# Analysis of a regularized, time-staggered discretization applied to a vertical slice model*

MARK DUBAL†, ANDREW STANIFORTH, NIGEL WOOD,
*Met Office, Exeter, U.K.*
and
SEBASTIAN REICH
*University of Potsdam, Germany*

16th August 2006

**Abstract**

A regularized and time-staggered discretization of the two-dimensional, vertical slice Euler equation set is described and analysed. A linear normal mode analysis of the time-discrete system indicates that unconditional stability is obtained, for appropriate values of the regularization parameters, for both the hydrostatic and non-hydrostatic cases. Furthermore, when these parameters take their optimal values, the stability behaviour of the normal modes is identical to that obtained from a semi-implicit discretization of the unregularized equations.

KEYWORDS: Euler equations; normal modes; dispersion; numerical stability; semi-implicit

SHORT TITLE: Analysis of a regularized, time-staggered vertical slice model

## 1 Introduction

A time-staggered discretization, combined with a regularization of the continuous governing equations, has recently been proposed as a solution method for the shallow water equations (Frank et al. 2005). It was shown by linear analysis that, for an appropriate regularization of the geopotential field, unconditional stability is obtained. An improved regularization procedure, which additionally preserves any balance present in the original unregularized equations, is described in Wood et al. (2006). This work was extended further to also include the effects of advection and spatial discretization on an Arakawa C-grid (Staniforth



†Corresponding author: Met Office, FitzRoy Road, Exeter, Devon EX1 3PB UK. e-mail: mark.dubal@metoffice.gov.uk

et al. 2006, Reich 2006). Linear analysis (Staniforth et al. 2006) showed that the regularized, time-staggered discretization remains unconditionally stable if the regularization parameters are chosen in the same way as for the non-advecting case.

Here the regularized, time-staggered discretization procedure is applied to the two-dimensional, vertical slice Euler equations. Whilst the shallow water studies, listed above, demonstrate the validity of the method for handling fast gravity waves, analysis of the vertical slice equations enables the handling of combined fast gravity and acoustic waves to be examined.

Following a presentation of the continuous vertical slice equations in section 2, they are written in a regularized form in section 3. The regularized equations are then linearized in section 4 and a normal mode analysis performed in section 5, in order to determine the dispersion effects of the regularization procedure. In section 6 the regularized and linearized equations are discretized in time using the time-staggered approach (the spatial representation remains continuous). A stability analysis of the linear normal modes of the equation set is then performed. Both hydrostatic and non-hydrostatic cases are considered. This analysis indicates that unconditional stability is assured, in either case, for appropriate choices of the regularization parameters. In section 7 a stability analysis of the unregularized equations, using a semi-implicit time-discretization, is performed and the results compared with those of section 6. Conclusions are presented in section 8.

# 2 The continuous governing equations

The continuous set of governing equations is written in Cartesian coordinates. The fully compressible, inviscid vertical slice $(x - z)$ Euler equation set, in the absence of diabatic forcing, is,

$$\frac{Du}{Dt} + c_p\theta\frac{\partial\pi}{\partial x} = 0, \tag{2.1}$$

$$\frac{Dv}{Dt} = 0, \tag{2.2}$$

$$\delta_V\frac{Dw}{Dt} + c_p\theta\frac{\partial\pi}{\partial z} + g = 0, \tag{2.3}$$

$$\left(\frac{1-\kappa}{\kappa}\right)\frac{D\pi}{Dt} + \pi\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 0, \tag{2.4}$$

$$\frac{D\theta}{Dt} = 0, \tag{2.5}$$

with the equation of state

$$\pi^{(1-\kappa)/\kappa} = \frac{R}{p_{00}}\rho\theta. \tag{2.6}$$

A non-hydrostatic/hydrostatic switch has been introduced. The equation set is fully compressible when $\delta_V = 1$, but reduces to the hydrostatic primitive equation set when $\delta_V = 0$.

In the above $u$, $v$, $w$ are components of the velocity in the $x$, $y$ and $z$-directions respectively, $\pi$ is the Exner pressure defined as $\pi \equiv (p/p_{00})^\kappa$, where $p$ is the pressure and $p_{00}$ is a reference pressure, $\theta$ is the potential temperature, $g$ is the acceleration due to gravity (assumed constant), $c_p$ is the specific heat at constant pressure for dry air, $\kappa \equiv R/c_p$ and $R$ is the gas constant for dry air. The material time derivative is,

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}. \tag{2.7}$$

Eqs. (2.1)-(2.6) are solved subject to appropriate boundary conditions. Herein, these are assumed to be periodicity in the horizontal together with rigid top and bottom boundaries, such that $w$ vanishes there. A consequence of the vertical boundary conditions is that $\pi$ is in hydrostatic balance at the top and bottom boundaries.

