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Analysis of a regularized, time-staggered discretization method and its link to the semi-implicit method†

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Abstract

A key aspect of the recently proposed Hamiltonian particle-mesh (HPM) method is its time-staggered discretization combined with a regularization of the continuous governing equations. In this article, the time discretization aspect of the HPM method is analysed for the linearized, rotating, shallow-water equations with orography, and the combined effect of time-staggering and regularization is compared analytically with the popular two-time-level semi-implicit time discretization of the unregularized equations. It is found that the two approaches are essentially equivalent, provided the regularization parameter is chosen appropriately in terms of the time step Δt. The article treats space as a continuum and, hence, its analysis is not limited to the HPM method. Copyright © 2005 Royal Meteorological Society

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1. Introduction

An important issue in numerical weather prediction is the treatment of poorly resolved inertia-gravity waves. To circumvent the strict limitations imposed via the CFL condition on the maximum time step of explicit integration methods, most operational codes make use of some implicitness. At each time step, fully implicit methods require the solution of a nonlinear system of equations, whereas linearly implicit methods require only the solution of a linear system. In this article, an alternative strategy is investigated, which is based on applying a regularization procedure to the continuous governing equations that renders them suitable for explicit integration. This approach has been proposed in the context of the Hamiltonian particle-mesh (HPM) method (see, e.g. Frank *et al*. (2002) and Frank & Reich (2004)).

The HPM method is based on the Lagrangian formulation of fluid dynamics and uses a conservative (Hamiltonian) version of the classical particle-mesh spatial truncation technique (Birdsall & Langdon, 1981; Hockney & Eastwood, 1988). Encouraging numerical results have been reported in a number of articles (Frank *et al*., 2002; Frank & Reich 2004; Cotter *et al*., 2004). However, with the exception of conservation properties (Frank & Reich, 2003; Bridges *et al*., 2005; Cotter & Reich, 2004), theoretical understanding of the HPM method is somewhat limited. In this article, the time-stepping aspect of the HPM method is investigated. It is applied to the two-dimensional shallow-water equations (henceforth referred to as the SWEs) and its linearized free and forced response is analysed and compared with the standard two-time-level semi-implicit approach (see, e.g. Staniforth (1997) and Durran (1998)).

In Section 2, the regularization procedure is discussed and applied to the orographically forced SWEs on an f-plane. These equations are then linearized and discretized in Section 3, where the two-time-level semi-implicit discretization of the linearized, unregularized equations is also given. The analytical properties of the regularized continuous equations are discussed in Section 4, which motivates a comparison of the non-rotating discrete system with its semi-implicit counterpart in Section 5. This comparison is extended to the rotating system in Sections 6 and 7 before conclusions are drawn in Section 8.

2. The regularization procedure of the HPM method applied to the SWEs

The numerical treatment of the SWEs has been the subject of extensive research as these equations serve as a model system for the more complex primitive equations and/or the non-hydrostatic Euler equations of three-dimensional atmospheric fluid dynamics (Durran, 1998). The orographically forced SWEs on an f-plane in an Eulerian framework are

$$\frac{Du}{Dt} = +fv - gh_x - gh_x^S, \qquad (1)$$

$$\frac{Dv}{Dt} = -fu - gh_y - gh_y^S, \quad (2)$$

$$\frac{Dh}{Dt} = -h(u_x + v_y). \quad (3)$$

Here $h^S = h^S(x, y)$ is the height of the orography above mean sea level and $h = h(x, y, t)$ is the fluid depth, i.e. the depth of the fluid between the orography and the fluid's free surface. Also, g is gravity (assumed constant), f is twice the (constant) angular velocity of the reference plane,

$$\frac{D}{Dt}(.) = (.)_t + u(.)_x + v(.)_y, \quad (4)$$

is the material time derivative, and subscripts denote partial differentiation with respect to that variable.

Alternatively, in a Lagrangian framework, the SWEs are given by

$$u_t = +fv - gh_x - gh_x^S, \quad (5)$$

$$v_t = -fu - gh_y - gh_y^S, \quad (6)$$

$$x_t = u, \quad (7)$$

$$y_t = v, \quad (8)$$

$$h = \left[\frac{\partial(x,y)}{\partial(a,b)}\right]^{-1} h^0. \quad (9)$$

Here $x = x(a, b, t)$ and $y = y(a, b, t)$ are now the coordinates of a fluid particle with initial coordinates $x = a$ and $y = b$,

$$\frac{\partial(x,y)}{\partial(a,b)} = x_a y_b - x_b y_a, \quad (10)$$

denotes the Jacobian of the transformation, and $h^0(a, b)$ is the initial fluid depth. Note that the independent variables in the Lagrangian framework are time t and labels (a, b). Furthermore, the Lagrangian partial time derivative $(.)_t$ corresponds to the material time derivative $D(.)/Dt$ in the Eulerian framework.

