Symbiotic Branching

Compact Interface Property

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Part I

(3-Step Motivation of Symbiotic Branching)

Part II

(Compact Support Property)

Part I -

3-Step Motivating of Symbiotic Branching

Step 1

A stochastic evolution equation (SPDE):

Heat Equation with Feller Noise

$$\frac{\partial u}{\partial t}(t,x) \quad = \quad \frac{1}{2}\triangle u(t,x) + \sqrt{u(t,x)}\dot{W}_t$$

with initial condition $u(0,x) = u_0(x)$ *and a white noise* $\dot{W}$*.*

u_0 is non-negative with subexponential growth. (Always: $t \geq 0$, $x \in \mathbb{R}$)

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In the following we want to talk about

- the definition of white noise/weak solutions
- the 'branching' term
- the 'diffusion' term

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White Noise/Weak Solutions

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White noise on a σ-finite measure space $(E, \mathcal{E}, \mu)$ is a mean zero Gaussian process indexed by measurable, finite measure subsets of E with covariance function

$$C_{A,B} = \mu(A \cap B).$$

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$\rightarrow$ weak solution satisfy for $\phi \in C_c^2$ the integral equation

$$\begin{aligned}\int u(t,x)\phi(x)\,dx &= \int u_0(x)\phi(x)\,dx + \frac{1}{2}\int_0^t \int u(s,x) \triangle \phi(x)\,dxds \\ &+ \int_0^t \int \sqrt{u(s,x)}\phi(x)\,\dot{W}(ds,dx),\ a.s.\end{aligned}$$

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From Galton-Watson to Feller's Diffusion

Recall Galton-Watson process $(Z_n)_{n\geq 0}$

- describes number of male (female) members of a family
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There are $Z_0 = x$ males at time 0 and

$$Z_{n+1} = \sum_{k=1}^{Z_n} X_n^k$$

at time $n+1$. The X_n^k are iid with mean 1 (critical) and variance $\sigma^2 < \infty$. X_n^k denotes the number of sons of male k at time n.

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$\rightarrow$ connection of branching and $\sqrt{\dots}$ (note: no space yet)

From SuperBrownianMotion to Stochastic PDEs

- SuperBrownianMotion (SBM) U_t: spatial generalization of Feller's Diffusion
- recall: $U_t(\omega)$ is a measure
- U_t has a density only in spatial dimension 1, i.e. there is $u(t,x)$ such that

$$U_t(A) = \int_A u(t,x)\,dx$$

- $u(t,x)$ is weak solution of

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$\rightarrow$ spatial movement yields additional $\frac{1}{2}\triangle$

Step 2

A stochastic evolution system:

Mutually Catalytic Branching Model

$$\begin{aligned}\frac{\partial u_1}{\partial t}(t,x) &= \frac{1}{2}\triangle u_1(t,x) + \sqrt{u_1(t,x)u_2(t,x)}\dot{W}_t^1 \\ \frac{\partial u_2}{\partial t}(t,x) &= \frac{1}{2}\triangle u_2(t,x) + \sqrt{u_2(t,x)u_1(t,x)}\dot{W}_t^2\end{aligned}$$

with initial condition $u_1(0,x) = u_1(x), u_2(0,x) = u_2(x)$.

$\dot{W}^1, \dot{W}^2$ are independent white noises, u_1, u_2 are non-negative with subexponential growth.

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- the 'interaction' terms

From Catalytic SBM to Mutually Catalytic Branching

Consider a SuperBrownianMotion U_1 where

- branching rate depends on spatial position ($\to$ catalytic)
- branching rate depends on a second species U_2: rate is proportional to the amount of the second species at this site
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- one-way interaction

$\rightarrow$ two-way interaction yields the terms

$$\sqrt{u_1(t,x) u_2(t,x)}, \sqrt{u_2(t,x) u_1(t,x)}$$

Step 3

A stochastic evolution system:

Symbiotic Branching Model (Etheridge/Fleischmann (2004))

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with initial condition $u_1(0,x) = u_1(x), u_2(0,x) = u_2(x)$.

u_1, u_2 are non-negative with subexponential growth and the white noises are now correlated:

$$E[\dot{W}^1(t,x)\dot{W}^2(t',x')] = \varrho\delta(t,t')\delta(x,x')$$

for $\varrho \in [-1,1]$.

