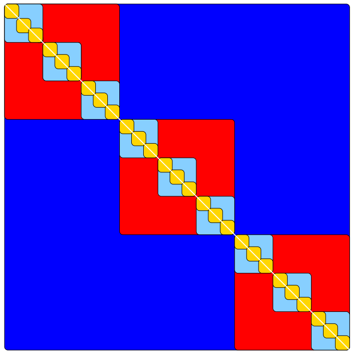
On a nonhierarchical version of the GREM



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Outline

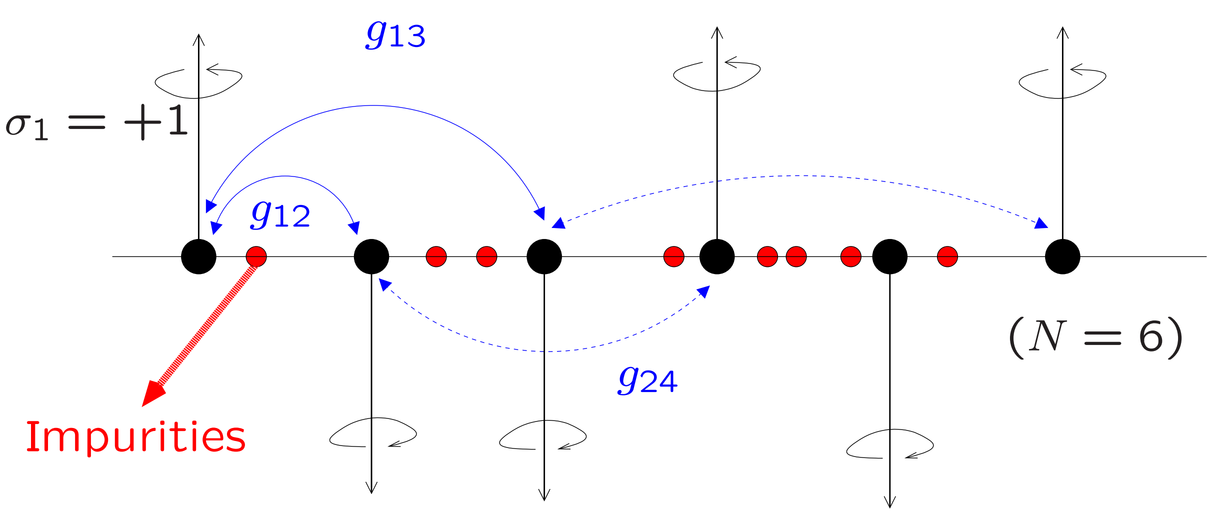
1. The SK-model and the Parisi Picture

2. Derrida's GREM

3. The nonhierarchical GREM

1. The Sherrington-Kirkpatrick Model.

Configurations $\sigma \in \Sigma_N = \{-1,+1\}^N$. On $(\Omega, \mathcal{F}, \mathbb{P})$ random interactions (g_{ij}) iid standard gaussians.



SK-Hamiltonian $H_{N,\omega}(\sigma) = \frac{1}{\sqrt{N}} \sum_{1 \leq i < j \leq N} g_{ij}(\omega) \sigma_i \sigma_j$.

Random Gibbs measure

$$\mathcal{G}_{\beta,N,\omega}(\sigma) = \frac{1}{Z_{N,\omega}(\beta)} 2^{-N} \exp\left[\beta H_{N,\omega}(\sigma)\right],$$

partition function $Z_{N,\omega}(\beta) = \sum_{\tau \in \Sigma_N} 2^{-N} \exp\left[\beta H_{N,\omega}(\tau)\right]$.

$$\lim_{N\to\infty} \left(\mathcal{G}_{\beta,N,\cdot}(\sigma);\ \sigma \in \Sigma_N\right) \text{ ? } \textbf{Hard Problem}$$

$\Rightarrow$ Study first **free energy**

$$f_{N,\omega}(\beta) = \frac{1}{N} \log Z_{N,\omega}(\beta)$$

Basics. Y_i iid, var$[Y_i] < \infty$.

Law of Large Numbers $$\frac{1}{n}\sum_{i=1}^{n} Y_i \to \mathbb{E}[Y_1] \qquad \textbf{(LLN)}$$

Extreme Value Theorem $$\max_{i=1\ldots n} \frac{Y_i}{\sqrt{2\log n}} \to 1 \qquad \textbf{(EVT)}$$

In the SK-model.