The HPM method applies to the Lagrangian framework and the HPM discretization consists essentially of three steps.

First, (5)–(9) are regularized by applying a modified inverse Helmholtz operator $\mathcal{A}$ to the fluid depth obtained from the continuity Equation (9). Denoting now the *unmodified* fluid depth by μ, this step leads to the replacement of h, as it appears in the momentum Equations (5) and (6), by the *modified* fluid depth h given as

$$h = \mathcal{A} * \mu, \qquad \mu = \left[\frac{\partial(x,y)}{\partial(a,b)}\right]^{-1} \mu^0, \quad (11)$$

where

$$\mathcal{A} * \mu \equiv \left(1 + \gamma^2 - \alpha^2\nabla^2\right)^{-1} \mu, \quad (12)$$

so that

$$\mu = \left(1 + \gamma^2 - \alpha^2\nabla^2\right) h. \quad (13)$$

Also, $\nabla^2 = \partial_x^2 + \partial_y^2$, $\alpha > 0$ is a prescribed 'smoothing length scale' and $\gamma > 0$ is a further smoothing parameter, which is set equal to zero in the standard implementations of the HPM method. Additionally, $\mu^0 = \left(1 + \gamma^2 - \alpha^2\nabla^2\right) h^0$.

Second, the resulting equations are discretized in time by a staggered leapfrog discretization

$$\frac{u^{n+1} - u^n}{\Delta t} = +f\frac{v^{n+1} + v^n}{2} - gh_x^{n+1/2} - gh_x^S, \quad (14)$$

$$\frac{v^{n+1} - v^n}{\Delta t} = -f\frac{u^{n+1} + u^n}{2} - gh_y^{n+1/2} - gh_y^S, \quad (15)$$

$$\frac{x^{n+1/2} - x^{n-1/2}}{\Delta t} = u^n, \quad (16)$$

$$\frac{y^{n+1/2} - y^{n-1/2}}{\Delta t} = v^n, \quad (17)$$

$$\mu^{n+1/2} = \left[\frac{\partial\left(x^{n+1/2}, y^{n+1/2}\right)}{\partial(a,b)}\right]^{-1} \mu^0, \quad (18)$$

together with $h^{n+1/2} = \mathcal{A} * \mu^{n+1/2}$. The staggered discretization is fully explicit for $f = 0$. For $f \neq 0$, the discrete momentum updates (14)–(15) lead to a 2×2 implicit equation in u^{n+1} and v^{n+1}.

Third, a spatial discretization, via a classical particle-mesh method (Birdsall & Langdon, 1981; Hockney & Eastwood, 1988), is applied, taking particular care that the resulting finite-dimensional differential equations are conservative (i.e. Hamiltonian) (Frank *et al.*, 2002; Frank & Reich, 2004).

The empirical rationale behind the introduction of a smoothing operator $\mathcal{A}$ into the HPM method is to control poorly resolved, high-frequency, inertia-gravity waves. It has been found that such waves can otherwise destabilize the HPM method. The particular form of the operator $\mathcal{A}$ is motivated by its success in numerical experiments. However, other 'smoothing' operators are conceivable.

In this article, only the analysis of the first two steps in the derivation of the HPM method is considered, and therefore the spatial discretization aspect of the HPM method is ignored.

3. Linearizing the Lagrangian fluid equations and the HPM discretization

The only non-linearity in the Lagrangian picture arises from Equation (11). Its linearization about a motionless basic state of constant free surface height H leads to the relation

$$\mu = H(1 - x_a' - y_b'), \quad (19)$$

provided the orography h^S is assumed to be a perturbation quantity, in the sense $|h^S| \ll H$. Here $x' = x - a$

and $y' = y - b$ denote small perturbations about the basic state. Since the basic state is assumed to be motionless, $x'_a \approx x'_x$ and $y'_b \approx y'_y$. Hence, the linear system of partial differential equations is

$$u_t = +fv - gh_x - gh^S_x, \tag{20}$$

$$v_t = -fu - gh_y - gh^S_y, \tag{21}$$

$$x'_t = u, \tag{22}$$

$$y'_t = v, \tag{23}$$

$$\mu = H(1 - x'_x - y'_y), \tag{24}$$

together with $h = \mathcal{A} * \mu$. Equations (22)–(24) can be simplified to the Eulerian form

$$\mu_t = -H(u_x + v_y), \tag{25}$$

since $x'_{xt} = u_x$ and $y'_{yt} = v_y$.