# 3 The regularized equations

The regularization of (2.1)-(2.5) is performed about a motionless, hydrostatically balanced basic state with potential temperature $\theta^* (z)$ and Exner pressure $\pi^* (z)$, so that

$$c_p\theta^* \frac{d\pi^*}{dz} + g = 0. \tag{3.1}$$

The regularized equations for the system are written as,

$$\frac{Du}{Dt} + c_p\theta\frac{\partial\tilde{\pi}}{\partial x} = 0, \tag{3.2}$$

$$\frac{Dv}{Dt} = 0, \tag{3.3}$$

$$\delta_V\frac{Dw}{Dt} + \frac{1}{1+\alpha^2}\left(c_p\theta\frac{\partial\tilde{\pi}}{\partial z} + g\right) = 0, \tag{3.4}$$

$$\frac{D\pi}{Dt} + \pi\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 0, \tag{3.5}$$

$$\frac{D\theta}{Dt} = 0, \tag{3.6}$$

together with the definition of the regularized Exner function, $\tilde{\pi}$:

$$\left\{1 - \left[\beta_x^2\frac{\partial^2}{\partial x^2} + \frac{\beta_z^2}{\theta^*}\left\{\frac{\partial}{\partial z} + \left(\frac{1-\kappa}{\kappa}\right)\frac{1}{\pi^*}\frac{d\pi^*}{dz}\right\}\frac{\theta^*}{1+\alpha^2}\frac{\partial}{\partial z}\right]\right\}(\tilde{\pi} - \pi) = -\frac{R_{\tilde{\pi}}}{c_p\theta^*}. \tag{3.7}$$

In (3.7) the quantities $\alpha$, $\beta_x$ and $\beta_z$ are regularization parameters, assumed real, and

$$R_{\tilde{\pi}} \equiv \beta_x^2\frac{\partial}{\partial x}R_u + \beta_z^2\left[\frac{\partial}{\partial z} + \left(\frac{1-\kappa}{\kappa}\right)\frac{1}{\pi^*}\frac{d\pi^*}{dz}\right]\left(\frac{R_w}{1+\alpha^2}\right), \tag{3.8}$$

with

$$R_u \equiv -c_p\theta^*\frac{\partial\pi}{\partial x}, \tag{3.9}$$

and

$$R_w \equiv -\frac{\theta^*}{\theta}\left(c_p\theta\frac{\partial\pi}{\partial z} + g\right). \tag{3.10}$$

Eq. (3.7) is solved subject to boundary conditions consistent with those applied to the unregularized equation set (2.1)-(2.6). As noted above, these are periodicity in the horizontal together with rigid top and bottom boundaries, such that $w$ vanishes there. A consequence of the vertical boundary conditions is that $\tilde{\pi}$ is also in hydrostatic balance at the top and bottom boundaries. Therefore, the boundary conditions for (3.7) are periodicity in the horizontal and $\partial\left(\tilde{\pi}-\pi\right)/\partial z=0$ at the top and bottom boundaries.

The definition of $\tilde{\pi}$ given by (3.7) is not unique for the non-linear case. However, the form presented is consistent with the normal mode structure of the underlying equations (Thuburn et al. 2002, Davies et al. 2003). Moreover, note that the form (3.7)-(3.10) has the important property that it maintains any motionless, horizontally homogeneous, hydrostatically balanced state, since then $R_u=R_w=0$ and therefore $\widetilde{\pi}=\pi$.

# 4 Linearization of the regularized equations

Now linearize the regularized equations (3.2)-(3.7) about the same stationary, motionless, hydrostatic basic state as in section 3 by writing $\theta=\theta^*+\theta'$, $\pi=\pi^*+\pi'$ and $\widetilde{\pi}=\pi^*+\widetilde{\pi}'$. Dropping primes for convenience, the linearized equations are then,

$$\frac{\partial u}{\partial t}+c_p\theta^*\frac{\partial\widetilde{\pi}}{\partial x}=0, \tag{4.1}$$

$$\frac{\partial v}{\partial t}=0, \tag{4.2}$$

$$\delta_V\frac{\partial w}{\partial t}+\frac{1}{1+\alpha^2}\left(c_p\theta^*\frac{\partial\widetilde{\pi}}{\partial z}-g\frac{\theta}{\theta^*}\right)=0, \tag{4.3}$$

$$\frac{\partial\pi}{\partial t}+\pi^*\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+\frac{d\pi^*}{dz}w=0, \tag{4.4}$$

$$\frac{\partial\theta}{\partial t}+\frac{d\theta^*}{dz}w=0, \tag{4.5}$$

$$\begin{aligned}\left\{1-\left[\beta_x^2\frac{\partial^2}{\partial x^2}+\frac{\beta_z^2}{\theta^*}\left\{\frac{\partial}{\partial z}+\left(\frac{1-\kappa}{\kappa}\right)\frac{1}{\pi^*}\frac{d\pi^*}{dz}\right\}\frac{\theta^*}{1+\alpha^2}\frac{\partial}{\partial z}\right]\right\}\widetilde{\pi}\\ =\pi-\frac{\beta_z^2}{c_p\theta^*}\left[\frac{\partial}{\partial z}+\left(\frac{1-\kappa}{\kappa}\right)\frac{1}{\pi^*}\frac{d\pi^*}{dz}\right]\frac{g\theta}{\left(1+\alpha^2\right)\theta^*},\end{aligned} \tag{4.6}$$

where (3.1) has been used in (4.3).

