Malliavin Calculus for Lévy processes and Applications in Finance

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Summary

- Financial Markets
- Classical Malliavin Calculus for Wiener Spaces and Applications in Finance.
- Financial Markets generated by Lévy processes
- Development of Malliavin Calculus for Lévy processes
- Applications

Financial Markets

- Financial market, with underlier

$$dX_t = \mu_t dt + \sigma_t dW_t$$

- Contingent claim, e.x.
  - European call option

$$f(X_T) = (X_T - K)^+$$

  - Digital Call option

$$f(X_T) = \mathbf{1}_{\{X_T > K\}}$$

Main Issues

- Pricing

$$u(x) = E[f(X_T)]$$

where $x = X_0$.

- Hedging

$$f(X_T) = u(x) + \int_0^T \psi_t dX_t$$

- Explicit calculation of hedging strategies

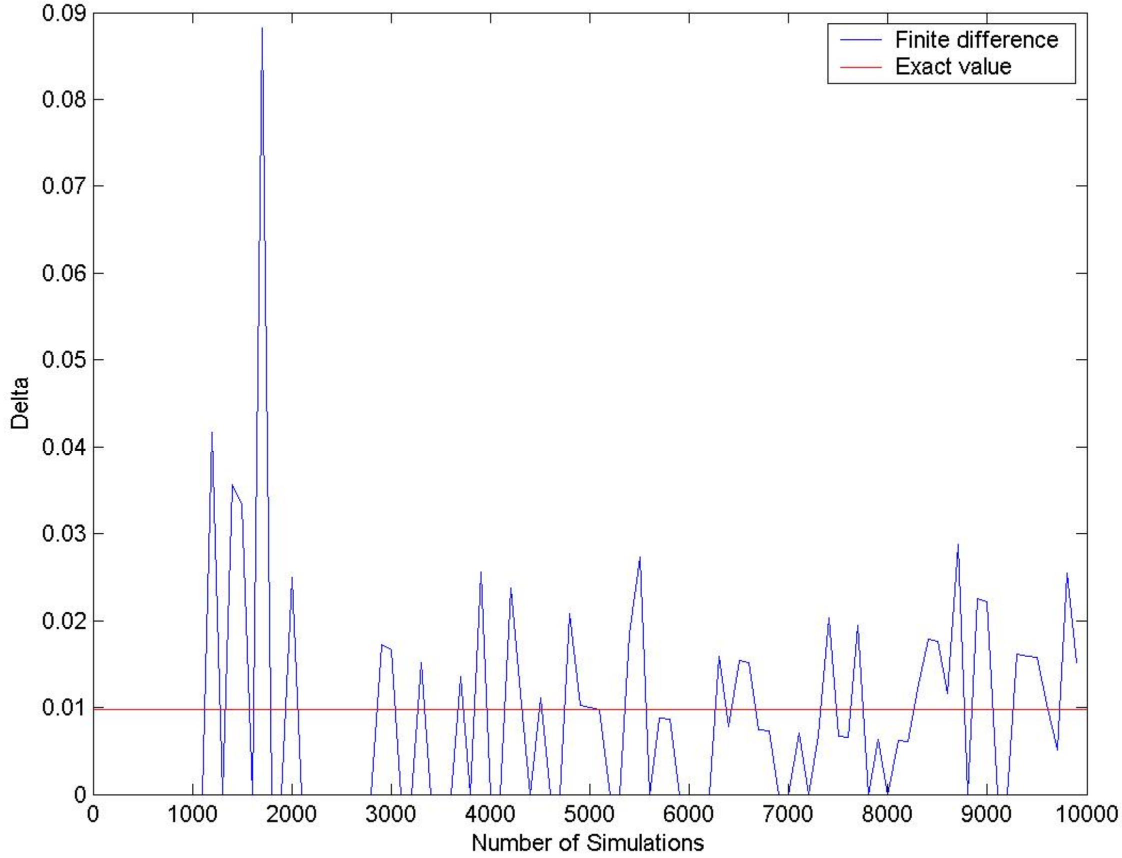
$$\psi_t = ?$$

- European call option $\Rightarrow$ delta hedge, i.e.

$$\psi_t = \Delta_t = \frac{\partial}{\partial x} E[f(X_T)|\mathcal{F}_t]$$

- Compute Δ though numerical methods e.g. finite differences

Δ of a digital option



- **Reason**

$$\frac{\partial}{\partial x}u(x) \;=\; E[f'(X_T)\frac{\partial}{\partial x}X_T]$$

- **Problem** How to calculate the delta for discontinuous options?