It can be verified that linearization and discretization are commutative Processes and, hence, the staggered leapfrog discretization applied to the linear Equations (20), (21) and (25) gives

$$\frac{u^{n+1} - u^n}{\Delta t} = +f\frac{v^{n+1} + v^n}{2} - gh^{n+1/2}_x - gh^S_x, \tag{26}$$

$$\frac{v^{n+1} - v^n}{\Delta t} = -f\frac{u^{n+1} + u^n}{2} - gh^{n+1/2}_y - gh^S_y, \tag{27}$$

$$\frac{\mu^{n+1/2} - \mu^{n-1/2}}{\Delta t} = -H(u^n_x + v^n_y), \tag{28}$$

together with $h^{n+1/2} = \mathcal{A} * \mu^{n+1/2}$.

In Section 4, the analytic solutions of (20), (21), and (25) are compared with the solutions of the standard linearized SWEs with $h = \mu$, and the impact of the filtering operator $\mathcal{A}$ on both the forced and free solutions is discussed.

The second part of the article, Sections 5–7, is devoted to the numerical discretization (26)–(28) and a comparison with the two-time-level semi-implicit discretization of (20), (21), and (25) with $h = \mu$, namely

$$\frac{u^{n+1} - u^n}{\Delta t} = +f\frac{v^{n+1} + v^n}{2} - g\frac{h^{n+1}_x + h^n_x}{2} - gh^S_x, \tag{29}$$

$$\frac{v^{n+1} - v^n}{\Delta t} = -f\frac{u^{n+1} + u^n}{2} - g\frac{h^{n+1}_y + h^n_y}{2} - gh^S_y, \tag{30}$$

$$\frac{h^{n+1} - h^n}{\Delta t} = -H\frac{u^{n+1}_x + u^n_x}{2} - H\frac{v^{n+1}_y + v^n_y}{2}. \tag{31}$$

Such a comparison is motivated by the fact that the semi-implicit time discretization is widely used in numerical weather prediction and climate modelling. Note that, for the linear equations, the two-time-level semi-implicit method is akin to the Crank–Nicolson, trapezoidal and implicit midpoint methods.

4. Analytic impact of regularizing the linear SWEs on an *f*-plane

In order to isolate the slow modes (here the stationary, degenerate Rossby modes) from the propagating fast (inertia-gravity) modes, the curl and divergence of (20) and (21) are formed, giving

$$\zeta_t = -fD, \tag{32}$$

and

$$D_t = f\zeta - g\nabla^2(h + h^S). \tag{33}$$

Equation (25) may be rewritten as

$$\mu_t = -HD. \tag{34}$$

Manipulation of (32) and (34) yields

$$Q_t = 0, \tag{35}$$

where

$$Q \equiv \zeta - \frac{f}{H}\mu, \tag{36}$$

is the linearized and scaled potential vorticity perturbation. Equations (33), (34) and (36) then lead to

$$\begin{aligned}\mu_{tt} &= -HD_t = -fH\zeta + gH\nabla^2 h + gH\nabla^2 h^S \\ &= -\left(fHQ + f^2\mu\right) + c_0^2\nabla^2(h + h^S),\end{aligned} \tag{37}$$

with μ and h related by (13) and $c_0 \equiv \sqrt{gH}$. It is convenient, using (13) and (12), to rewrite (37) as an equation for h, i.e. as

$$\begin{aligned}h_{tt} &+ f^2 h - f^2 L_R^2 \nabla^2 \mathcal{A} * h \\ &= -f^2\mathcal{A} * \left(\frac{HQ}{f} - L_R^2\nabla^2 h^S\right),\end{aligned} \tag{38}$$

where $L_R \equiv c_0/f$ denotes the Rossby radius of deformation.

Equations (35) and (38) govern the evolution of Q and h respectively, and ζ can be diagnosed from these using (36).

