An improved regularization for time-staggered discretization and its link to the semi-implicit method*

NIGEL WOOD,† ANDREW STANIFORTH,
Met Office, Exeter, UK
SEBASTIAN REICH
University of Potsdam, Germany

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Abstract

A regularized time-staggered discretization of the shallow-water equations has recently been proposed. Here, a new form of the regularization operator is presented. This form addresses a weakness in the original formulation so that now the discretization preserves the analytic forced response. Since the reformulation takes account of the forcing terms in the equations, if the unregularized equations are in balance, in the sense that $\nabla.\left(D\mathbf{u}/Dt\right)$ vanishes, then the regularized equations maintain this balance.

1 Introduction

Based on the Hamiltonian Particle-Mesh method, Frank et al. (2005) (henceforth referred to as F05) proposed a regularization of the orographically forced shallow-water equations (SWEs). The effect of the regularization is governed by two parameters: α, which is a smoothing length scale; and γ, which is a further smoothing parameter.

Among other results, linear analysis of the continuous equations showed that the forced response of the regularized equations is close to that of the unregularized equations provided γ is chosen so that $\gamma \ll 1$ and α is chosen to be much smaller than the Rossby radius of deformation, i.e., $\alpha^2 \ll L_R^2$. The regularized equations were then discretized using an explicit, time-staggered leapfrog scheme. Again, linear analysis of the discrete equation set showed that the regularized, time-staggered leapfrog discretization only yields a similar result to the analytic forced response when $(\alpha/L_R)^2 \ll 1$ and $\gamma^2 \ll 1$. (In contrast the semi-implicit discretization yields the exact analytic forced response.) Additionally, the discussion of the forced response was in terms of the *regularized* depth field. Further details of the scheme and its advantages can be found in F05.

Here a further development of the regularization procedure is proposed which addresses the shortcomings of the forced response of the regularized equations (both the continuous and the explicit, time-staggered discrete equations). The resulting forced response, now in terms of the *unregularized* depth field, maintains exactly the analytic response. The new scheme has the advantage over the original one, that the regularization only impacts the *unbalanced* components of the flow.

The structure of the paper is similar to that of F05. The new regularization procedure for both the continuous equations and the discrete ones is set out in section 2. Those equations are linearized in section 3 enabling the effect of the regularization on the continuous equations to be analyzed in section 4. In section 5 the discrete equations are similarly analyzed and compared with results for the semi-implicit scheme. Conclusions are drawn in section 6.

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†Corresponding author: Met Office, FitzRoy Road, Exeter, Devon EX1 3PB, UK. e-mail: nigel.wood@metoffice.gov.uk

2   The new regularization procedure applied to the SWEs

The orographically forced SWEs on an f-plane are

$$\frac{Du}{Dt} = +fv - g\mu_x - g\mu_x^S, \tag{2.1}$$

$$\frac{Dv}{Dt} = -fu - g\mu_y - g\mu_y^S, \tag{2.2}$$

$$\frac{D\mu}{Dt} = -\mu\left(u_x + v_y\right). \tag{2.3}$$

Here $\mu^S = \mu^S(x, y)$ is the height of the orography above mean sea level and $\mu = \mu(x, y, t)$ is the fluid depth, i.e., the depth of the fluid between the orography and the fluid's free surface. Also, g is the gravitational constant, f is twice the (constant) angular velocity of the reference plane,

$$\frac{D}{Dt}(.) = (.)_t + u\,(.)_x + v\,(.)_y\,, \tag{2.4}$$

is the Lagrangian or material time derivative, and subscripts denote partial differentiation with respect to that variable.

Obtaining the regularized time-staggered leapfrog equations consists of two steps.