# 5 Normal mode analysis

Normal modes of the time-continuous, linearized equations are examined in this section to prepare the way for the analysis, in section 6, of the time-staggered discrete equation set. For analytical tractability, rigid boundary conditions are assumed at the lower boundary

$z = 0$ and at the upper boundary $z = z_T =$ constant. Furthermore, choose the stationary basic state to be isothermal (this also greatly facilitates analytic tractability) so that,

$$\pi^* = \pi_S^* e^{-\kappa z/H}, \qquad \theta^* = \theta_S^* e^{\kappa z/H}, \tag{5.1}$$

where the subscript $S$ denotes evaluation at the surface, $H \equiv RT^*/g$ is the scale height of the atmosphere and the constant temperature is $T^* \equiv \pi^*\theta^*$. The normal modes of (4.1)-(4.5) for this isothermal basic state atmosphere and the assumed boundary conditions have been determined following section 3 of Thuburn et al. (2002) - see also Davies et al. (2003). They fall into two classes; external modes and internal modes.

## 5.1 External modes

In the case of external modes, seek solutions of the form (cf. (5.4)-(5.11) of Davies et al. (2003) with $\delta_A \equiv \delta_B \equiv 1$),

$$\begin{array}{rcl} u & = & u_0 \exp\left[i\left(k_x x + \omega t\right)\right] \exp\left(\frac{\kappa z}{H}\right), \\ v & = & 0, \\ \widetilde{\pi} & = & \widetilde{\pi}_0 \exp\left[i\left(k_x x + \omega t\right)\right], \\ \pi & = & \mathcal{B}^{-1}\widetilde{\pi}, \\ w & = & 0, \\ \theta & = & 0, \end{array} \tag{5.2}$$

where

$$\mathcal{B}^{-1} \equiv \left[1 - \beta_x^2 \frac{\partial^2}{\partial x^2} - \frac{\beta_z^2}{1+\alpha^2}\left(\frac{\partial^2}{\partial z^2} - 2\Gamma\frac{\partial}{\partial z}\right)\right], \tag{5.3}$$

and $\Gamma \equiv (1 - 2\kappa)/(2H)$.

Inserting (5.2) into (4.1)-(4.6) leads to,

$$u_0 = -\frac{c_p \theta_S^* k_x}{\omega}\widetilde{\pi}_0, \qquad \mathcal{B}^{-1} = \mathcal{B}_{ext}^{-1} \equiv 1 + \beta_x^2 k_x^2, \qquad \omega^2 = \mathcal{B}_{ext}\left(c_s^*\right)^2 k_x^2, \tag{5.4}$$

where $\left(c_s^*\right)^2 \equiv \kappa c_p T^*/(1-\kappa)$ is the square of the sound speed for the basic state. When $\beta_x^2 \equiv 0$ no regularization occurs since $\mathcal{B}_{ext}^{-1} = 1$ and so $\widetilde{\pi} = \pi$. It can be seen that, for a fixed value of $\beta_x^2 > 0$, the propagation of the external modes is increasingly slowed as the horizontal wave number, $k_x$, becomes larger. Therefore, in terms of accuracy, $\beta_x^2$ should have the smallest (positive) value consistent with the numerical stability of any discretization scheme combined with the regularization procedure (see section 6 below).

## 5.2 Internal modes

For the internal modes, seek solutions of the form (cf. (5.15)-(5.20) of Davies et al. (2003) with $\delta_B \equiv \delta_C \equiv \delta_D \equiv 1$),

$$\begin{aligned}
u &= u_0 \exp\left[i\left(k_x x + \omega t\right)\right]\left[\Gamma \sin\left(k_z z\right) - k_z \cos\left(k_z z\right)\right] \exp\left(\tfrac{z}{2H}\right), \\
v &= 0, \\
\widetilde{\pi} &= \widetilde{\pi}_0 \exp\left[i\left(k_x x + \omega t\right)\right]\left[\Gamma \sin\left(k_z z\right) - k_z \cos\left(k_z z\right)\right] \exp\left[\left(\tfrac{1-2\kappa}{2H}\right) z\right], \\
\pi &= \mathcal{B}^{-1}\widetilde{\pi}, \\
w &= w_0 \exp\left[i\left(k_x x + \omega t\right)\right] \sin\left(k_z z\right) \exp\left(\tfrac{z}{2H}\right), \\
\theta &= \theta_0 \exp\left[i\left(k_x x + \omega t\right)\right] \sin\left(k_z z\right) \exp\left[\left(\tfrac{1+2\kappa}{2H}\right) z\right].
\end{aligned} \tag{5.5}$$