- Solution Transform it such a way that we take out the derivative.

- If the density p_x of X exists then: $\frac{\partial}{\partial x}u(x) = E[f(X_T)\pi]$ where $\pi = \frac{\partial}{\partial x}(\log p_x(z))$

- Theoritical approach great! Practical?

- Introduction of Malliavin Calculus by Fournie et al.(1999) in Finance and Stochastics.
  Aim Calculate the weight π without any knowledge of the density p_x.

Classical Malliavin Calculus $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$ Wiener Space.

- Ito's Chaos expansion
- Malliavin derivative
- Properties

Ito's Chaos Expansion (Intuition)

- **Example 1**

$$W_T^2 = \int_0^T \int_0^{t_2} f_2(t_1, t_2) dW_{t_1} dW_{t_2} + T$$

where $f_2(t_1, t_2) = 2$

- **Example 2**

$$W_T^3 - 3TW_T = \int_0^T \int_0^{t_3} \int_0^{t_2} f_3(t_1, t_2, t_3) dW_{t_1} dW_{t_2} dW_{t_3}$$

where $f_3(t_1, t_2, t_3) = 6$

Ito's Chaos Expansion

- $J_n(f_n) = \int_0^T \int_0^{t_n} \dots \int_0^{t_2} f_n(t_1, \dots, t_n) dW_{t_1} \dots dW_{t_n}$

- **Theorem** Every $F \in L^2(\Omega, \mathcal{F}_T)$ random variable can be uniquely written as a constant plus a sum of integrals J_n of deterministic functions f_n.

$$F = E[F] + \sum_{n=1}^{\infty} J_n(f_n)$$

Malliavin Derivative

$$
\begin{aligned}
D_t J_n(f_n) \;=\; & \sum_{k=1}^{n} \int_0^T \int_0^{t_n} \cdots \int_0^{t_{k+1}} \int_0^{t_{k-1}} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_{k-1}, t, t_{k+1}, t_n) \\
\times \;\; & \mathbf{1}_{\{t_k - 1 \le t \le t_{k+1}\}} dW_{t_1} \ldots dW_{t_{k-1}} dW_{t_{k+1}} dW_{t_n}
\end{aligned}
$$

Example 1

$$
\begin{aligned}
D_t W_T^2 &= D_t \Big(\int_0^T \int_0^{t_2} f_2(t_1, t_2) dW_{t_1} dW_{t_2} + T \Big) \\
&= \int_t^T 2 dW_{t_2} + \int_0^t 2 dW_{t_1} \\
&= 2 W_T
\end{aligned}
$$

- **Malliavin Derivative** If F is in the domain of D then

$$D_t F = \sum_{n=1}^{\infty} D_t J_n(f_n)$$

- **Chain Rule**

  If $f \in C_b(\mathbb{R})$, then

$$D_t(f(F)) = f'(F) D_t F$$

SDE

$$dX_t = \alpha(t, X_{t-})dt + \sigma(t, X_{t-})dW_t$$

- α, σ continuously differentiable functions with bounded derivatives in x

- Then if $Y_t = \frac{\partial}{\partial x} X_t$ the first variation of X_t

$$D_r X_t = Y_t Y_r^{-1} \sigma(r, X_r), \quad \forall r \leq t$$

Sensitivities

$$\frac{\partial}{\partial x}u(X_T) = E[f'(X_T)\frac{\partial}{\partial x}X_T]$$
$$= E[f(X_T)\int_0^T \frac{Y_t}{\alpha(t)\sigma(t, X_t)}dW_t],$$

where $\alpha \in L^2([0, T])$ such that $\int_0^T \alpha(t)dt = 1$.