- Exploding variances

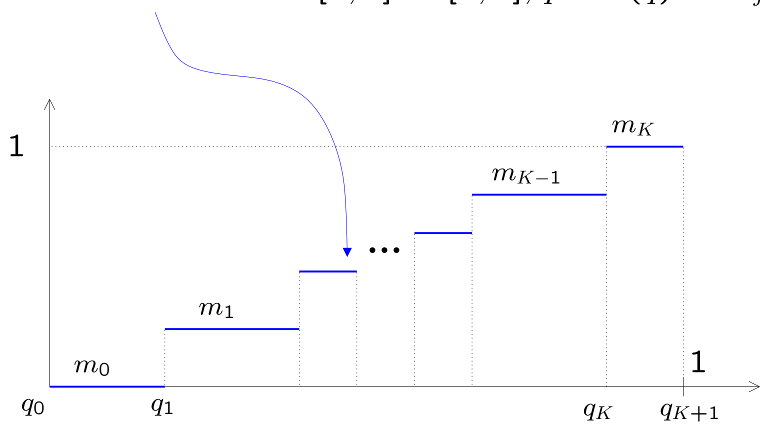
$$\mathbb{E}\left[H_{N,\cdot}(\sigma)^2\right] \simeq N/2$$

- Correlations through overlap $q_N(\sigma,\tau)$

$$\mathbb{E}[H_{N,\cdot}(\sigma)H_{N,\cdot}(\tau)] \simeq \frac{N}{2}\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i\tau_i\right)^2 .$$

Parisi Picture for SK-model, Free Energy

Order Parameter $x : [0,1] \to [0,1], q \mapsto x(q) = m_j,\ q \in [q_{j-1}, q_j)$

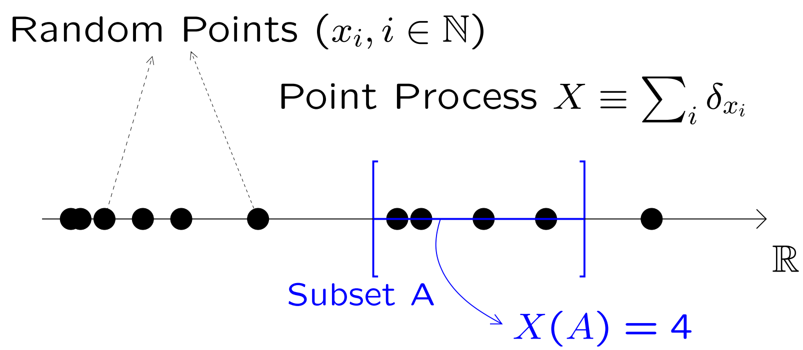


Let $f = f(q, y; x, \beta)$ solve Parisi-PDE

$$\partial_q f + \frac{1}{2}\left[\partial_y^2 f + x(q)\,(\partial_y f)^2\right] = 0, \quad f(1, y) = \log\cosh(\beta y).$$

$$\mathbb{P} - \lim_N f_{N,\omega}(\beta) = \inf_x \left\{ f(0,0;x,\beta) - \frac{\beta^2}{2}\int q x(q) dq \right\}$$

Random Probabilities, Poisson Point Processes.



Poisson Point Process (PPP) of density $g(x)dx$

$$\circ\ \mathbb{P}\left[X(A) = n\right] = \exp\left[-\mu(A)\right]\frac{\mu(A)^n}{n!}, \quad \mu(A) = \int_A g(x)dx.$$

$\circ\ A_1, \ldots, A_n \subset \mathbb{R}$ disjoint $\Rightarrow X(A_1), \ldots, X(A_n)$ **independent**

If $x_i \geq 0$ and $\sum_i x_i < \infty$ $\mathbb{P}$-a.s. define Normalization

$$x_i \mapsto \overline{x}_i = \frac{x_i}{\sum_j x_j}, \quad \mathcal{N}((x_i, i \in \mathbb{N})) = (\overline{x}_i, i \in \mathbb{N}).$$

Derrida-Ruelle Structures with $0 < m_1 < ... < m_K < 1$

$$\text{PP}\left(\eta_{\boldsymbol{i}}; \boldsymbol{i} \in \mathbb{N}^K\right), \ \eta_{\boldsymbol{i}} = \eta^1_{i_1} \eta^2_{i_1,i_2} \cdots \eta^K_{i_1,\dots,i_K}$$

