n+1/2}, && (35)\\
\zeta^{n+1/2} &= \zeta^n - \frac{\Delta t f}{2}D^{n+1/2}, && (36)\\
\Phi^{n+1/2} &= \Phi^n - \frac{\Delta t \Phi_0}{2}D^{n+1/2}. && (37)
\end{aligned}$$

We combine (35) and (36) to obtain

$$\left[1+\frac{\Delta t^2 f^2}{4}\right]D^{n+1/2} = D^n + \frac{\Delta t f}{2}\zeta^n - \frac{\Delta t}{2}\nabla^2\Phi^{n+1/2}. \tag{38}$$

We now use (37) to eliminate $D^{n+1/2}$ from (38). Rearranging terms finally yields

$$\left[1+\frac{\Delta t^2 f^2}{4}-\frac{\Delta t^2\Phi_0}{4}\nabla^2\right]\Phi^{n+1/2}=\left[1+\frac{\Delta t^2 f^2}{4}\right]\widehat{\Phi}^{n+1/2}-\frac{\Delta t^2\Phi_0}{4}f\widehat{\zeta}^{n+1/2}. \tag{39}$$

Here $\widehat{\zeta}^{n+1/2}$ is defined by (29) and

$$\widehat{\Phi}^{n+1/2}\equiv\Phi^n-\frac{\Delta t\Phi_0}{2}D^n \tag{40}$$

is the explicit midpoint approximation (predictor) to the geopotential, Φ, obtained by approximating $D^{n+1/2}$ in (37) by D^n. Furthermore, the continuity equation (33) becomes equivalent to

$$\frac{\widehat{\Phi}^{n+1/2}-\widehat{\Phi}^{n-1/2}}{\Delta t}=-\Phi_0 D^n. \tag{41}$$

Hence, we have shown that the semi-implicit method (31)-(33) is algebraically equivalent to a staggered formulation consisting of (31)-(32), (41), and (39) when the identification (34) is made. This formulation in turn is equivalent to (25)-(28) under the identification $\widehat{\Phi}^{n+1/2}\to\Phi^{n+1/2}$, $\Phi^{n+1/2}\to\widetilde{\Phi}^{n+1/2}$, and α given by (30).

4 The staggered scheme for nonlinear SWEs

Numerical implementations of the regularized equations for the fully nonlinear SWEs have been proposed by Staniforth et al. (2006) and Reich (2006). It is the purpose of this section to investigate the relation between a semi-implicit method and the regularized staggered time-stepping method in more detail. For simplicity of presentation we assume that advection terms can be treated in a grid-based (non-SL) manner and the semi-implicit method will be analyzed from an Eulerian perspective.

4.1 A semi-implicit method

We repeat the analysis of §3.2 and consider a simple version of the semi-implicit method of the form

$$\frac{u^{n+1}-u^n}{\Delta t}+\mathcal{A}(u^n)-fv^{n+1/2} = -\frac{\partial\Phi^{n+1/2}}{\partial x}, \tag{42}$$

$$\frac{v^{n+1}-v^n}{\Delta t}+\mathcal{A}(v^n)+fu^{n+1/2} = -\frac{\partial\Phi^{n+1/2}}{\partial y}, \tag{43}$$

$$\frac{\Phi^{n+1}-\Phi^n}{\Delta t}+\mathcal{A}(\Phi^n) = -\Phi_0\left(\frac{\partial u^{n+1/2}}{\partial x}+\frac{\partial v^{n+1/2}}{\partial y}\right)-(\Phi^n-\Phi_0)\left(\frac{\partial u^n}{\partial x}+\frac{\partial v^n}{\partial y}\right). \tag{44}$$

This method is first-order for any choice of the linearly implicit part and second-order when the linearization is performed about the current (time-level t_n) state (see, e.g., Hundsdorfer & Verwer (2003)). The specific nature of the spatial approximation is not relevant for the subsequent discussion and we formally treat spatial variables as continuous.