Equation (35) essentially governs the geostrophic ($f = \text{const.}$) degenerate Rossby mode and has the solution $Q = Q^0$, where Q^0 is the initial value of Q. Equation (38) is a forced, second-order-in-time partial differential equation for h. Both $Q = Q^0$ and h^S are independent of time. Therefore, the forced response of h is stationary. It is the free, time-dependent response of h that governs the propagation of the inertia-gravity modes.

The behaviour of the free and forced responses of the regularized and unregularized equations are now compared.

4.1. Forced solutions

After application of $(1+\gamma^2-\alpha^2\nabla^2)$ to (38), the time-independent, forced solution $h=h^{\text{forced}}$ is related to Q^0 and h^S by

$$h^{\text{forced}} = -\left[1+\gamma^2-\left(L_R^2+\alpha^2\right)\nabla^2\right]^{-1} \times \left(\frac{H}{f}Q^0 - L_R^2\nabla^2 h^S\right), \quad (39)$$

where superscript '*forced*' denotes the forced solution. Furthermore, noting that the forced solution is time-independent, the forced solution for ζ is found from (33) to be related to that of h by

$$\zeta^{\text{forced}} = \frac{g}{f}\nabla^2\left(h^{\text{forced}}+h^S\right), \quad (40)$$

which does not introduce any further dependence on the regularization parameters α and γ. Hence, only the forced response of the fluid depth, h^{forced}, need be considered.

Comparison of (39) with the unregularized result (i.e. (39) with $\alpha \equiv 0$ and $\gamma \equiv 0$), shows that, provided $\alpha \ll L_R$ and $\gamma \ll 1$, the regularization does not significantly influence the forced response of h to the initial potential vorticity perturbation Q^0 and to the orography h^S. It is found later (see discussion in Section 8) that such a choice of α and γ is not only justified but also practicable.

4.2. Free solutions

Using (12) and $c_0^2 = f^2 L_R^2$, the free response of (38), which represents the inertia-gravity waves, is governed by the regularized wave equation

$$h_{tt}^{\text{free}} + f^2 h^{\text{free}} - c_0^2\left(1+\gamma^2-\alpha^2\nabla^2\right)^{-1}\nabla^2 h^{\text{free}} = 0, \quad (41)$$

where superscript '*free*' denotes the free solution.

Comparison of (41) with the unregularized wave equation

$$h_{tt}^{\text{free}} + f^2 h^{\text{free}} - c_0^2\nabla^2 h^{\text{free}} = 0, \quad (42)$$

reveals that the impact of the regularization procedure is to artificially reduce the frequency of linear inertia-gravity waves from $\omega = \pm\sqrt{f^2+c_0^2\left(k^2+l^2\right)}$ to

$$\omega = \pm\sqrt{f^2 + \frac{c_0^2\left(k^2+l^2\right)}{1+\gamma^2+\alpha^2\left(k^2+l^2\right)}}. \quad (43)$$

This is an analytic result, independent of any discretization procedure. It means that a spurious numerical dispersion is introduced into the continuous problem such that the highest wavenumber components are increasingly retarded as a function of increasing wavenumber (i.e. decreasing scale).

The result is qualitatively reminiscent of the impact of a semi-implicit discretization of the original, unregularized equations, which also progressively retards the propagation of gravity modes as a function of decreasing scale (see e.g. Staniforth (1997)). Thus, in qualitative terms at least, the regularization procedure does at an analytic level what the semi-implicit method is known to do at a discrete level. This aspect is now investigated.

5. An equivalence between explicit time-staggered discretization of the regularized equations and semi-implicit time discretization of the unregularized equations

For simplicity, throughout this section, the non-rotating, linear SWEs are considered, i.e. it is assumed that $f \equiv 0$. The more general case of $f \neq 0$ is addressed in Sections 6 and 7.