First, the unregularized depth, μ, in (2.1) and (2.2) is replaced by the regularized depth field h:

$$\frac{Du}{Dt} = +fv - gh_x - g\mu_x^S, \tag{2.5}$$

$$\frac{Dv}{Dt} = -fu - gh_y - g\mu_y^S, \tag{2.6}$$

where h is determined from the Helmholtz equation

$$\left(1 - \alpha^2\nabla^2\right)(gh - g\mu) = -\alpha^2\left(R_x^u + R_y^v\right), \tag{2.7}$$

with

$$R^u \equiv +fv - g\mu_x - g\mu_x^S, \tag{2.8}$$

$$R^v \equiv -fu - g\mu_y - g\mu_y^S. \tag{2.9}$$

Also, $\nabla^2 \equiv \partial_x^2 + \partial_y^2$, and $\alpha > 0$ is a regularization parameter.

There are a number of points to note. (i) Eqs. (2.5) and (2.6) differ from their counterparts in F05 (i.e., their (5) and (6) with h given by (11) and (12)) in that the orographic terms are not explicitly regularized. Instead they appear in the R^u and R^v terms. However, the net effect, in terms of the orography, is the same in that the present scheme is algebraically equivalent to regularizing the sum $\mu + \mu^S$ and replacing μ^S in (2.5) and (2.6) by the implied h^S. (ii) There is now only one regularization parameter, α (as will be shown, its optimal value is the same as that of the F05 scheme). (iii) Crucially, and inspired both by considerations of balance and also by how the semi-implicit scheme operates, the Coriolis terms now appear as forcing terms for the regularization operator, i.e.,

$$R_x^u + R_y^v = -g\nabla^2\left(\mu + \mu^S\right) + f\left(v_x - u_y\right). \tag{2.10}$$

(iv) A consequence of (iii) is that if the unregularized equations are in balance, in the sense that $\nabla.\left(D\mathbf{u}/Dt\right)$ vanishes, then the regularization maintains that balance.

Second, the resulting equations are discretized in time (along trajectories) by an explicit, time-staggered leapfrog discretization

$$\frac{u^{n+1} - u^n}{\Delta t} = +f\frac{v^{n+1} + v^n}{2} - gh_x^{n+1/2} - g\mu_x^S, \tag{2.11}$$

$$\frac{v^{n+1} - v^n}{\Delta t} = -f\frac{u^{n+1} + u^n}{2} - gh_y^{n+1/2} - g\mu_y^S, \tag{2.12}$$

$$\frac{\mu^{n+1/2} - \mu^{n-1/2}}{\Delta t} = -\left(\frac{\mu^{n+1/2} + \mu^{n-1/2}}{2}\right)\left(u_x^n + v_y^n\right), \tag{2.13}$$

(where the orography terms are evaluated at trajectory midpoints) together with

$$\left[1 - \left(\frac{\alpha^2}{1+F^2}\right)\nabla^2\right]\left(gh^{n+1/2} - g\mu^{n+1/2}\right) = -\alpha^2\left(\mathcal{R}_x^u + \mathcal{R}_y^v\right), \tag{2.14}$$

(where $F \equiv f\Delta t/2$, Δt being the time step) and the coupled equations

$$\mathcal{R}^u \equiv +f\left(v^n + \frac{\Delta t}{2}\mathcal{R}^v\right) - g\mu_x^{n+1/2} - g\mu_x^S, \tag{2.15}$$

$$\mathcal{R}^v \equiv -f\left(u^n + \frac{\Delta t}{2}\mathcal{R}^u\right) - g\mu_y^{n+1/2} - g\mu_y^S. \tag{2.16}$$

(The factor $\left(1 + F^2\right)^{-1}$ on the left-hand side of (2.14) arises due to the implicit appearance of the same factor on the right-hand side of the equation when (2.15) and (2.16) are solved for $\mathcal{R}^u$ and $\mathcal{R}^v$.) Noting that (2.11)-(2.12) can be rewritten as

$$\left(\frac{u^{n+1} - u^n}{\Delta t}\right) = +f\left[v^n + \frac{\Delta t}{2}\left(\frac{v^{n+1} - v^n}{\Delta t}\right)\right] - gh_x^{n+1/2} - g\mu_x^S, \tag{2.17}$$

$$\left(\frac{v^{n+1} - v^n}{\Delta t}\right) = -f\left[u^n + \frac{\Delta t}{2}\left(\frac{u^{n+1} - u^n}{\Delta t}\right)\right] - gh_y^{n+1/2} - g\mu_y^S, \tag{2.18}$$

and comparing the form of these equations with the form of (2.15)-(2.16), it is seen that $\mathcal{R}^u$ and $\mathcal{R}^v$ are the *unregularized* tendencies of u and v, i.e., those that would result from (2.11)-(2.12) if h were replaced by μ. From this it is seen that (2.15)-(2.16) provide a consistent midpoint approximation to (2.8)-(2.9).