Using the midpoint approximation (34) it follows from equations (42)-(44) that

$$D^{n+1/2} = D^n+\frac{\Delta tf}{2}\zeta^{n+1/2}-\frac{\Delta t}{2}\nabla^2\Phi^{n+1/2}-\frac{\Delta t}{2}\left[\frac{\partial}{\partial x}\mathcal{A}(u^n)+\frac{\partial}{\partial y}\mathcal{A}(v^n)\right], \tag{45}$$

$$\zeta^{n+1/2} = \zeta^n-\frac{\Delta tf}{2}D^{n+1/2}-\frac{\Delta t}{2}\left[\frac{\partial}{\partial x}\mathcal{A}(v^n)-\frac{\partial}{\partial y}\mathcal{A}(u^n)\right], \tag{46}$$

$$\Phi^{n+1/2} = \Phi^n-\frac{\Delta t\Phi_0}{2}D^{n+1/2}-\frac{\Delta t}{2}\left[(\Phi^n-\Phi_0)D^n+\mathcal{A}(\Phi^n)\right]. \tag{47}$$

We combine (45) and (46) to obtain

$$\left[1+\frac{\Delta t^2 f^2}{4}\right] D^{n+1/2} = D^n + \frac{\Delta t f}{2}\zeta^n - \frac{\Delta t}{2}\nabla^2\Phi^{n+1/2} - \frac{\Delta t}{2}\left[\frac{\partial}{\partial x}\mathcal{A}(u^n) + \frac{\partial}{\partial y}\mathcal{A}(v^n)\right] - \frac{\Delta t^2 f}{4}\left[\frac{\partial}{\partial x}\mathcal{A}(v^n) - \frac{\partial}{\partial y}\mathcal{A}(u^n)\right]. \tag{48}$$

We now apply (47) to eliminate $D^{n+1/2}$ from (48). Rearranging terms finally yields

$$\left[1+\frac{\Delta t^2 f^2}{4} - \frac{\Delta t^2 \Phi_0}{4}\nabla^2\right]\Phi^{n+1/2} = \left[1+\frac{\Delta t^2 f^2}{4}\right]\widehat{\Phi}^{n+1/2} - \frac{\Delta t^2\Phi_0}{4}\left\{f\widehat{\zeta}^{n+1/2} - \frac{\partial\mathcal{A}(u^n)}{\partial x} - \frac{\partial\mathcal{A}(v^n)}{\partial y}\right\}. \tag{49}$$

Here $\widehat{\Phi}^{n+1/2}$ is now the explicit midpoint approximation (predictor) defined by

$$\widehat{\Phi}^{n+1/2} \equiv \Phi^n - \frac{\Delta t}{2}\left[\Phi^n D^n + \mathcal{A}(\Phi^n)\right] \tag{50}$$

and, similarly,

$$\widehat{\zeta}^{n+1/2} = \zeta^n - \frac{\Delta t}{2}\left\{f D^n + \left[\frac{\partial}{\partial x}\mathcal{A}(v^n) - \frac{\partial}{\partial y}\mathcal{A}(u^n)\right]\right\}. \tag{51}$$

These approximations are obtained from a forward Euler discretization of (1)-(3) over half a time-step.

4.2 Regularization

We now turn (49) into a regularization procedure for the staggered approach. We rewrite (49) as

$$\left[1-\alpha^2\nabla^2\right]\left(\Phi^{n+1/2} - \widehat{\Phi}^{n+1/2}\right) = \alpha^2\left\{\nabla^2\widehat{\Phi}^{n+1/2} - f\widehat{\zeta}^{n+1/2} + \left[\frac{\partial}{\partial x}\mathcal{A}(u^n) + \frac{\partial}{\partial y}\mathcal{A}(v^n)\right]\right\}, \tag{52}$$

where α is given by (30).

