Stochastic correlation in exponential utility indifference valuation

Christoph Frei

cfrei@math.ethz.ch
Department of Mathematics
ETH Zurich

Joint work with Martin Schweizer

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Motivation: Valuation of contingent claims

Given: Discounted share price S (cont. semimartingale); payoff $H \in L^\infty(\mathcal{F}_T)$.

Question: Fair value $h(t)$ for H at $t < T$?

H attainable $\left(H = x + \int_0^T \vartheta_s \, dS_s\right)$ → unique arbitrage-free price $\left(h(t) = x + \int_0^t \vartheta_s \, dS_s\right)$

H nonattainable, many values consistent with no-arbitrage → use additional criterion

Criterion: Exponential utility indifference

$$\sup_\vartheta \mathbb{E}\left[U\left(\int_0^T \vartheta_s \, dS_s\right)\right] = \sup_\vartheta \mathbb{E}\left[U\left(\int_0^T \vartheta_s \, dS_s + H - h(0)\right)\right]$$

Motivation: Exponential utility indifference valuation

- Exponential utility indifference valuation
  - explicit formula (Brownian setting)
    - PDE-approach (Markovian setting) Musiela and Zariphopoulou
    - martingale arguments (stochastic correlation) Frei and Schweizer
    - ad hoc technics (constant corr.) Tehranchi
  - general properties (dual methods)

Outline

1. The nontradable asset model
    - Financial market
    - Optimization problem
2. Bounds for the value process
    - Proposition
    - First steps of the proof
3. An explicit indifference valuation formula
    - Main result
    - Basic idea of the proof

The nontradable asset model

Financial market:

- Tradable stock S

$$\frac{dS_s}{S_s} = \mu_s\, ds + \sigma_s\, dW_s, \quad 0 \leqslant s \leqslant T, \qquad S_0 > 0;$$

$\updownarrow$ correlation ρ

- Contingent claim H is $\tilde{W}$-measurable.

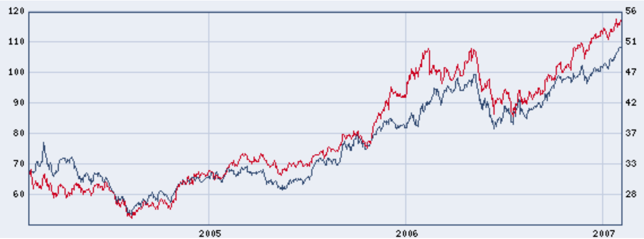
Stochastic framework:

- $[0, T]$ finite time horizon and $(\Omega, \mathcal{F}, \mathbb{P})$ probability space supporting two independent Brownian motions $\tilde{W}$ and $\tilde{W}^\perp$;
- $\mathbb{F} = (\mathcal{F}_s)$ filtration of $\left(\tilde{W}, \tilde{W}^\perp\right)$, $\tilde{\mathbb{F}} = \left(\tilde{\mathcal{F}}_s\right)$ filtration of $\tilde{W}$;
- ρ process valued in $[-1, 1]$;
- $\mathbb{F}$-Brownian motion W is then defined by

$$W := \int \rho\, d\tilde{W} + \int \sqrt{1-\rho^2}\, d\tilde{W}^\perp.$$

Example: Executive stock options

- Manager receives call options on the stock of her company.
- She must not trade the company stock because of legal restrictions.
- She may trade a correlated stock, e.g., shares of another company in the same line of business.



– Deutsche Bank (left-hand scale, in €)
– Credit Suisse (right-hand scale, in €)

Source: www.finanzen.net

Assumptions:

- Correlation ρ bounded away from 1 and -1;
- Drift μ bounded;
- Volatility σ bounded away from 0 and ∞;
- Contingent claim H bounded;
- Zero interest rate;
- Sharpe ratio $\lambda := \frac{\mu}{\sigma}$ and correlation ρ $\tilde{\mathbb{F}}$-optional;
- Utility function $U(x) = -\exp(-\gamma x)$, $x \in \mathbb{R}$, fixed $\gamma > 0$.