- Global weight $\pi = \frac{Y_t}{\alpha(t)\sigma(t,X_t)}dW_t$

Models with Lévy processes

1. Are we able to extend the results of Fournie et al?

2. Working in Incomplete markets.
   What is the form of the hedging strategies?

Lévy Processes

A Lévy process $Z \in L^2(\Omega)$ is a right continuous process:

- has independent increments; $Z_t - Z_s, Z_s$ are independent for $s \leq t$
- is stationary; the law of $Z_{t+h} - Z_t$ does not depend on t
- is Stochastic continous; $\lim_{h \to 0} P(|Z_{t+h} - Z_t| \geq \epsilon) = 0$

Decomposition Theorem

$$Z_t = \sigma W_t + "compensated \quad Jump \quad part"$$

Jump part:

- a compensated Poisson process $N_t - \lambda t$
- a com. compound Poisson process $J_t = \sum_{i=1}^{N_t} Y_i - E[Y_i]\lambda t$, where Y_i are i.i.d
- an infinite activity jump process, e.x. Normal Inverse Gaussian

Representation Property

$$Z_t = \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z(\mu - \pi)(dz, ds)$$

- W_t: standard Wiener process
- $\mu(\cdot, \cdot)$: Poisson random measure
- $\pi(dz, dt)$ is the compensator of μ
- $\tilde{\mu}(dz, dt) = \mu(dz, dt) - \pi(dz, dt)$

Malliavin Calculus for Lévy processes

Theorem 1. *Every square integrable $\mathcal{F}_T$ random variable F can represented as*

$$F = \sum_{n=0}^{\infty} \sum_{j_1,\dots,j_n=0,1} J_n^{(j_1,\dots,j_n)}(f_{j_1,\dots,j_n})$$

Example 3

$$J_3^{(0,1,0)}(f_{0,1,0}) =$$
$$\int_0^T \int_0^{t_3} \int_{\mathbb{R}_0} \int_0^{t_2} f_{0,1,0}(t_1, t_2, z, t_3) dW_{t_1} \tilde{\mu}(dt_2, dz) dW_{t_3}$$

Directional Derivative

- Define derivatives in the Wiener and the Poisson Random measure direction, denote respectively as $D^{(0)}$ and $D^{(1)}$

- Reduce the order of integration as in the classical Wiener case.

Directional Derivative

Let $F \in \mathbb{D}^{(l)}$. Then the derivative on the l-th direction is:

$$D^{(l)}_{u^l} F = \sum_{n=1}^{\infty} \sum_{j_1,\dots,j_n=0,1} \sum_{i=1}^{n} \mathbf{1}_{\{j_i=l\}} D^l J_n^{(j_1,\dots,\widehat{j}_i,\dots,j_n)} \left(f_{j_1,\dots,j_n}\right)$$

Example 3

$$\begin{aligned}
& D_t^0 J_3^{(0,1,0)}(f_{0,1,0}) \\
= \ & D_t^0 \int_0^T \int_0^{t_3} \int_{\mathbb{R}_0} \int_0^{t_2} f_{0,1,0}(t_1, t_2, z, t_3) dW_{t_1} \tilde{\mu}(dt_2, dz) dW_{t_3} \\
= \ & \int_0^t \int_{\mathbb{R}_0} \int_0^{t_2} f_{0,1,0}(t_1, t_2, z, t) dW_{t_1} \tilde{\mu}(dt_2, dz) \\
+ \ & \int_0^T \int_0^{t_3} \int_{\mathbb{R}_0} f_{0,1,0}(t, t_2, z, t_3) \tilde{\mu}(dt_2, dz) dW_{t_3}
\end{aligned}$$

Chain Rule

Theorem 2. *Let $F \in \mathbb{D}^{(0)}$ and f be a continuously differentiable function with bounded derivative. Then $f(F) \in \mathbb{D}^{(0)}$ and the following chain rule holds:*

$$D^{(0)} f(F) = f'(F) D^{(0)} F.$$

1. *Let $F \in \mathbb{D}^{(1)}$ then*

$$D^{(1)}_{(t,z)} F = F \circ \epsilon^{+}_{(t,z)} - F,$$

where ϵ^{+} is a transformation on Ω that implies that we have a jump of size z at time t.