- For $i_1, \dots, i_{j-1}, \left(\eta^j_{i_1,\dots,i_{j-1},k}, k\right)$ PPP$\left(m_j t^{-m_j-1} dt\right)$ on $\mathbb{R}_+$
- (η^j) independent for different j
- $\left(\eta^j_{i_1,\dots,i_{j-1},l}, l\right)$ independent for different $i_1, \dots, i_{j-1}$

Remarkable properties

- $\max_i \eta_i < \infty, \quad \sum_i \eta_i < \infty, \quad \mathbb{P}-$a.s.

- $\sum_i \overline{\eta}_i = 1 \rightsquigarrow (\overline{\eta}_i)$ Random Probability on $\mathbb{N}^K$

- $\sum_i \delta_{\overline{\eta}_i} \overset{\text{(law)}}{=} \mathcal{N}\left(\text{PPP}(m_K t^{-m_K-1} dt)\right)$

$$\Rightarrow \sum_i \delta_{\overline{\eta}_i} \text{ forgets structure}$$

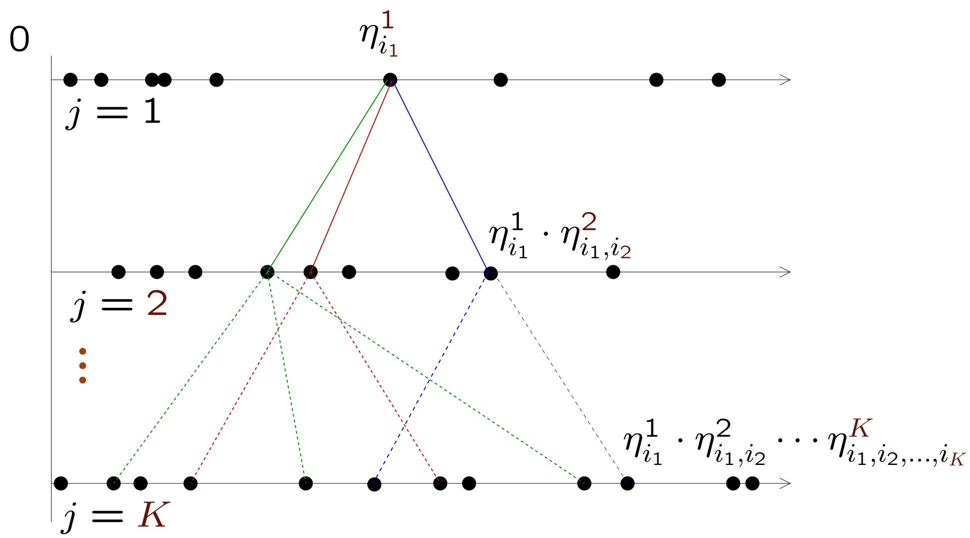
Retaining the K-levels tree structure

Order $(\eta_{\boldsymbol{i}})$ downwards, random bijection

$$\pi : \mathbb{N} \to \mathbb{N}^K, \qquad k \mapsto \pi(k),$$

with $\eta_{\pi(k)}$ k^{th}-largest element in $\{\eta_{\boldsymbol{i}}\}$. Overlaps

$$q(i,i') = \max_{\boldsymbol{r}} \left\{ \pi(i)_1, \ldots \pi(i)_{\boldsymbol{r}} = \pi(i')_1, \ldots \pi(i')_{\boldsymbol{r}} \right\}$$



Equivalence relations on $\mathbb{N}$, $\quad i \sim_k i' \Longleftrightarrow q(i,i') \geq k.$

Keep track of equivalence classes for $k = 0, 1, ..., K$

$\rightsquigarrow$ clustering of random partitions $(\mathcal{Z}_K, \mathcal{Z}_{K-1}, \ldots, \mathcal{Z}_1, \mathcal{Z}_0)$

$\mathcal{Z}_K =$ 'singletons', $\mathcal{Z}_0$= 'one single set'.