5.1. Explicit time-staggered discretization of the regularized SWEs

Consider the explicit time-staggered discretization of the regularized SWEs (26)–(28) with $f \equiv 0$. Taking the divergence of (26) and (27), and using (13), yields the formulation

$$\frac{D^{n+1}-D^n}{\Delta t} = -g\nabla^2\left(h^{n+1/2}+h^S\right), \quad (44)$$

and

$$\frac{h^{n+1/2}-h^{n-1/2}}{\Delta t} = -H\mathcal{A}*D^n, \quad (45)$$

in terms of the variables $h^{n+1/2}$ and $D^n \equiv u_x^n + v_y^n$. Algebraic manipulation of (44) and (45) then yields

$$h^{n+3/2} - 2h^{n+1/2} + h^{n-1/2} = c_0^2\Delta t^2\nabla^2\mathcal{A}*\left(h^{n+1/2}+h^S\right). \quad (46)$$

Averaging successive time steps of (46), defining the integer time-level approximations

$$h^n \equiv \frac{1}{2}\left(h^{n+1/2}+h^{n-1/2}\right), \quad (47)$$

and using (12) leads to the equivalent formulation

$$h^{n+1} = 2h^n - h^{n-1} + (c_0\Delta t)^2\left(1+\gamma^2-\alpha^2\nabla^2\right)^{-1}\nabla^2\left(h^n+h^S\right). \quad (48)$$

Algorithmically, most of the computational cost of the time-staggered discretization of the regularized equations is the overhead, when applying the smoothing operator in (46), of solving a modified Helmholtz problem whose Helmholtz coefficient is $\left(1+\gamma^2\right)\alpha^{-2}$.

5.2. Semi-implicit time discretization of the unregularized SWEs

Repeating the manipulations from the previous section, but for (29)–(31), the semi-implicit discretization of the unregularized SWEs with $f \equiv 0$ becomes equivalent to

$$h^{n+1} - 2h^n + h^{n-1} = (c_0\Delta t)^2 \times \nabla^2 \left[\frac{\left(h^{n+1} + 2h^n + h^{n-1}\right)}{4} + h^S \right], \quad (49)$$

which can be considered to be a time-centred discretization of the wave Equation (42) for $f = 0$.

Algorithmically, most of the computational cost of the semi-implicit discretization is the overhead of solving the modified Helmholtz problem defined by (5049) whose Helmholtz coefficient is $(c_0\Delta t/2)^{-2}$. The inversion of this modified Helmholtz operator yields the equivalent 'explicit' recursion relation

$$h^{n+1} = 2h^n - h^{n-1} + (c_0\Delta t)^2 \left[1 - \left(\frac{c_0\Delta t}{2}\right)^2 \nabla^2 \right]^{-1} \times \nabla^2 \left(h^n + h^S\right). \quad (50)$$

Comparing now (48) with (50), it is seen that they are equivalent if α is set to $c_0\Delta t/2$ and γ to zero, which, for $h^S \equiv 0$, can be seen as a numerical approximation to (42). The two Helmholtz coefficients are then also identical. *This means that, in the non-rotating case, the time-staggered discretization of the regularized, linear SWEs is precisely equivalent to a semi-implicit time discretization of the unregularized linear SWEs when* $\alpha = c_0\Delta t/2$ *and* $\gamma = 0$. (As will be found in Section (7), γ plays a crucial role when $f \neq 0$.)

6. Time-staggered discretization of the forced regularized SWEs on an *f*-plane

Consider now the time-staggered discretization of the regularized linear SWEs on an *f*-plane, i.e. Equations (26)–(28).

6.1. Derivation of an equivalent difference equation for the fluid depth

Defining

$$\zeta^n \equiv v_x^n - u_y^n, \quad (51)$$

and assuming a continuous representation in space, (26)–(28), together with (13), may be equivalently rewritten as

$$\frac{\zeta^{n+1} - \zeta^n}{\Delta t} = -\frac{f}{2}\left(D^{n+1} + D^n\right), \quad (52)$$

$$\frac{D^{n+1} - D^n}{\Delta t} = \frac{f}{2}\left(\zeta^{n+1} + \zeta^n\right) - g\nabla^2\left(h^{n+1/2} + h^S\right), \quad (53)$$

$$\frac{h^{n+1/2} - h^{n-1/2}}{\Delta t} = -H\mathcal{A} * D^n. \quad (54)$$

Subtracting (54) from its index increment gives

$$\frac{h^{n+3/2} - 2h^{n+1/2} + h^{n-1/2}}{\Delta t^2} = -H\mathcal{A} * \left(\frac{D^{n+1} - D^n}{\Delta t}\right), \quad (55)$$

and using (53) gives

$$\frac{h^{n+3/2} - 2h^{n+1/2} + h^{n-1/2}}{\Delta t^2} = \mathcal{A} * \left[-\frac{Hf}{2}\left(\zeta^{n+1} + \zeta^n\right) + c_0^2\nabla^2\left(h^{n+1/2} + h^S\right)\right]. \quad (56)$$