We have seen so far that

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- comes from the branching behaviour
- comes from the movement in space
- comes from the mutually catalytic behaviour
- additional correlation

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goals: any properties that change their behaviour with the correlation parameter ϱ

Part II -

Compact Interface Property for Symbiotic Branching

From now on we restrict to $u_1 = \mathbb{1}_{\mathbb{R}^+}$, $u_2 = \mathbb{1}_{\mathbb{R}^-}$. For a realization of (u_1, u_2) the interface is defined as:

Interface

$$Int_t(\omega) = cl\{x | u_1(t,x)(\omega) u_2(t,x)(\omega) \neq 0\}$$

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Question

Does the compact interface property hold?

i.e. is

$$\bigcup_{t \leq T} Int_t$$

a.s. compact?

Answer (Etheridge/Fleischmann (2004))

- *For each* $\varrho \in [-1,1]$ *there is a set* Ω_1 *of measure 1 such that for each* $\omega \in \Omega_1$ *and* $T > 0$

$$\bigcup_{t \leq T} Int_t(\omega)$$

is compact.

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is compact.

- *More precisely, there is a positive constant* c *and a random time* T_0 *such that*

$$\bigcup_{t \le T} Int_t(\omega) \subset [-cT, cT]$$

for each $T \geq T_0(\omega)$.

Conjecture

There is a constant c, a random time T_0 and a set Ω_1 of measure 1 such that for $\varrho < 0$ each $\omega \in \Omega_1$

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(for $\varrho = -1$ and $u_1(t,x) + u_2(t,x) = 1$ (heat-equation with 'Wright-Fisher noise') Tribe (1995) proved propagation with $\sqrt{T}$.)

How to prove the Conjecture

The proof of Etheridge/Fleischmann is based on estimates of

$$\mathbb{E}[u_1(t,x)u_2(t,x)]^q.$$

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- they only considered a uniform estimate in ϱ
- if we can manage to give much better bounds, the rest of their proof implies the conjecture

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techniques to estimate the moments:

- perturbed duality
- moment equations

Using Generators to Prove Duality

Given markov processes X_t, N_t with statespaces X, N, generators Ω^X, Ω^N and a duality function $H : X \times N \to \mathbb{R}$ bounded.

Proposition (duality)

$$\begin{aligned} & \Omega^X H(\cdot, n)(x) = \Omega^N H(x, \cdot)(n) \\ \Rightarrow \quad & \mathbb{E}^{X_0}[H(X_t, N_0)] = \mathbb{E}^{N_0}[H(X_0, N_t)] \end{aligned}$$

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Further, for bounded f

Proposition (perturbed duality)

$$\begin{aligned} & \Omega^X H(\cdot, n)(x) = \Omega^N H(x, \cdot)(n) + f(n) H(x, \cdot)(n) \\ \Rightarrow \quad & \mathbb{E}^{X_0}[H(X_t, N_0)] = \mathbb{E}^{N_0}\left[H(X_0, N_t) exp(\int_0^t f(N_s)\, ds)\right] \end{aligned}$$

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$\to$ nice generators are nice to prove duality

A Perturbated Duality in Symbiotic Branching

Etheridge/Fleischmann:

- used a dual process to estimate the moments
- used generators to prove perturbed duality

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where

$$(\mathbb{1}_{\mathbb{R}^+},\mathbb{1}_{\mathbb{R}^-})^{N_t}=\begin{cases}1, \text{ all } B_t^{red}\geq 0, \text{ all } B_t^{blue}\leq 0\\ 0, \textit{otherwise}\end{cases}$$

goal: use the dual process to prove better moment bounds

Thank You For Listening

Existence and Uniqueness

Existence

There is a weak solution with paths a.s. in $C(\mathbb{R}^+, C_{tem})$.

Uniqueness in Law

Uniqueness in law is known.

Pathwise Uniqueness

Pathwise uniqueness is unknown. But: known for certain coloured noises