Inserting (5.5) into (4.1)-(4.6) produces (cf. (5.21)-(5.25) of Davies et al. (2003) with $\delta_A \equiv 1$),

$$u_0 = -\frac{c_p \theta_S^* k_x}{\omega} \widetilde{\pi}_0, \tag{5.6}$$

$$w_0 = -\frac{i\omega c_p \theta_S^* \left(\Gamma^2 + k_z^2\right)}{\left(\mathcal{N}^*\right)^2 - \delta_V \left(1+\alpha^2\right)\omega^2} \widetilde{\pi}_0, \tag{5.7}$$

$$\theta_0 = \frac{\kappa c_p \left(\theta_S^*\right)^2 \left(\Gamma^2 + k_z^2\right)/H}{\left(\mathcal{N}^*\right)^2 - \delta_V \left(1+\alpha^2\right)\omega^2} \widetilde{\pi}_0, \tag{5.8}$$

$$\mathcal{B}^{-1} = \mathcal{B}_{int}^{-1} \equiv 1 + \beta_x^2 k_x^2 + \frac{\beta_z^2}{1+\alpha^2}\left(\Gamma^2 + k_z^2\right) \frac{\delta_V \left(1+\alpha^2\right)\omega^2}{\delta_V\left(1+\alpha^2\right)\omega^2 - \left(\mathcal{N}^*\right)^2}, \tag{5.9}$$

and the dispersion relation,

$$\begin{aligned}
\left[\delta_V \left(1+\alpha^2\right)\omega^2 - \left(\mathcal{N}^*\right)^2\right] \left\{\left[1 + \beta_x^2 k_x^2 + \frac{\beta_z^2}{1+\alpha^2}\left(\Gamma^2 + k_z^2\right)\right]\omega^2 - \left(c_s^*\right)^2 k_x^2\right\} \\
- \left(\Gamma^2 + k_z^2\right)\left[\left(c_s^*\right)^2 - \frac{\beta_z^2}{1+\alpha^2}\left(\mathcal{N}^*\right)^2\right]\omega^2 = 0,
\end{aligned} \tag{5.10}$$

where $\left(\mathcal{N}^*\right)^2 \equiv \kappa g^2 / \left(R T^*\right)$ is the square of the buoyancy frequency of the basic state.

### 5.2.1 Hydrostatic case, $\delta_V = 0$

When $\delta_V = 0$, the acoustic modes are eliminated (because (5.10) reduces from a quartic equation to a quadratic one, with two fewer solutions). Then (5.10) simplifies to,

$$\omega^2 = \frac{\left(\mathcal{N}^*\right)^2 k_x^2}{\left(\Gamma^2 + k_z^2\right) + \left(1 + \beta_x^2 k_x^2\right)\left(\mathcal{N}^*\right)^2 / \left(c_s^*\right)^2}. \tag{5.11}$$

It is clear from (5.11) that the modes are increasingly slowed, for a fixed $\beta_x^2 > 0$, as $k_x \to \infty$. Regularization in the vertical plays no role for the hydrostatic case.

### 5.2.2 Non-hydrostatic case, $\delta_V = 1$

If $\delta_V = 1$ there are four solutions to (5.10). It can be shown that the dispersion relation is then,

$$\omega^2 = \frac{\mathcal{K} \pm \sqrt{\mathcal{K}^2 - 4\mathcal{L}\left(\mathcal{N}^*\right)^2 \left(c_s^*\right)^2 k_x^2}}{2\mathcal{L}}, \tag{5.12}$$

where

$$\mathcal{K} \equiv \left[\left(1+\alpha^2\right) k_x^2 + \Gamma^2 + k_z^2\right] \left(c_s^*\right)^2 + \left(1 + \beta_x^2 k_x^2\right) \left(\mathcal{N}^*\right)^2, \tag{5.13}$$

$$\mathcal{L} \equiv \left(1+\alpha^2\right) \left[1 + \beta_x^2 k_x^2 + \frac{\beta_z^2}{1+\alpha^2} \left(\Gamma^2 + k_z^2\right)\right]. \tag{5.14}$$

The role of the regularization is to scale-selectively slow down the modes with the highest frequency. For the acoustic modes (the positive root in (5.12)) this occurs, for fixed regularization parameters, when $k_x \to \infty$ or $k_z \to \infty$ (and $k_z$ or $k_x$ fixed respectively). If $\omega_U$ denotes the unregularized frequency (obtained by setting $\alpha = \beta_x = \beta_z = 0$ in (5.13) and (5.14)), an examination of (5.12) indicates that $(\omega/\omega_U) \to 0$ when $k_x \to \infty$ or $k_z \to \infty$, i.e. the regularization increasingly slows the propagation of these fast acoustic modes. In the case of the gravity waves (the negative root in (5.12)), for a given $k_z$, the frequency increases as $k_x \to \infty$, the highest frequency occurring for the deepest internal mode. For this case $(\omega/\omega_U) \to 1/\left(1+\alpha^2\right)$, i.e. the high frequency modes are slowed and increasingly so as the regularization parameter $\alpha$ increases.

The smallest (positive) values of $\alpha^2$, $\beta_x^2$ and $\beta_z^2$, consistent with numerical stability, provide the most accurate solutions when the regularization procedure is applied in the context of the discretized equations, as in the following section.