Using (47), (56) then leads to

$$\frac{h^{n+1} - 2h^n + h^{n-1}}{\Delta t^2} = \mathcal{A} * \left[-\frac{Hf}{4}\left(\zeta^{n+1} + 2\zeta^n + \zeta^{n-1}\right) + c_0^2\nabla^2\left(h^n + h^S\right)\right]. \quad (57)$$

Next, using (36) and (13), the discrete linear potential vorticity perturbation is defined as

$$Q^n \equiv \zeta^n - \frac{f}{H}\mu^n = \zeta^n - \frac{f}{H}\left(1 + \gamma^2 - \alpha^2\nabla^2\right) \times \frac{h^{n+1/2} + h^{n-1/2}}{2}. \quad (58)$$

It can be verified that Q^n is constant and equal to its initial value Q^0 under the Equations (52)–(54).

Hence, ζ^n can be replaced in (57) by

$$\zeta^n = Q^0 + \frac{f}{H}\left(1 + \gamma^2 - \alpha^2\nabla^2\right)h^n, \quad (59)$$

with corresponding replacements for the integer shifted values. Finally, the governing second-order difference equation for h is derived as

$$\frac{h^{n+1} - 2h^n + h^{n-1}}{\Delta t^2} = -\frac{f^2}{4}\left(h^{n+1} + 2h^n + h^{n-1}\right) - \mathcal{A} * \left[fHQ^0 - c_0^2\nabla^2\left(h^n + h^S\right)\right]. \quad (60)$$

6.2. Stability of the free solution

Using the definition (12), the free solution to (60) is governed by the equation

$$\frac{h^{n+1} - 2h^n + h^{n-1}}{\Delta t^2} = -\frac{f^2}{4}\left(h^{n+1} + 2h^n + h^{n-1}\right) + \left(1 + \gamma^2 - \alpha^2\nabla^2\right)^{-1} c_0^2\nabla^2 h^n. \quad (61)$$

Seeking solutions of the form

$$h^n \propto \lambda^n e^{i(kx+ly)}, \quad (62)$$

yields

$$\left(\lambda^2 - 2B\lambda + 1\right) = 0, \tag{63}$$

where

$$B \equiv \frac{b\left(1-F^2\right) - c_0^2\Delta t^2\left(k^2+l^2\right)/2}{b\left(1+F^2\right)},$$
$$b \equiv 1 + \gamma^2 + \alpha^2\left(k^2+l^2\right),$$
$$F \equiv \frac{f\Delta t}{2}, \tag{64}$$

with solutions

$$\lambda = B \pm i\sqrt{1-B^2}. \tag{65}$$

Thus, the requirement for stability that $|\lambda| \le 1$ gives the necessary and sufficient condition

$$B^2 \le 1, \tag{66}$$

in which case $|\lambda| = 1$ and the solutions are neutrally stable. Substituting the definitions (64) into (66) then gives

$$\left(\frac{c_0^2\Delta t^2}{2}\right)^2\left(k^2+l^2\right)^2 \le 4b^2F^2 + b\left(1-F^2\right)c_0^2\Delta t^2\left(k^2+l^2\right), \tag{67}$$

which may be rewritten as

$$0 \le \left(c_0^2\Delta t^2 m^2 + 4F^2 b\right) \times \left(1 + \gamma^2 + \alpha^2 m^2 - \frac{c_0^2\Delta t^2}{4}m^2\right), \tag{68}$$

where $m = \left(k^2+l^2\right)^{1/2}$ is the horizontal wave number. For this inequality to be satisfied for any horizontal wavenumber thus requires

$$\alpha^2 \ge \left(\frac{c_0\Delta t}{2}\right)^2. \tag{69}$$

In order that the regularized continuous governing equations are as close as possible to the unregularized ones, as small a value of α as possible, consistent with numerical stability, should be chosen. Therefore, from (69), the optimal choice for the smoothing length scale is $\alpha = c_0\Delta t/2$. Note also that, for fixed γ, increasing α beyond this lower limit for stability anyway decreases the coefficient of the associated Helmholtz problem and, hence, decreases the efficiency of an iterative solver.