Optimization problem:

- Value process

$$V(x_t, q, t) := \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_t(x_t)} \underbrace{\mathbb{E}\Big[-\exp\big(-\gamma(\pi^0_T + \pi^1_T) - \gamma q H\big)\Big|\mathcal{F}_t\Big]}_{=: \varphi(\pi, q, t)}$$

  with π^0 = amount invested in bank account,
  π^1 = amount invested in tradable stock S;

- Indifference value $h(x_t, q, t)$ implicitly defined by

$$V(x_t, 0, t) = V\big(x_t - h(x_t, q, t), q, t\big);$$

- Admissible strategies on $[t, T]$ with initial capital x_t

$$\mathcal{A}_t(x_t) = \left\{ \pi \left| \begin{array}{l} \pi \ \mathbb{F}\text{-optional, self-financing, } \pi^0_t + \pi^1_t = x_t, \\ \int_t^T \left|\pi^0_s\right| ds < \infty \text{ and } \int_t^T \left|\pi^1_s\right|^2 ds < \infty \text{ a.s.,} \\ \Big(\exp\big(-\gamma\big(\pi^0_s + \pi^1_s\big)\big)\Big)_{t \leqslant s \leqslant T} \text{ of class } (D) \end{array} \right. \right\}.$$

Proposition (Bounds for the value process V)

Fix $q \in \mathbb{R}$, $t \in [0,T]$ and x_t bounded $\mathcal{F}_t$-measurable. For every $\pi \in \mathcal{A}_t(x_t)$,

$$\varphi(\pi, q, t) \leqslant -e^{-\gamma x_t} \mathbb{E}_{\mathbb{P}'}\left[\exp\left(-\gamma q H - \frac{1}{2}\int_t^T \lambda_s^2\, ds\right)^{\frac{1}{\overline{\delta}(t)}} \middle| \tilde{\mathcal{F}}_t\right]^{\overline{\delta}(t)}.$$

There exists a $\pi^\star \in \mathcal{A}_t(x_t)$ such that

$$\varphi(\pi^\star, q, t) = -e^{-\gamma x_t} \mathbb{E}_{\mathbb{P}'}\left[\exp\left(-\gamma q H - \frac{1}{2}\int_t^T \lambda_s^2\, ds\right)^{\frac{1}{\underline{\delta}(t)}} \middle| \tilde{\mathcal{F}}_t\right]^{\underline{\delta}(t)},$$

$$\overline{\delta}(t) := \sup_{s \in [t,T]} \left\| \frac{1}{1-\rho_s^2} \right\|_{L^\infty(\mathbb{P})}, \quad \underline{\delta}(t) := \inf_{s \in [t,T]} \frac{1}{\|1-\rho_s^2\|_{L^\infty(\mathbb{P})}}.$$

First step of the proof

$$\varphi(\pi, q, t) = -e^{-\gamma x_t} \mathbb{E}_{\mathbb{P}'}\Bigg[\underbrace{\exp\bigg(\int_t^T (\lambda_s - \gamma \pi_s^1 \sigma_s)\, dW_s'\bigg)}_{\text{controllable by } \pi} \underbrace{\Psi(q,t)}_{\tilde{\mathcal{F}}_T\text{-meas.}} \Bigg| \mathcal{F}_t\Bigg],$$

$$\Psi(q,t) := \exp\bigg(-\gamma q H - \frac{1}{2}\int_t^T \lambda_s^2\, ds\bigg),$$

$$\frac{d\mathbb{P}'}{d\mathbb{P}} := \exp\bigg(-\int_0^T \lambda_s\, dW_s - \frac{1}{2}\int_0^T \lambda_s^2\, ds\bigg),$$