# 6 Stability of the time-staggered discretization

## 6.1 Time-staggered discretization

The linearized, regularized equations, (4.1)-(4.6), with the assumption of an isothermal basic state (5.1), are discretized in a time-staggered manner, analogous to Frank et al. (2005), to produce,

$$\frac{u^{n+1} - u^n}{\Delta t} + c_p \theta^* \frac{\partial \tilde{\pi}^{n+1/2}}{\partial x} = 0, \tag{6.1}$$

$$\frac{v^{n+1} - v^n}{\Delta t} = 0, \tag{6.2}$$

$$\delta_V \left(1+\alpha^2\right) \frac{w^{n+1} - w^n}{\Delta t} + c_p \theta^* \frac{\partial \tilde{\pi}^{n+1/2}}{\partial z} - g \frac{\theta^{n+1/2}}{\theta^*} = 0, \tag{6.3}$$

$$\frac{\pi^{n+1/2} - \pi^{n-1/2}}{\Delta t} + \pi^* \left[\left(\frac{\kappa}{1-\kappa}\right) \left(\frac{\partial u^n}{\partial x} + \frac{\partial w^n}{\partial z}\right) - \frac{\kappa}{H} w^n\right] = 0, \tag{6.4}$$

$$\frac{\theta^{n+1/2} - \theta^{n-1/2}}{\Delta t} + \frac{\kappa}{H} \theta^* w^n = 0, \tag{6.5}$$

$$\left[1-\beta_x^2\frac{\partial^2}{\partial x^2}-\frac{\beta_z^2}{1+\alpha^2}\left(\frac{\partial^2}{\partial z^2}-2\Gamma\frac{\partial}{\partial z}\right)\right]\widetilde{\pi}^{n+1/2}=\pi^{n+1/2}-\frac{g\beta_z^2}{c_p\left(1+\alpha^2\right)\left(\theta^*\right)^2}\left(\frac{\partial}{\partial z}-\frac{1}{H}\right)\theta^{n+1/2}, \tag{6.6}$$

where $\Delta t$ is the time-step interval. The variables $u$, $v$ and $w$ are stored at integer temporal levels $n\Delta t$, while $\theta$ and $\pi$ are stored at half-integer levels $(n-1/2)\,\Delta t$. Given a state $\left(u^n, v^n, w^n, \theta^{n-1/2}, \pi^{n-1/2}\right)$ the system is advanced a time interval $\Delta t$ by:

- Using (6.4) and (6.5) to obtain $\pi^{n+1/2}$ and $\theta^{n+1/2}$ respectively.
- Solving (6.6) to obtain $\widetilde{\pi}^{n+1/2}$.
- Using (6.1)-(6.3) to obtain $u^{n+1}$, $v^{n+1}$ and $w^{n+1}$.

## 6.2 Stability of external modes

Since (6.1)-(6.6) keep the spatial representation continuous, the expansions (5.2) are valid solutions of this set of equations. Substituting (5.2) into (6.1)-(6.6) yields,

$$u_0\mathcal{S}=-\frac{\Delta t}{2}c_pk_x\theta_S^*\tilde{\pi}_0, \qquad \mathcal{B}_{ext}^{-1}\tilde{\pi}_0\mathcal{S}=-\frac{\Delta t}{2}k_x\pi_S^*\left(\frac{\kappa}{1-\kappa}\right)u_0, \qquad \mathcal{B}_{ext}=\frac{1}{1+\beta_x^2k_x^2}, \tag{6.7}$$

where $\mathcal{S}\equiv\sin\left(\omega\Delta t/2\right)$. From (6.7),

$$\mathcal{S}^2=\left(\frac{c_s^*\Delta t}{2}\right)^2\frac{k_x^2}{1+\beta_x^2k_x^2}. \tag{6.8}$$

Note that (6.8) is equivalent to the time-continuous expression (5.4) if $\omega$, in the latter, is replaced by its discrete analogue $2\mathcal{S}/\Delta t$.

For stable solutions $\mathcal{S}$ is required to be real and $-1\le\mathcal{S}\le 1$. Equivalently $0\le\mathcal{S}^2\le 1$, and then, using (6.8),

$$0\le\left(\frac{c_s^*\Delta t}{2}\right)^2k_x^2\le 1+\beta_x^2k_x^2. \tag{6.9}$$

Stability is obtained, independently of $\Delta t$, $c_s^*$ and $k_x$, if $\beta_x^2\ge\left(c_s^*\Delta t/2\right)^2$.

## 6.3 Stability of internal modes

Following a similar procedure to that used for the external modes leads to the corresponding discrete dispersion relation,

$$\left[\delta_V\left(1+\alpha^2\right)\left(\frac{2}{\Delta t}\right)^2\mathcal{S}^2-\left(\mathcal{N}^*\right)^2\right]\left\{\left[1+\beta_x^2k_x^2+\frac{\beta_z^2}{1+\alpha^2}\left(\Gamma^2+k_z^2\right)\right]\left(\frac{2}{\Delta t}\right)^2\mathcal{S}^2-\left(c_s^*\right)^2k_x^2\right\}$$
$$-\left(\Gamma^2+k_z^2\right)\left[\left(c_s^*\right)^2-\frac{\beta_z^2}{1+\alpha^2}\left(\mathcal{N}^*\right)^2\right]\left(\frac{2}{\Delta t}\right)^2\mathcal{S}^2=0. \tag{6.10}$$

Again an equivalence can be made between (6.10) and its time-continuous analogue (5.10) if $\omega$ is replaced by its discrete analogue $2\mathcal{S}/\Delta t$.