3 Linearizing the equations

The equations are linearized about a motionless basic state of constant free surface depth H assuming that the orography μ^S is a perturbation quantity, in the sense $|\mu^S| \ll H$.

Linearizing (2.5)-(2.6), (2.3) and (2.7)-(2.9) gives

$$u_t = +fv - gh_x - g\mu_x^S, \tag{3.1}$$
$$v_t = -fu - gh_y - g\mu_y^S, \tag{3.2}$$
$$\mu_t = -H\left(u_x + v_y\right), \tag{3.3}$$

together with (2.7) and

$$R^u \equiv +fv - g\mu_x - g\mu_x^S, \tag{3.4}$$
$$R^v \equiv -fu - g\mu_y - g\mu_y^S. \tag{3.5}$$

Both μ and h now, and henceforth, represent perturbations from H.

It can be verified that linearization and discretization are commutative processes and, hence, the time-staggered leapfrog discretization applied to the linear equations (3.1)-(3.5) gives

$$\frac{u^{n+1} - u^n}{\Delta t} = +f\frac{v^{n+1} + v^n}{2} - gh_x^{n+1/2} - g\mu_x^S, \tag{3.6}$$

$$\frac{v^{n+1} - v^n}{\Delta t} = -f\frac{u^{n+1} + u^n}{2} - gh_y^{n+1/2} - g\mu_y^S, \tag{3.7}$$

$$\frac{\mu^{n+1/2} - \mu^{n-1/2}}{\Delta t} = -H\left(u_x^n + v_y^n\right), \tag{3.8}$$

together with (2.14)-(2.16), but where μ and h now represent perturbations from H.

4 Analytic impact of regularizing the linear SWEs on an f-plane

Following the procedure of section 4 of F05 applied to (3.1)-(3.3), it is found that

$$\mu_{tt} = -\left(HfQ + f^2\mu\right) + gH\nabla^2\left(h + \mu^S\right), \tag{4.1}$$

where

$$Q \equiv \zeta - \frac{f}{H}\mu, \tag{4.2}$$

is the linearized and scaled potential vorticity perturbation and $\zeta \equiv (v_x - u_y)$ is relative vorticity. Using (3.4)-(3.5) and (4.2), (2.7) can be written as

$$\left(1 - \alpha^2\nabla^2\right)gh = g\mu - \alpha^2\left(fQ^0 + \frac{f^2}{H}\mu - g\nabla^2\mu^S\right), \tag{4.3}$$

where the fact that $dQ/dt = 0$, so that $Q = Q^0$, has been used. Applying the operator $\left(1 - \alpha^2\nabla^2\right)$ to both sides of (4.1) and using (4.3) leads to

$$\left(1 - \alpha^2\nabla^2\right)\mu_{tt} + \left(f^2 - c_0^2\nabla^2\right)\mu = -f^2\left(\frac{H}{f}Q^0 - L_R^2\nabla^2\mu^S\right), \tag{4.4}$$

where $c_0 \equiv \sqrt{gH}$ and $L_R \equiv c_0/f$ denotes the Rossby radius of deformation.

The behaviour of the free and forced responses of the regularized and unregularized equations are now considered. Note that here forced is used to mean forced by the orography, μ^S, and/or by the initial potential vorticity perturbation, Q^0.