Clark-Ocone-Haussman (COH) formula

Theorem 3. *Let* $F \in L^2(\mathcal{F}_T) \cap (\cap_{l=0,1} \mathbb{D}^{(l)})$. *Then*

$$F = E[F] + \int_0^T E[D_t^{(0)} F | \mathcal{F}_{t^-}] dW_t + \int_0^T \int_{\mathbb{R}_0} E[D_{(t,z)}^{(1)} F | \mathcal{F}_{t^-}] \tilde{\mu}(dz, st)$$

Properties of SDEs

- $\{X_t\}_{t\in[0,T]}$: a square integrable process satisfying the following stochastic differential equation:

$$dX_t = \alpha(t, X_{t^-})dt + \sigma(t, X_{t^-})dW_t + \int_{\mathbb{R}_0} \gamma(t, z, X_{t^-})\tilde{\mu}(dz, dt)$$

- α, σ and γ are "nice enough" so there exists a unique solution

- The directional derivatives of X exist and for the Wiener directional derivative

$$D_r^{(0)} X_t = Y_t Y_{r^-}^{-1} \sigma(r, X_{r^-}), \quad \forall r \leq t$$

Sensitivities

$$\begin{aligned} dX_t &= \alpha(t, X_{t^-})dt + \sigma(t, X_{t^-})dW_t + \int_{\mathbb{R}_0} \gamma(t, z, X_{t^-})\tilde{\mu}(dz, dt), \\ X_0 &= x \end{aligned}$$

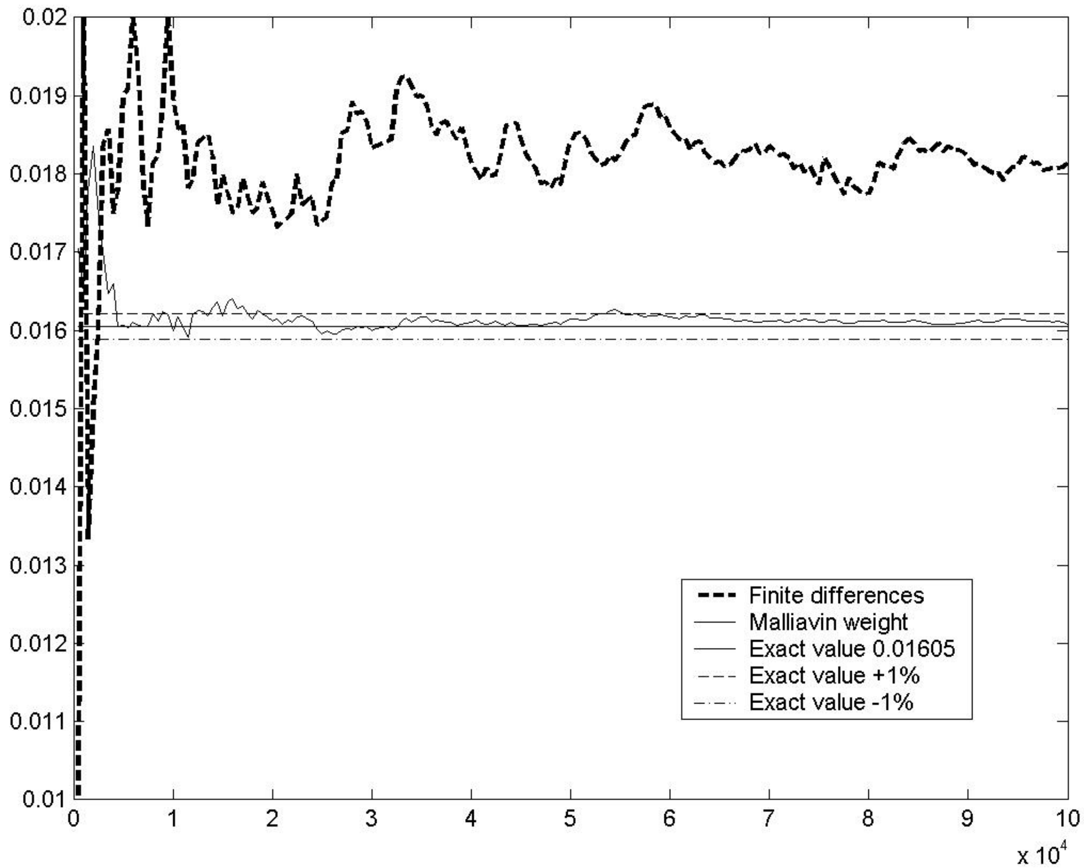
$$\Delta = E\Big[f(X_T) \int_0^T \alpha(t)\sigma^{-1}(t, X_{t^-})Y_{t^-}dW_t \Big]$$