Our analysis of the semi-implicit method (42)-(44) and the results from §3.2 suggest that (21) should be modified to

$$\left[1-\alpha^2\nabla^2\right]\left(\widetilde{\Phi} - \Phi\right) = \alpha^2\left(\nabla^2\Phi - f\zeta + \left\{\frac{\partial}{\partial x}\mathcal{A}(u) + \frac{\partial}{\partial y}\mathcal{A}(v)\right\}\right), \tag{53}$$

where α is still given by (30). Scaling arguments, as used in §2.2, imply that this expression can be simplified further to obtain

$$\left[1-\alpha^2\nabla^2\right]\left(\widetilde{\Phi} - \Phi\right) = \alpha^2\left(\nabla^2\Phi - f\zeta - 2J(u,v)\right). \tag{54}$$

We conclude that $\widetilde{\Phi} = \Phi$ under the nonlinear balance relation (17). It is now also transparent that (54) is a proper "nonlinear" generalization of (21).

4.3 A numerical implementation of nonlinear balance

The methods of Staniforth et al. (2006) and Reich (2006) can be used as published with linear balance replaced by nonlinear balance. The only changes required come from a necessary approximation of the Jacobian $J(u,v)$. Following the corresponding approximation (52) for the semi-implicit method, such an approximation can be found by using current (time-level t_n) velocity values and direct evaluation of $J(u^n, v^n)$. This is the approach used for the numerical experiments of §5. Alternatively, one can make use of the approximation

$$\begin{aligned} J(u,v) &= \frac{1}{2}\left\{D^2 + \mathcal{A}(D) - \left[\frac{\partial}{\partial x}\mathcal{A}(u) + \frac{\partial}{\partial y}\mathcal{A}(v)\right]\right\} \\ &\approx \frac{1}{2}\left\{\mathcal{A}(D) - \left[\frac{\partial}{\partial x}\mathcal{A}(u) + \frac{\partial}{\partial y}\mathcal{A}(v)\right]\right\}. \end{aligned} \tag{55}$$

The RHS of (55) can be evaluated conveniently by a semi-Lagrangian approximation (see, e.g., Staniforth & Coté (1991) or Durran (1998)).

We stress that any staggered semi-Lagrangian implementation will *not* be entirely equivalent to a semi-implicit semi-Lagrangian implementation. However it is expected that nonlinear balance will reduce the level of spurious gravity wave activities. This aspect will now be assessed by a simple numerical experiment.

5 Numerical experiment

The semi-Lagrangian regularized time-staggered method of Reich (2006) is implemented for both linear and nonlinear balance. The spatial discretization uses the Arakawa C-grid over a doubly periodic domain with $L_x = L_y = 3840$ km (see Staniforth et al. (2006) for specific implementation details). The grid size is $\Delta x = \Delta y = 60$ km. The time step is $\Delta t = 20$ min and the value of f corresponds to an f-plane at 14.5^o latitude. The reference geopotential is set to $\Phi_0 = 9.50 \times 10^4$ m^2 s^{-2}. The resulting smoothing length, determined by (30), satisfies $\alpha \approx 3.3\,\Delta x$. The Rossby radius of deformation is $L_R \equiv \sqrt{\Phi_0}/f \approx 8500$ km. The initial configuration consists of two counterclockwise rotating vortices with a spatial length-scale of $L \approx 500$ km. The maximum initial wind speed is approximately 30 m s^{-1}. The initial wind fields and the initial geopotential, Φ, satisfy the nonlinear balance relation (17). This implies $\widetilde{\Phi} = \Phi$ at initial time for the regularized method with the improved balance relation (54) while $\widetilde{\Phi} \neq \Phi$ under linear balance (see Figure 1). Results are compared to a semi-implicit semi-Lagrangian scheme over a time period of two days with identical initial wind field and geopotential. Numerical results for the geopotential fields at initial time and after two days can be found in Figure 1 for all three methods. Displayed is the regularized geopotential, $\tilde{\Phi}$, for the case of the regularized time-staggered discretization methods. It is evident that linear balance leads to relatively large amplitude deviations in the regularized geopotential, $\tilde{\Phi}$, while both the semi-implicit semi-Lagrangian method and the regularized method with nonlinear balance lead to nearly identical geopotential fields.