Bolthausen-Sznitman coalescent

Parisi Picture for SK-model, Gibbs measure $(\beta > 1)$

- Distance $d_N = (1 - q_N)$ becomes **ultrametric**,

$$\lim_N \mathbb{P} \otimes \mathcal{G}_{\beta,N,\cdot}\left[d_N(\sigma,\sigma') \leq \textbf{max}_{\sigma''}\left\{d_N(\sigma,\sigma'), d_N(\sigma',\sigma'')\right\}\right] = 1$$

$\Rightarrow$ **Configuration space hierarchically organized**

- **Pure states** $(\eta_i)_{i\in\mathbb{N}}$:

$$\langle\cdot\rangle_{\beta,N,\omega} = \sum(\cdot)\mathcal{G}_{\beta,N,\omega}(\sigma) \to \sum_i \eta_i(\omega)\,\langle\cdot\rangle_\omega^{(i)}\,,$$

$$(\eta_i) \stackrel{\text{(d)}}{=} \mathcal{N}(\text{PPP}(m_K t^{-m_K-1}dt))$$

- **Tree structure** $(\mathcal{Z}_K, \mathcal{Z}_{K-1}, \ldots, \mathcal{Z}_0)$ on pure states, possibly $K = \infty$, **independent** of Gibbs measure.

Claim: Picture universal,

valid for "every" Hamiltonian

$\Rightarrow$ Applications to Physics (condensed matter), **Probability**, Computer Science, Combinatorial Optimization, Neural Networks, Coding Theory, etc.

$$\lim_{\beta\to\infty} (1/\beta) \log Z_N(\beta) = \max_{\sigma} H_N(\sigma)$$

État de la Recherche

- Free Energy predicted by Parisi is correct. [Guerra-Toninelli ’02, Guerra ’03, Talagrand ’04]

- Gibbs measure in high temperature ($\beta < 1,\ f(\beta) = \beta^2/4$) fully understood [Talagrand, ’98-’04]

- Low temperature: Big Mystery. **Ultrametricity?** Pure States?

- Universality out of reach.

2. Derrida's GREM ($n \in \mathbb{N}$ fixed).

Configurations $\sigma = (\sigma_1..\sigma_n)$, $\sigma_i = 1..2^{\gamma_i N}$, and $\sum_{i\leq n}\gamma_i = 1$.

Tree-like Hamiltonian

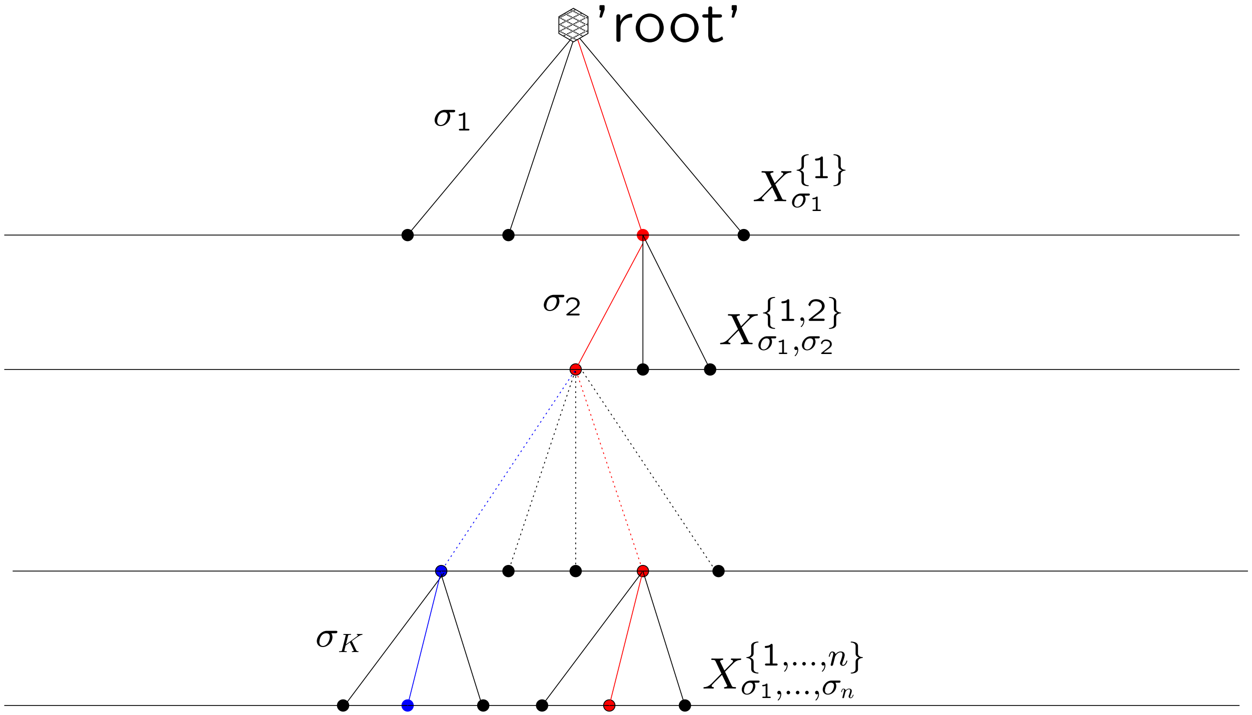
$$X_\sigma = X_{\sigma_1}^{\{1\}} + X_{\sigma_1,\sigma_2}^{\{1,2\}} + \cdots + X_{\sigma_1,\sigma_2,\ldots,\sigma_n}^{\{1,\ldots,n\}},$$