Since stable solutions require $\omega$ to be real, it is convenient to write (6.10) in terms of $\mathcal{T} \equiv \tan(\omega\Delta t/2)$, so that $\mathcal{S}^2 \equiv \mathcal{T}^2/(1+\mathcal{T}^2)$, and therefore

$$a\mathcal{T}^4 + b\mathcal{T}^2 + c = 0, \tag{6.11}$$

where

$$a \equiv pq - r, \qquad b \equiv -\left[p\left(c_s^*\right)^2 k_x^2 + q\left(\mathcal{N}^*\right)^2 + r\right], \qquad c \equiv \left(\mathcal{N}^*\right)^2\left(c_s^*\right)^2 k_x^2, \tag{6.12}$$

with

$$p \equiv \delta_V\left(1+\alpha^2\right)\left(\frac{2}{\Delta t}\right)^2 - \left(\mathcal{N}^*\right)^2, \tag{6.13}$$

$$q \equiv \left[1 + \beta_x^2 k_x^2 + \frac{\beta_z^2}{1+\alpha^2}\left(\Gamma^2 + k_z^2\right)\right]\left(\frac{2}{\Delta t}\right)^2 - \left(c_s^*\right)^2 k_x^2, \tag{6.14}$$

$$r \equiv \left(\Gamma^2 + k_z^2\right)\left[\left(c_s^*\right)^2 - \frac{\beta_z^2}{1+\alpha^2}\left(\mathcal{N}^*\right)^2\right]\left(\frac{2}{\Delta t}\right)^2. \tag{6.15}$$

### 6.3.1 Hydrostatic case, $\delta_V = 0$

If $\delta_V = 0$, then (6.10) may be manipulated to give,

$$-\left[q\left(\mathcal{N}^*\right)^2 + r\right]\mathcal{T}^2 + c = 0. \tag{6.16}$$

Stability is assured if $\mathcal{T}^2$ is both real and positive, i.e. it is assured for the hydrostatic internal modes if

$$\mathcal{T}^2 = \frac{c}{q\left(\mathcal{N}^*\right)^2 + r} \geq 0. \tag{6.17}$$

From above, $c \geq 0$ is always true and $q\left(\mathcal{N}^*\right)^2 + r \geq 0$ leads to,

$$1 + \beta_x^2 k_x^2 + \left(\frac{c_s^*}{\mathcal{N}^*}\right)^2\left(\Gamma^2 + k_z^2\right) \geq \left(\frac{c_s^*\Delta t}{2}\right)^2 k_x^2. \tag{6.18}$$

Condition (6.18) will be satisfied independently of $\Delta t$, $k_x$, $k_z$ and the basic state if,

$$\beta_x^2 \geq \left(\frac{c_s^*\Delta t}{2}\right)^2. \tag{6.19}$$

Note that there are no conditions on $\alpha$ or $\beta_z$, so that it is possible to set $\alpha^2 = \beta_z^2 = 0$, reducing (3.7) to a one-dimensional equation. This could provide efficiency advantages over the standard semi-implicit approach for a hydrostatic model.

### 6.3.2 Non-hydrostatic case, $\delta_V = 1$

For stability $\mathcal{T}^2$ is again required to be both real and positive.

If $\mathcal{T}^2$ is to be real, then $b^2 - 4ac \geq 0$ is required. From (6.12)-(6.15) it can be shown that

$$b^2 - 4ac = \left[p\left(c_s^*\right)^2 k_x^2 - q\left(\mathcal{N}^*\right)^2 + r\right]^2 + 4r\left(\mathcal{N}^*\right)^2\left[q + \left(c_s^*\right)^2 k_x^2\right]. \tag{6.20}$$

This is guaranteed to be positive if $r \geq 0$. From (6.15) this implies that

$$(c_s^*)^2 \left(1+\alpha^2\right) \geq \beta_z^2 \left(\mathcal{N}^*\right)^2, \tag{6.21}$$

which is satisfied, independently of the basic state, if

$$\alpha^2 \geq \beta_z^2 \left(\frac{\mathcal{N}^*}{c_s^*}\right)^2. \tag{6.22}$$

Since $c$ is guaranteed to be positive, $\mathcal{T}^2$ will additionally be positive (and stability will be assured) if $a \geq 0$ and $b \leq 0$. Condition (6.22) guarantees that $r \geq 0$. Then inspection of (6.12) shows that $b$ will be negative or zero if $p \geq 0$ and $q \geq 0$. Taking $\delta_V = 1$ these inequalities lead to,

$$1+\alpha^2 \geq \left(\frac{\mathcal{N}^* \Delta t}{2}\right)^2, \tag{6.23}$$

and (6.18), respectively. Thus,

$$\alpha^2 \geq \left(\frac{\mathcal{N}^* \Delta t}{2}\right)^2, \tag{6.24}$$

and (6.19) guarantee $b \leq 0$ for the non-hydrostatic case, independently of $\Delta t$, $k_x$, $k_z$, $\beta_z$ and the isothermal basic state. If $a \geq 0$ then $pq \geq r$, which implies,

$$p\left\{1+\left[\beta_x^2-\left(\frac{c_s^*\Delta t}{2}\right)^2\right]k_x^2\right\} \geq -\left(\Gamma^2+k_z^2\right)\left[\left(\frac{2}{\Delta t}\right)^2\beta_z^2-(c_s^*)^2\right]. \tag{6.25}$$