The effect of nonlinear balance should become less pronounced for increasing values of the parameter $F = (L/L_R)^2$ (Haltiner, 1971; Pedlosky, 1987). To verify this statement we conduct a sequence of experiments with identical initial conditions except for changing the value of the reference geopotential Φ_0. Specifically, given the initial velocity field (u,v) and the initial geopotential Φ with reference value Φ_0 as described above, we introduce $\Phi_0' = \kappa\Phi_0$, $\kappa = 1, 1/4, 1/9, 1/16$, and define associated initial geopotentials

$$\Phi_\kappa = \Phi - \Phi_0 + \Phi_0'. \tag{56}$$

For each value of κ we compute a reference solution $\Phi^R(t)$ using the SISL implementation with $\Phi^R = \Phi_\kappa$ at initial time and compare it to simulations of the regularized equations with linear and nonlinear balance again using $\Phi = \Phi_\kappa$ at initial time. We compute the following normalized l_2-norm

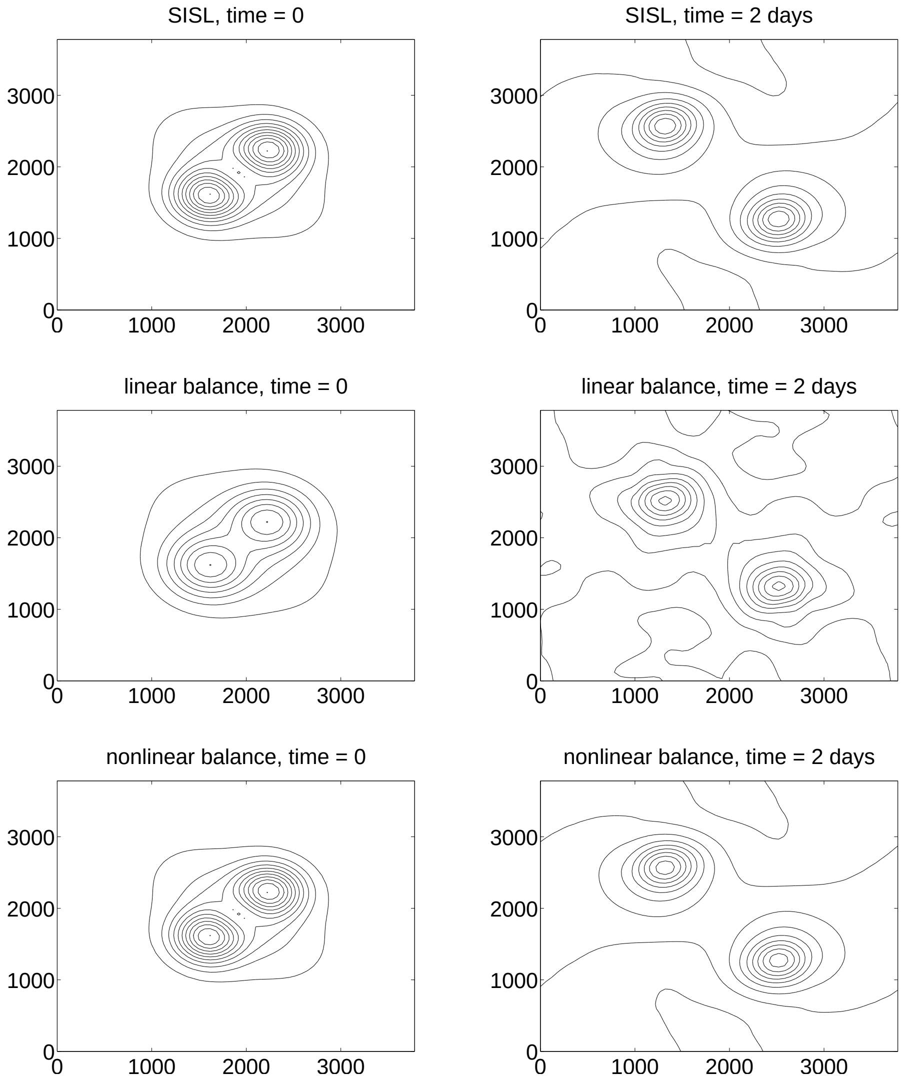

Figure 1: Computed time evolution of geopotential fields over the domain $(x, y) \in [0, 3840\,\text{km}] \times [0, 3840\,\text{km}]$ using (a) a semi-implicit semi-Lagrangian method, (b) a regularized time-staggered method with linear balance and (c) a regularized time-staggered method with nonlinear balance. For (b) and (c) the regularized geopotential $\widetilde{\Phi}$ is displayed. All methods use a timestep $\Delta t = 20\,\text{min}$. Contours plotted between $9.35 \times 10^4\,\text{m}^2\text{s}^{-2}$ and $9.50 \times 10^4\,\text{m}^2\text{s}^{-2}$ with contour interval $1.50 \times 10^2\,\text{m}^2\text{s}^{-2}$.

of the difference between $\Phi(t)$ and $\widetilde{\Phi}(t)$ over the grid:

$$\text{Res}(t) = \frac{\left\{\sum_{i,j}\left[\widetilde{\Phi}_{i+1/2,j+1/2}(t) - \Phi_{i+1/2,j+1/2}(t)\right]^2\right\}^{1/2}}{\left\{\sum_{i,j}\left[\Phi^R_{i+1/2,j+1/2}(0) - \Phi'_0\right]^2\right\}^{1/2}} \tag{57}$$