where $\alpha \in L^2([0, T))$ such that $\int_0^T \alpha(t)dt = 1$

Bates Model

$$
\begin{aligned}
dX_t^1 &= rX_{t-}^1 dt + \sqrt{X_{t-}^2} X_{t-}^1 dW_t^1 \\
&+ X_{t-} dJ_t \\
dX_t^2 &= k(m - X_{t-}^2)dt + \sigma \sqrt{X_{t-}^2} dW_t
\end{aligned}
$$

△ of a digital option



Hedging strategies in Lévy markets

- $dX_t = \sigma(t, X_{t-})dW_t + \int_{\mathbb{R}_0} \gamma(t, z, X_{t-})\tilde{\mu}(dz, dt)$

- Contingent claim $f(X_T)$

- Hedging portfolio $V_T = V_0 + \int_0^T \phi_t dX_t$

- **(Kunita-Watanabe decomposition)**
  Let $(X_t)_{t\in[0,T]}$ be a martingale, then

$$f(X_T) = E[f(X_T)] + \int_0^T \theta_t dX_t + N,$$

  where $(\theta_t)_{t\in[0,T]}$ is a predictable process and N is a random variable orthogonal to all integrals with respect to X.

  – Minimize Variance of the Hedging error; i.e. minimize N

$$\inf_\theta E[|f(X_T) - V_T|^2]$$

Question

Is Δ hedging the optimal hedging strategy?

Answer

NO!!!

Contingent Claim representation

$$\begin{aligned}
f(X_T) &= E[f(X_T)] + \int_0^T E[D_t^{(0)} f(X_T)|\mathcal{F}_{t-}]dW_t \\
&+ \int_0^T \int_{\mathbb{R}_0} E[D_{(t,z)}^{(1)} f(X_T)|\mathcal{F}_{t-}]\tilde{\mu}(dz, st) \\
&= E[f(X_T)] + \int_0^T \Delta_t Y_{t^-}^{-1} \sigma(t, X_{t^-}) dW_t \\
&+ \int_0^T \int_{\mathbb{R}_0} E[f(X_T \circ \epsilon_{t,z}^{+}) - f(X_T)]\tilde{\mu}(dz, dt)
\end{aligned}$$

- f differentiable then

$$\theta_t := \frac{1}{\sigma_t'}\Big[\sigma(t,X_{t-})E[D_t^{(0)}f(X_T)|\mathcal{F}_{t-}] + \int_{\mathbb{R}_0}\gamma(t,z,X_{t-})E[D_{t,z}^{(1)}f(X_T)|\mathcal{F}_{t-}]\nu(dz)\Big],$$

- f is not differentiable

$$\theta_t = \frac{1}{\sigma_t'}\Big[\sigma(t,X_{t-})\Delta_t Y_{t^-}^{-1}\sigma(t,X_{t-}) + \int_{\mathbb{R}_0}\gamma(t,z,X_{t-})E\Big[f(X_T\circ\epsilon^{+}_{(t,z)}) - f(X_T)\Big|\mathcal{F}_{t^-}\Big]\nu(dz)\Big]$$

- where $\sigma_t' = \sigma^2(t,X_{t-}) + \int_{\mathbb{R}_0}\gamma^2(t,z,X_{t-})\nu(dz)$.

- **Merton's model**

$$X_t = X_0 e^{\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i},$$

λ, Y_i $N(a, b^2)$ are i.i.d and $\mu = \frac{\sigma^2}{2} - \lambda E[e^{Y_i} - 1]$