with $\quad X_{\sigma_1,\ldots,\sigma_i}^{\{1,\ldots,i\}} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, N a_{1,\ldots,i}\right), \quad \sum_{i\leq n} a_{1,\ldots,i} = 1.$

$$Z_N(\beta) = 2^{-N}\sum_\sigma \exp\left[\beta X_\sigma\right], \qquad f_N(\beta) = \frac{1}{N}\log Z_N(\beta),$$

$$\mathcal{G}_{\beta,N}(\sigma) = \frac{1}{Z_N(\beta)} 2^{-N}\exp\left[\beta X_\sigma\right]$$

Construction of the GREM-Hamiltonian



$$X_\sigma = X^{\{1\}}_{\sigma_1} + \cdots + X^{\{1,\dots,n\}}_{\sigma_1,\dots,\sigma_n}$$

Thm. Parisi Picture correct for GREM.

- *[Capocaccia et.al. '87] FE self-averaging. Limit with K phase transitions $\beta_1 < \beta_2 < ... < \beta_K$.*

- *[Bovier-Kurkova '04] Energy Levels*

$$\sum_\sigma \delta_{X_\sigma - a_N} \to \sum_{\boldsymbol{i}} \delta_{\xi^1_{i_1} + \cdots + \xi^j_{i_1,\dots,i_K}}$$

with ξ^j PPP($\beta_j e^{-\beta_j t} dt$). For $\beta > \beta_K, m_K = \beta_K/\beta$,

$$\triangleright \sum_\sigma \delta_{\mathcal{G}_{\beta,N}(\sigma)} \to \mathcal{N}(PPP(m_K t^{-m_K - 1} dt)),$$

$$\triangleright (\mathcal{Z}^N_K, \mathcal{Z}^N_{K-1}, ..., \mathcal{Z}^N_1, \mathcal{Z}^N_0) \to \text{ BS-coalescent}$$

- *Similar results for $\beta_{j-1} < \beta < \beta_j$.*

But what the origin of ultrametricity?

3. The nonhierarchical GREM - GGREM

Replace tree by **arbitrary graph**,

$$X_\sigma = \sum_{J \subset I = \{1 \dots n\}} X^J_{\sigma_J},$$

$\sigma_i \leq 2^{\gamma_i N}$, $\sum_{i \leq n} \gamma_i = 1$, $\sum_J a_J = 1$, $X^J_{\sigma_J} \overset{\text{iid}}{\sim} \mathcal{N}(0, a_J N)$.

Coarse graining.

For $A \subset I$, $\mathcal{P}_A = \{J \subset A : a_J > 0\}$, $\gamma(A) = \sum_{i \in A} \gamma_i$.

For **chain**

$$\mathbf{T} = \{\emptyset = A_0 \subset A_1 \dots \subset A_K = I\}$$

define **GREM** $X_\sigma(\boldsymbol{T})$ with parameters $\widehat{\gamma}_j \equiv \gamma(A_j \setminus A_{j-1})$, variances $\widehat{a}_{A_j} = \sum_{\mathcal{P}_{A_j} \setminus \mathcal{P}_{A_{j-1}}} a_J$, and limiting FE $f(\beta, \boldsymbol{T})$.