and the normalized l_2-norm of the difference between $\Phi(t)$ and the reference solution $\Phi^R(t)$:

$$\text{Diff}(t) = \frac{\left\{\sum_{i,j}\left[\Phi^R_{i+1/2,j+1/2}(t) - \Phi_{i+1/2,j+1/2}(t)\right]^2\right\}^{1/2}}{\left\{\sum_{i,j}\left[\Phi^R_{i+1/2,j+1/2}(0) - \Phi'_0\right]^2\right\}^{1/2}}. \tag{58}$$

The results can be found in Table 1 for $t = 0$ and $t = 2$ days. Note that the denominator in (57) and (58), respectively, is the same for all simulations and, hence, it becomes a constant normalization

κ	linear balance Res(0)	Res(2)	Diff(2)	nonlinear balance Res(0)	Res(2)	Diff(2)
1	0.2215	0.5512	0.6104	5.6832e-05	0.0332	0.0411
1/4	0.1047	0.1219	0.1536	1.6157e-05	0.0056	0.0181
1/9	0.0573	0.0523	0.0813	4.5814e-06	0.0035	0.0200
1/16	0.0352	0.0291	0.0587	3.0460e-06	0.0028	0.0169

Table 1: Difference in geopotentials, as measured by (57) and (58), for simulations using linear and nonlinear balance as a function of the parameter κ with the parameter $F = (L/L_R)^2$ being proportional to $1/\kappa$.

factor. It can be seen that the solutions with nonlinear balance are always closer to the reference solution $\Phi^R(t)$ as provided by the SISL scheme. However, the difference between linear and nonlinear balance diminishes for decreasing values of κ (and, hence, for increasing values of $F = (L/L_R)^2 \propto 1/\kappa$).

6 Conclusion

Based on an analysis of a semi-implicit method a further (inexpensive) refinement of the regularization procedure has been suggested. The relation $\widetilde{\Phi} = \Phi$ is now also maintained under nonlinear balance (17). The dramatic improvement on the geopotential fields has been demonstrated numerically for a simple nonlinear shallow-water problem.

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