- $f(X_T) = (X_T - K)^+$

-

$$D_t^{(0)} f(X_T) = \sigma X_T \mathbf{1}_{\{X_T > K\}}$$

$$D_{(t,z)}^{(1)} f(X_T) = (e^z X_T - K)^+ - (X_T - K)^+$$

$$
\begin{aligned}
&E[D_t^{(0)} f(X_T)|\mathcal{F}_t] \\
= \;& \sigma \frac{X_t}{X_0} \sum_{n=0}^{\infty} e^{\lambda'(T-t)} \frac{(\lambda'(T-t))}{n!} \Phi\left(\frac{-\log K' + (\mu_n + \frac{\sigma_n^2}{2})(T-t)}{\sigma_n \sqrt{(T-t)}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&E[D_{(t,z)}^{(1)} f(X_T)|\mathcal{F}_t] \\
= \;& BS(e^z \frac{X_t}{X_0}, \sigma_n, T-t, \mu_n, K'') - BS(\frac{X_t}{X_0}, \sigma_n, T-t, \mu_n, K')\Big\}
\end{aligned}
$$

where $K' = \frac{X_t}{X_0}$ and $K'' = e^z \frac{X_t}{X_0}$.

- $f(X_T) = (X_T - K)^+$

$$\theta_t = \frac{\sigma E[D_t^{(0)} f(X_T)|\mathcal{F}_{t^-}] + \int_{\mathbb{R}_0} (e^z - 1) E[D_{(t,z)}^{(1)} f(X_T)|\mathcal{F}_{t_-}]\nu(dz)}{X_{t^-}(\sigma^2 + \int_{\mathbb{R}_0} (e^z - 1)^2 \nu(dz))}$$

- $f(X_T) = \mathbf{1}_{\{X_T > K\}}$

$$\theta_t = \frac{\sigma^2 \Delta_t + \int_{\mathbb{R}_0} (e^z - 1) E[D_{(t,z)}^{(1)} f(X_T)|\mathcal{F}_{t_-}]\nu(dz)}{X_{t^-}(\sigma^2 + \int_{\mathbb{R}_0} (e^z - 1)^2 \nu(dz))}$$

where $\Delta_t = E[f(X_T) \frac{W_T}{X_0 \sigma T} |\mathcal{F}_{t^-}]$

Hedging with other derivatives

- Infinite number of risk factors need infinite number of assets to complete the market

- **Example** BNS model

$$
\begin{aligned}
dX_t &= \sigma_t X_{t-} dW_t \\
X_0 &= x \\
d\sigma_t^2 &= -\kappa \sigma_t^2 dt + dZ_t \\
\sigma_0^2 &> 2, \kappa > 0
\end{aligned}
$$

- $f^1(X_T) = \mathbf{1}_{\{X_T > K^1\}}$

- Hedging strategy with respect to the underlier is given by

$$\theta_t^1 = \frac{\sigma_{t-} E[f^1(X_T) \frac{1}{x} \left(\int_0^T \frac{dW_t}{\sigma_{t-}} \right) |\mathcal{F}_{t-}]}{\sigma_t'}$$

- The underlier only hedges the Wiener part.

- Hedge with X and $f^2(X_T) = \mathbf{1}_{\{X_T > K^2\}}$
- What is the sde that f^2 satisfies?
- Applying Ito's formula will impose conditions on the jump part of X
- Use COH-formula to achieve a representation

- Using COH-formula

$$f^2(X_T) = E[f^2(X_T)] + \int_0^T E[f^2(X_T)\frac{1}{x}\left(\int_0^T \frac{dW_t}{\sigma_{t^-}}\right)|\mathcal{F}_{t-}]dW_t$$

$$+ \int_0^T \int_{\mathbb{R}_0} E[\mathbf{1}_{\left\{S_{t^-} \exp(\int_t^T \sqrt{\sigma_s^2+z} dW_s)>K\right\}} - \mathbf{1}_{\{S_T>K\}}|\mathcal{F}_{t-}]\tilde{\mu}(dz,dt)$$

- Hedging strategy

$$\theta_t^2 = \frac{1}{\sigma_t'}\Big(\sigma_{t^-}E[f^2(X_T)\frac{1}{x}\left(\int_0^T \frac{dW_t}{\sigma_{t^-}}\right)|\mathcal{F}_{t-}]$$

$$+ \int_{\mathbb{R}_0} E[\mathbf{1}_{\left\{S_{t^-} \exp(\int_t^T \sqrt{\sigma_s^2+z} dW_s)>K\right\}} - \mathbf{1}_{\{S_T>K\}}|\mathcal{F}_{t-}]\nu(dz)\Big)$$

- Optimal portfolio
  - θ_t^1 of the underlier X
  - θ_t^1 of the of the digital option f^2