E.g. "Triangle- GREM".

$$X_\sigma = X^{\{1,2\}}_{\sigma_1,\sigma_2} + X^{\{1,3\}}_{\sigma_1,\sigma_3} + X^{\{2,3\}}_{\sigma_2,\sigma_3},$$

$$\sigma_i \le 2^{\gamma_i N}, \text{var}\left(X^{\{i,j\}}_{\sigma_i,\sigma_j}\right) = N a_{ij}.$$

Given chain $\boldsymbol{T} = \{\{1,2\},\{1,2,3\}\}$, define GREM

$$X_\sigma(\boldsymbol{T}) = X^{\{1\}}_{\sigma_1}(\boldsymbol{T}) + X^{\{1,2\}}_{\sigma_1,\sigma_2}(\boldsymbol{T}),$$

where

$$\sigma_1 \le 2^{(\gamma_1+\gamma_2)N}, \qquad \sigma_2 \le 2^{(\gamma_3)N},$$

$$\text{var}\left(X^{\{1\}}_{\sigma_1}(\boldsymbol{T})\right) = N a_{12}, \quad \text{var}\left(X^{\{1,2\}}_{\sigma_1,\sigma_2}(\boldsymbol{T})\right) = N\left(a_{13} + a_{23}\right).$$

Thm (Bolthausen-K. '04). GREM universality.

- *GGREM-Free Energy exists, self-averaging.*

- $\exists$ *chain* $\boldsymbol{T} = \{A_1, A_2, \ldots, A_K\}$

$$f(\beta) = f(\beta, \boldsymbol{T})$$

- $\boldsymbol{T}$ *is minimal,*

$$f(\beta) = \min_{\boldsymbol{S}} f(\beta, \boldsymbol{S})$$

But Ultrametricity $\iff$ Hamiltonian irreducible

Addressing the Ultrametricity for GGREM

▶ [Finite N] Marked Point Process on $\mathbb{R} \times \mathbb{R} \times 2^I$

$$\Xi_N \stackrel{\text{def}}{=} \sum_{\sigma \neq \tau} \delta_{X_\sigma - a_N, X_\tau - a_N; \widehat{q}(\sigma,\tau)}$$

with 'overlap'

$$\widehat{q}(\sigma, \tau) = J \iff \sigma_i = \tau_i, \; i \in J.$$

▶ [Limiting Object] Marked Point Process on $\mathbb{R} \times \mathbb{R} \times 2^I$

$$\Xi \stackrel{\text{def}}{=} \sum_{\boldsymbol{i} \neq \boldsymbol{i}'} \delta_{\xi_{\boldsymbol{i}}, \xi_{\boldsymbol{i}'}; \breve{q}(\boldsymbol{i}, \boldsymbol{i}')}$$

$$\xi_{\boldsymbol{i}} = \xi^1_{i_1} + \cdots + \xi^K_{i_1,\ldots,i_K} \quad \text{and} \quad \left(\xi^j\right) \; \text{PPP}(\kappa_j \beta_j e^{-\beta_j t} dt)$$

and 'overlap'

$$\breve{q}(\boldsymbol{i}, \boldsymbol{i}') = A_j \iff \max_r \left\{(i_1, \ldots, i_r) = (i'_1, \ldots, i'_r)\right\} = j.$$

Thm (B-K '06). X_σ *irreducible.*

- $\Xi_N \to \Xi$ *weakly.*
- *Let* $\beta > \beta_K, m_K = \beta_K/\beta$, *and* $\eta_{\boldsymbol{i}} = \exp[\beta\xi_{\boldsymbol{i}}]$.

$$\sum \delta_{\mathcal{G}_{\beta,N}(\sigma),\, \mathcal{G}_{\beta,N}(\tau);\, \widehat{q}(\sigma,\tau)} \longrightarrow \sum \delta_{\overline{\eta_{\boldsymbol{i}}},\, \overline{\eta_{\boldsymbol{i}'}};\, \breve{q}(\boldsymbol{i},\boldsymbol{i}')}$$

$$\stackrel{(d)}{=} \mathcal{N}\left(PPP^{(2)}(m_K t^{-m_K-1})\right) \otimes \text{"BS-coalescent"}.$$

In particular, **ultrametricity**:

$$\lim_{N\to\infty} \mathbb{E}\left[\left\langle \widehat{q}(\sigma,\sigma') \cap \widehat{q}(\sigma',\sigma'') \subset \widehat{q}(\sigma,\sigma'') \right\rangle_{\beta,N,\cdot}\right] = 1.$$

- *Similar result for* $\beta_{j-1} < \beta < \beta_j, j = 1, \ldots, K$