6.3. Forced solution

Seeking solutions of the form $h^n = h^{n\pm1} = h^{\text{forced}}$ in (60) and using the definition (12) gives the relation

$$0 = -f^2\left(1+\gamma^2-\alpha^2\nabla^2\right)h^{\text{forced}} - fHQ^0 + gH\nabla^2\left(h^{\text{forced}} + h^S\right). \tag{70}$$

Solving for h^{forced} in (70) shows that the numerical forced fluid depth h^{forced} is given exactly as in (39), and this itself reduces to the unregularized result under the assumption that $\alpha \ll L_R$ and $\gamma^2 \ll 1$.

6.4. Free solution

From (61), the free solution (corresponding to the inertia-gravity waves) is governed by the explicit recursion relation

$$h^{n+1} = 2h^n - h^{n-1} - \frac{f^2\Delta t^2}{1+f^2\Delta t^2/4}\left[1 - L_R^2\left(1+\gamma^2-\alpha^2\nabla^2\right)^{-1}\nabla^2\right]h^n$$
$$= 2h^n - h^{n-1} - \frac{f^2\Delta t^2}{1+f^2\Delta t^2/4}\left(1+\gamma^2-\alpha^2\nabla^2\right)^{-1} \times \left\{1 - L_R^2\nabla^2 + \gamma^2\left[1 - \left(\frac{\alpha^2}{\gamma^2}\right)\nabla^2\right]\right\}h^n. \tag{71}$$

This is compared to the corresponding expression for the semi-implicit discretization in the next section.

7. Semi-implicit time discretization of the forced SWEs on an *f*-plane

In this section, the semi-implicit time discretization (29)–(31) is considered on an *f*-plane and the resulting discretization is compared with the time-staggered discretization of the regularized equations.

7.1. Derivation of an equivalent difference equation for the fluid depth

Manipulating (29)–(31) in a similar way as in Section 6 leads to the equivalent formulation

$$\frac{h^{n+1} - 2h^n + h^{n-1}}{\Delta t^2} = -\frac{fH}{4}\left(\zeta^{n+1} + 2\zeta^n + \zeta^{n-1}\right) + gH\nabla^2\left[\frac{\left(h^{n+1}+2h^n+h^{n-1}\right)}{4} + h^S\right], \tag{72}$$

in terms of the fluid depth and the vorticity alone. Next, the discrete linear potential vorticity perturbation is defined as

$$Q^n \equiv \zeta^n - \frac{f}{H}h^n. \tag{73}$$

It can be verified that Q^n is constant and equal to its initial value Q^0 under the semi-implicit time discretization. Hence, ζ^n in (72) may be replaced by

$$\zeta^n = Q^0 + \frac{f}{H}h^n, \tag{74}$$

with corresponding replacements for the integer shifted values. Finally, the governing second-order difference equation for h is derived as

$$\frac{h^{n+1} - 2h^n + h^{n-1}}{\Delta t^2} = -\frac{f^2}{4}\left(h^{n+1} + 2h^n + h^{n-1}\right) - fHQ^0 + gH\nabla^2\left[\frac{\left(h^{n+1} + 2h^n + h^{n-1}\right)}{4} + h^S\right]. \tag{75}$$

7.2. Stability of the free solution

It can be verified that the semi-implicit method is unconditionally (neutrally) stable.

7.3. Forced solution

Seeking solutions of the form $h^n = h^{n\pm1} = h^{\text{forced}}$ in (75) gives the relation

$$0 = -f^2 h^{\text{forced}} - fHQ^0 + gH\nabla^2\left(h^{\text{forced}} + h^S\right). \tag{76}$$

Solving for h^{forced}, it is found that the numerical forced fluid depth h^{forced} is given exactly as in the unregularized case (i.e. (39) with $\alpha \equiv 0$ and $\gamma \equiv 0$).

7.4. Free solution

The free solutions to (75) are governed by the explicit recursion

$$h^{n+1} = 2h^n - h^{n-1} - f^2\Delta t^2 \times \left[1 + \left(\frac{f\Delta t}{2}\right)^2 - \left(\frac{c_0\Delta t}{2}\right)^2\nabla^2\right]^{-1} \times \left(1 - L_R^2\nabla^2\right)h^n. \tag{77}$$

This is to be compared with the corresponding recursion (71) for the time-staggered discretization of the regularized equations. *Noting that $L_R \equiv c_0/f$, it is found that the two recursions are precisely equivalent for $\alpha = c_0\Delta t/2$ if, additionally, the choice $\gamma = f\Delta t/2$ is made.* This choice is consistent with the non-rotating ($f \equiv 0$) case where γ was required to be zero.