4.1 Forced solutions

The time-independent, forced solution $\mu = \mu^{forced}$ to (4.4) (which for $\mu^S \equiv 0$ corresponds to the stationary, degenerate Rossby mode) is related to Q^0 and μ^S by

$$\mu^{forced} = -\left(1 - L_R^2\nabla^2\right)^{-1}\left(\frac{H}{f}Q^0 - L_R^2\nabla^2\mu^S\right), \tag{4.5}$$

where superscript "$forced$" denotes the forced solution. This is exactly the analytic forced response for the unregularized depth field (i.e., (39) of F05 with $\alpha = \gamma = 0$ and h replaced by μ). This is in contradistinction to the forced response of F05 (their (39)) which: a) is in terms of the regularized depth field, h; and b) only reduces to the analytic response when $\gamma \ll 1$ and $\alpha^2 \ll L_R^2$.

4.2 Free solutions

The free response of (4.4), which represents the inertia-gravity waves, is governed by the regularized wave equation

$$\left(1 - \alpha^2\nabla^2\right)\mu_{tt}^{free} + \left(f^2 - c_0^2\nabla^2\right)\mu^{free} = 0, \tag{4.6}$$

where superscript "$free$" denotes the free solution.

Comparison of (4.6) with the unregularized wave equation

$$\mu_{tt}^{free} + \left(f^2 - c_0^2\nabla^2\right)\mu^{free} = 0, \tag{4.7}$$

reveals that the impact of the regularization procedure is to artificially reduce the frequency, ω, of linear inertia-gravity waves, with wavenumbers k and l in the x- and y-directions respectively, from $\omega = \pm\sqrt{f^2 + c_0^2\left(k^2 + l^2\right)}$ to

$$\omega = \pm\sqrt{\frac{f^2 + c_0^2\left(k^2 + l^2\right)}{1 + \alpha^2\left(k^2 + l^2\right)}}. \tag{4.8}$$

5 Explicit time-staggered discretization of the forced regularized SWEs on an f-plane

Consider now the explicit, time-staggered discretization of the regularized linear SWEs on an f-plane, i.e., equations (3.6)-(3.8) and (2.14)-(2.16).

5.1 Derivation of an equivalent difference equation for the fluid depth

Here the same procedure as in section 4 is followed but applied to the discrete equations (3.6)-(3.8) and (2.14)-(2.16), in a manner exactly analogous to section 6.1 of F05 except that now the unregularized depth is maintained as the primary variable. Then the governing second-order difference equation for μ is found to be

$$\frac{\mu^{n+1} - 2\mu^n + \mu^{n-1}}{\Delta t^2} + \frac{f^2}{4}\left(\mu^{n+1} + 2\mu^n + \mu^{n-1}\right) =$$
$$\left[1 - \left(\frac{\alpha^2}{1+F^2}\right)\nabla^2\right]^{-1}\left[c_0^2\nabla^2\left(\mu^n + \mu^S\right) - fHQ^0 - \left(\frac{\alpha^2}{1+F^2}\right)f^2\nabla^2\mu^n\right], \quad (5.1)$$

where

$$\mu^n \equiv \frac{\mu^{n+1/2} + \mu^{n-1/2}}{2}, \quad (5.2)$$

and

$$Q^n \equiv \zeta^n - \frac{f}{H}\mu^n = Q^0. \quad (5.3)$$

5.2 Stability of the free solution

The free solution to (5.1) is governed by the equation

$$\frac{\mu^{n+1} - 2\mu^n + \mu^{n-1}}{\Delta t^2} = -\frac{f^2}{4}\left(\mu^{n+1} + 2\mu^n + \mu^{n-1}\right) + \left[1 - \left(\frac{\alpha^2}{1+F^2}\right)\nabla^2\right]^{-1}\left[c_0^2 - \left(\frac{\alpha^2}{1+F^2}\right)f^2\right]\nabla^2\mu^n.$$
$$(5.4)$$

The same procedure as in section 6.2 of F05 is followed, i.e., seek solutions of the form $\mu^n \propto \lambda^n \exp\left[i\left(kx + ly\right)\right]$, and require $|\lambda| \leq 1$ for stability. From the resulting quadratic equation for λ, the necessary and sufficient condition for (neutral) stability is that $B^2 \leq 1$ where here

$$B \equiv \frac{2\left(1 - F^2\right) - \left[1 + \alpha^2\left(k^2 + l^2\right)/\left(1 + F^2\right)\right]^{-1}\Delta t^2\left[c_0^2 - \alpha^2 f^2/\left(1 + F^2\right)\right]\left(k^2 + l^2\right)}{2\left(1 + F^2\right)}. \quad (5.5)$$