The LHS of (6.25) will be positive, given the previous conditions (6.19) and (6.24), while the RHS will be zero or negative if,

$$\beta_z^2 \geq \left(\frac{c_s^*\Delta t}{2}\right)^2. \tag{6.26}$$

## 6.4 Summary

Drawing together the above results, stable solutions of the linearized and regularized time-staggered scheme are obtained, in the hydrostatic case, if,

$$\beta_x^2 \geq \left(\frac{c_s^*\Delta t}{2}\right)^2, \tag{6.27}$$

(there are no conditions on $\beta_z$ and therefore $\alpha = \beta_z = 0$ satisfies (6.22)), whilst for the non-hydrostatic case the conditions,

$$\alpha^2 \geq \beta_z^2\left(\frac{\mathcal{N}^*}{c_s^*}\right)^2, \qquad \beta_x^2 \geq \left(\frac{c_s^*\Delta t}{2}\right)^2, \qquad \beta_z^2 \geq \left(\frac{c_s^*\Delta t}{2}\right)^2, \tag{6.28}$$

are required.

The values of the regularization parameters obtained by taking equality in (6.27) and (6.28) are optimal in the sense that they minimize the distortion of the equation set, due to regularization, whilst providing unconditional stability.

# 7 Comparison with a semi-implicit scheme

The system (2.1)-(2.5) may be linearized about the isothermal basic state (5.1) and differenced in a semi-implicit manner, e.g. Staniforth (1997), to give,

$$\frac{u^{n+1}-u^n}{\Delta t}+\frac{1}{2}c_p\theta^*\frac{\partial}{\partial x}\left(\pi^{n+1}+\pi^n\right)=0, \tag{7.1}$$

$$\frac{v^{n+1}-v^n}{\Delta t}=0, \tag{7.2}$$

$$\delta_V\frac{w^{n+1}-w^n}{\Delta t}+\frac{1}{2}c_p\theta^*\frac{\partial}{\partial z}\left(\pi^{n+1}+\pi^n\right)-\frac{1}{2}g\frac{\left(\theta^{n+1}+\theta^n\right)}{\theta^*}=0, \tag{7.3}$$

$$\frac{\pi^{n+1}-\pi^n}{\Delta t}+\frac{1}{2}\pi^*\left\{\left(\frac{\kappa}{1-\kappa}\right)\frac{\partial}{\partial x}\left(u^{n+1}+u^n\right)+\left[\left(\frac{\kappa}{1-\kappa}\right)\frac{\partial}{\partial z}-\frac{\kappa}{H}\right]\left(w^{n+1}+w^n\right)\right\}=0, \tag{7.4}$$

$$\frac{\theta^{n+1}-\theta^n}{\Delta t}+\frac{1}{2}\frac{\kappa}{H}\theta^*\left(w^{n+1}+w^n\right)=0. \tag{7.5}$$

## 7.1 External modes

Inserting the expansions (5.2) into (7.1)-(7.5), setting $\mathcal{B}_{ext}^{-1}\equiv 1$, and using the definition of $\mathcal{T}$, then leads to,

$$u_0\mathcal{T}=-\frac{\Delta t}{2}c_pk_x\theta_S^*\pi_0, \qquad \pi_0\mathcal{T}=-\frac{\Delta t}{2}k_x\pi_S^*\left(\frac{\kappa}{1-\kappa}\right)u_0. \tag{7.6}$$

From (7.6), the dispersion relation is,

$$\mathcal{T}^2=\left(\frac{c_s^*\Delta t}{2}\right)^2k_x^2. \tag{7.7}$$

Comparing (7.7) with (6.8), after use of the identity $\mathcal{S}^2\equiv\mathcal{T}^2/\left(1+\mathcal{T}^2\right)$, they are identical when $\beta_x^2=\left(c_s^*\Delta t/2\right)^2$. This is the optimal value (in the sense discussed in sub-section 6.4 above) for this parameter.