$$W' := W + \int \lambda_s\, ds.$$

Second step of the proof

Write

$$\begin{aligned}
&\Psi(q,t)\\
&= \left(\Psi(q,t)^{\frac{1}{\overline{\delta}(t)}}\right)^{\overline{\delta}(t)}\\
&= \left(\mathbb{E}_{\mathbb{P}'}\left[\Psi(q,t)^{\frac{1}{\overline{\delta}(t)}}\middle|\tilde{\mathcal{F}}_t\right]\exp\left(\int_t^T \zeta_s\,d\tilde{W}'_s - \frac{1}{2}\int_t^T \zeta_s^2\,ds\right)\right)^{\overline{\delta}(t)}
\end{aligned}$$

for the $(\tilde{\mathbb{F}},\mathbb{P}')$-Brownian motion $\tilde{W}' := \tilde{W} + \int \lambda_s\rho_s\,ds$. Then plug this into

$$\varphi(\pi,q,t) = -e^{-\gamma x_t}\mathbb{E}_{\mathbb{P}'}\left[\exp\left(\int_t^T(\lambda_s - \gamma\pi_s^1\sigma_s)\,dW'_s\right)\Psi(q,t)\middle|\mathcal{F}_t\right].$$

Theorem (Explicit indifference valuation formula)

Fix $q \in \mathbb{R}$ and $t \in [0, T]$. There exist $\mathcal{F}_t$-measurable random variables $\delta^{(q)}(t,.), \delta^{(0)}(t,.) : \Omega \to \left[\underline{\delta}(t), \overline{\delta}(t)\right]$ such that we have, for almost all $\omega \in \Omega$ and every bounded $\mathcal{F}_t$-measurable x_t,

$$V(x_t, q, t)(\omega) = -e^{-\gamma x_t(\omega)} \left(\mathbb{E}_{\mathbb{P}'} \left[\Psi(q,t)^{1/\delta} \middle| \tilde{\mathcal{F}}_t \right](\omega) \right)^{\delta} \Bigg|_{\delta = \delta^{(q)}(t,\omega)},$$

$$h(q,t)(\omega) = \frac{1}{\gamma} \log \frac{\left(\mathbb{E}_{\mathbb{P}'} \left[\Psi(0,t)^{1/\delta'} \middle| \tilde{\mathcal{F}}_t \right](\omega) \right)^{\delta'} \Big|_{\delta' = \delta^{(0)}(t,\omega)}}{\left(\mathbb{E}_{\mathbb{P}'} \left[\Psi(q,t)^{1/\delta} \middle| \tilde{\mathcal{F}}_t \right](\omega) \right)^{\delta} \Big|_{\delta = \delta^{(q)}(t,\omega)}},$$

$$\Psi(q,t) := \exp\left(-\gamma q H - \frac{1}{2} \int_t^T \lambda_s^2 \, ds \right).$$

Basic idea of the proof: An interpolation argument

We already know that

$$f\big(\overline{\delta}(t),\omega\big) \leqslant -e^{\gamma x_t(\omega)} V(x_t,q,t)(\omega) \leqslant f\big(\underline{\delta}(t),\omega\big),$$

where the stochastic process $f(.,.) : \big[\underline{\delta}(t),\overline{\delta}(t)\big] \times \Omega \to \mathbb{R}$ is defined by

$$f(\delta,\omega) := \Big(\mathbb{E}_{\mathbb{P}'}\Big[\Psi(q,t)^{1/\delta}\Big|\tilde{\mathcal{F}}_t\Big](\omega)\Big)^{\delta}, \quad (\delta,\omega) \in \big[\underline{\delta}(t),\overline{\delta}(t)\big] \times \Omega.$$

The basic idea is now to apply the intermediate value theorem.

References

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3. Tehranchi, M. (2004). Explicit solutions of some utility maximization problems in incomplete markets. *Stochastic Process. Appl.* **114** 109–125.