8. Conclusions

A spatial regularization of the orographically forced SWEs, as used in the recently proposed HPM method, has been analysed in terms of linear perturbations. The effect of the regularization is governed by two parameters: α, which measures the length scale at which the scale-dependent smoothing becomes significant; and γ, which is a measure of the scale-independent smoothing. Provided γ is chosen so that $\gamma \ll 1$ and α is chosen to be much smaller than the Rossby radius of deformation, i.e. such that $\alpha^2 \ll L_R^2$, then the forced response of the regularized equations is close to that of the unregularized equations. Further, and as expected, the free response (the inertia-gravity waves) of the regularized equations approaches that of the unregularized ones as $\alpha \to 0$ and $\gamma \to 0$. For non-zero values of α, the inertia-gravity waves are increasingly retarded as their wavenumber increases (reminiscent of the effect on discrete inertia-gravity waves of the semi-implicit scheme). Increasing γ away from zero also retards the inertia-gravity waves, but the effect, in isolation from α, is independent of the wavenumber.

The regularized equations have then been discretized using a time-staggered leapfrog scheme. It is found that numerical stability of the free solution of this scheme requires

$$\alpha \geq \frac{c_0\Delta t}{2}. \tag{78}$$

Therefore, noting the above requirement for accuracy that α be as small as possible the optimal choice for the smoothing length is $\alpha = c_0\Delta t/2$. Increasing α beyond this value will unnecessarily reduce the accuracy of the free and forced responses. Additionally, for a fixed value of γ, it would also decrease the coefficient of the associated Helmholtz problem and, hence, decrease the efficiency of an iterative solver.

The regularized, time-staggered leapfrog discretization has been compared with the popular two-time-level semi-implicit time discretization. It is found that, for the linearized equations, if α assumes its optimal value and the choice $\gamma = f\Delta t/2$ is made, then the two schemes give exactly the same numerical dispersion relation for the free response, i.e. for the inertia-gravity waves. Additionally, the regularized, time-staggered leapfrog discretization yields a very similar result to the analytic forced response (which is obtained exactly by the semi-implicit discretization) provided $(\alpha/L_R)^2 \ll 1$ and $\gamma^2 \ll 1$. With $\alpha = c_0\Delta t/2$ and $\gamma = f\Delta t/2$, then $\alpha/L_R = \gamma$, and the two conditions reduce to the same requirement, namely that the time step should be chosen such that $f\Delta t/2 \ll 1$. This is generally the case for models of the Earth's atmosphere.

References

Birdsall CK, Langdon AB. 1981. *Plasma Physics via Computer Simulations*, McGraw-Hill: New York.

Bridges TJ, Hydon PE, Reich S. 2005. Vorticity and symplecticity in Lagrangian fluid dynamics. *Journal of Physics A-Mathematical and General* **38**: 1403–1418.

Cotter CJ, Frank J, Reich S. 2004. Hamiltonian particle-mesh method for two-layer shallow-water equations subject to the rigid-lid approximation. *SIAM Journal on Applied Dynamical Systems* **3**: 69–83.

Cotter CJ, Reich S. 2004. Adiabatic invariance and applications: from molecular dynamics to numerical weather prediction. *BIT* **44**: 439–455.

Durran DR. 1998. *Numerical Methods for Wave Equations in Geophysical Fluid Dynamics*, Springer-Verlag: Berlin, Heidelberg.

Frank J, Gottwald G, Reich S. 2002. The Hamiltonian particle-mesh method. In *Meshfree Methods for Partial Differential Equations, Lecture Notes in Computational Science and Engineering*, vol. 26. Griebel M, Schweitzer MA (eds). Springer-Verlag: Berlin, Heidelberg, 131–142.

Frank J, Reich S. 2003. Conservation properties of smoothed particle hydrodynamics applied to the shallow-water equations. *BIT* **43**: 40–54.

Frank J, Reich S. 2004. The Hamiltonian particle-mesh method for the spherical shallow water equations. *Atmospheric Science Letters* **5**: 89–95.

Hockney RW, Eastwood JW. 1988. *Computer Simulations Using Particles*. Institute of Physics Publisher: Bristol and Philadelphia.

Staniforth A. 1997. André Robert (1929–1993): His pioneering contributions to numerical modelling. In *Numerical Methods in Atmospheric and Oceanic Modelling, The André J. Robert memorial volume*, Lin CA, Laprise R, Ritchie H (eds). CMOS/NRC Press: Ottawa, 25–54.