## 7.2 Internal modes

Inserting (5.5) into (7.1)-(7.5), setting $\mathcal{B}_{int}^{-1}\equiv 1$, and using the definition of $\mathcal{T}$, leads to,

$$u_0\mathcal{T}=-\frac{\Delta t}{2}c_pk_x\theta_S^*\pi_0, \tag{7.8}$$

$$\delta_Vw_0\mathcal{T}=-i\frac{\Delta t}{2}\left[g\frac{\theta_0}{\theta_S^*}-c_p\theta_S^*\left(\Gamma^2+k_z^2\right)\pi_0\right], \tag{7.9}$$

$$\left[\frac{2i}{\Delta t\pi_S^*}\left(\frac{1-\kappa}{\kappa}\right)\pi_0\left(\frac{\mathcal{S}}{\mathcal{C}}\right)+ik_xu_0-w_0\right]=0, \tag{7.10}$$

$$\theta_0\mathcal{T}=i\frac{\Delta t}{2}\frac{\kappa\theta_S^*}{H}w_0. \tag{7.11}$$

The system (7.8)-(7.11) may be combined to give the dispersion relation in the form of a quartic for $\mathcal{T}$;

$$a\mathcal{T}^4 + b\mathcal{T}^2 + c = 0, \tag{7.12}$$

where

$$a \equiv \delta_V \left(\frac{2}{\Delta t}\right)^4, \qquad b \equiv -\left(\frac{2}{\Delta t}\right)^2 \left[(c_s^*)^2 \left(\delta_V k_x^2 + \Gamma^2 + k_z^2\right) + (\mathcal{N}^*)^2\right], \qquad c \equiv (\mathcal{N}^*)^2 \,(c_s^*)^2\, k_x^2. \tag{7.13}$$

### 7.2.1 Hydrostatic case, $\delta_V = 0$

If $\delta_V = 0$ then the dispersion relation is,

$$\mathcal{T}^2 = \frac{c}{\left(\frac{2}{\Delta t}\right)^2 \left[(c_s^*)^2 \,(\Gamma^2 + k_z^2) + (\mathcal{N}^*)^2\right]}. \tag{7.14}$$

Note that (6.17) is identical to (7.14) when $\beta_x^2 = (c_s^* \Delta t/2)^2$.

### 7.2.2 Non-hydrostatic case, $\delta_V = 1$

Using (6.12)-(6.15) it is found that, if $\alpha^2 = (\mathcal{N}^* \Delta t/2)^2$ and $\beta_x^2 = \beta_z^2 = (c_s^* \Delta t/2)^2$ (their optimal values), then the coefficients (6.12) are identical to (7.13) for $\delta_V = 1$.

# 8 Conclusions

A regularized and time-staggered discretization procedure has been applied to the two-dimensional, vertical slice Euler equations. A linear normal mode stability analysis of the system shows that the scheme is unconditionally stable for appropriate choices of the regularization parameters $\alpha$, $\beta_x$ and $\beta_z$ for both hydrostatic and non-hydrostatic formulations. Furthermore, when these parameters take their optimal values, the stability behaviour of the normal modes is identical to that obtained from a semi-implicit discretization of the linear unregularized Euler equations.

For the hydrostatic case it is possible to set the regularization parameters $\alpha$ and $\beta_z$ to zero, whilst retaining unconditional stability. This allows for a much simplified solution of the regularized Exner pressure, $\widetilde{\pi}$, which could offer efficiency advantages over a standard semi-implicit scheme. When non-hydrostatic flows are considered the regularized scheme allows more control and flexibility of the form of the Helmholtz equation which must be solved. For the semi-implicit approach the Helmholtz equation must be derived from, and be consistent with, the discrete form of the governing equations. Thus, the regularized system is less tightly coupled than the semi-implicit one, which may also be advantageous.

The results presented here indicate the exciting possibility of applying the regularized, time-staggered approach to the full Euler equations. If, additionally, it can be successfully coupled to a semi-Lagrangian advection scheme, this would then lead to a viable unconditionally stable scheme for the solution of these equations.

# Acknowledgments

The authors gratefully acknowledge the thorough and helpful reviews of the two anonymous referees.

# References

Davies, T., Staniforth, A., Wood, N. & Thuburn, J. 2003 , Validity of anelastic and other equation sets as inferred from normal-mode analysis, *Q. J. R. Meteorol. Soc.* **129**, 2761–2775. DOI: 10.1256/qj.02.195.

Frank, J., Reich, S., Staniforth, A., White, A. & Wood, N. 2005 , Analysis of a regularized, time-staggered discretization metod and its link to the semi-implicit method, *Atmos. Sci. Letters* **6**, 97–104. DOI: 10.1002/asl.97.

Reich, S. 2006 , Linearly implicit timestepping methods for numerical weather prediction, *B.I.T.* to appear.

Staniforth, A. 1997 , André Robert (1929 - 1993): His Pioneering Contributions to Numerical Modelling, *in* C. A. Lin, R. Laprise & H. Ritchie, eds, 'Numerical Methods in Atmospheric and Oceanic Modelling, The André J. Robert memorial volume', CMOS/ NRC Press, Ottawa, pp. 25–54.

Staniforth, A., Wood, N. & Reich, S. 2006 , A time-staggered semi-Lagrangian discretization of the rotating shallow-water equations, *Q. J. R. Meteorol. Soc.* (submitted).

Thuburn, J., Wood, N. & Staniforth, A. 2002 , Normal modes of deep atmospheres. II: $f - F-$plane geometry, *Q. J. R. Meteorol. Soc.* **128**, 1793–1806.

Wood, N., Staniforth, A. & Reich, S. 2006 , An improved regularization for time-staggered discretization and its link to the semi-implicit method, *Atmos. Sci. Letters* **7**, 21–25. DOI: 10.1002/asl.124.