This requires

$$0 \leq 1 + \left[\alpha^2 - \left(\frac{c_0\Delta t}{2}\right)^2\right]\left(k^2 + l^2\right). \quad (5.6)$$

For this inequality to be satisfied for any horizontal wavenumber $\left(k^2 + l^2\right)^{1/2}$ requires

$$\alpha^2 \geq \left(\frac{c_0\Delta t}{2}\right)^2. \quad (5.7)$$

This is the same condition as that found in F05, see their (69).

In order that the regularized continuous governing equations are as close as possible to the unregularized ones, as small a value of α as possible, consistent with numerical stability, should be chosen. Therefore, from (5.7), the optimal choice for the smoothing length scale is

$$\alpha^2 = \left(\frac{c_0\Delta t}{2}\right)^2. \quad (5.8)$$

As noted in F05, increasing α beyond this lower limit for stability, anyway decreases the coefficient of the associated Helmholtz problem and, hence, decreases the efficiency of an iterative solver.

5.3 Forced solution

Seeking solutions of the form $\mu^n = \mu^{n\pm1} = \mu^{forced}$ in (5.1) leads to

$$\mu^{forced} = -\left(1 - L_R^2\nabla^2\right)^{-1}\left(\frac{H}{f}Q^0 - L_R^2\nabla^2\mu^S\right),$$ (5.9)

i.e., the exact analytical forced response, independent of the choice of α. Again, this is in contradistinction to the forced response of F05 (their (70)) which: a) is in terms of the regularized depth field, h; and b) only reduces to the analytic response when $\gamma \ll 1$ and $\alpha^2 \ll L_R^2$.

5.4 Free solution

From (5.4) the free solution (corresponding to the inertia-gravity waves) is governed by the explicit recursion relation

$$\mu^{n+1} = 2\mu^n - \mu^{n-1} - 4\left(\frac{F^2}{1+F^2}\right)\left[1 - \left(\frac{1}{1+F^2}\right)\alpha^2\nabla^2\right]^{-1}\left(1 - L_R^2\nabla^2\right)\mu^n.$$ (5.10)

This can be compared to the corresponding expression for the semi-implicit discretization that was given by (77) of F05 as (noting that $h \equiv \mu$ for the semi-implicit discretization):

$$\mu^{n+1} = 2\mu^n - \mu^{n-1} - 4\left(\frac{F^2}{1+F^2}\right)\left[1 - \left(\frac{1}{1+F^2}\right)\left(\frac{c_0\Delta t}{2}\right)^2\nabla^2\right]^{-1}\left(1 - L_R^2\nabla^2\right)\mu^n.$$ (5.11)

It is seen that the two recursions are equivalent provided that α is given by its optimal choice for stability, i.e., by (5.8). Thus the new regularization, with this choice for α, is equivalent to the semi-implicit scheme not only for the free response, as for the regularization of F05, *but also for the forced (exact) response.*

6   Conclusions

A revised form of the regularization operator of F05 has been proposed and applied to an explicit time-staggered leapfrog discretization of the shallow-water equations. An important aspect of the new scheme is that it maintains balanced solutions (i.e., solutions for which $\nabla . (D\mathbf{u}/Dt)$ vanishes).

Linear analysis of the discrete equations shows that they are neutrally stable provided the regularization parameter, α, satisfies (5.7), the same condition as found for the regularization of F05. The optimal choice, in terms of accuracy of the continuous equations and computational efficiency for the discrete equations, is to choose equality in (5.7) so that α takes its smallest value permitted for unconditional stability.

Importantly, the analytic forced response is exactly captured by the scheme independently of the choice of α, thereby addressing a weakness of the F05 formulation, whilst the equivalence of the free response to that of the semi-implicit form is still maintained when α takes its optimal value.

References

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